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# BASIC EQUATIONS FOR SUSPENSION FLOWS WITH INTERPHASE MASS TRANSFER*

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ABSTRACT

In the case of suspension flows, the rate of interphase momentum transfer $\boldsymbol{M}_k$ and that of interphase energy transfer $E_k$, which were expressed as a sum of infinite discontinuities by Ishii, have been reduced to the sum of several terms which have concise physical significance. $\boldsymbol{M}_k$ is composed of the following terms: (i) the momentum carried by the interphase mass transfer; (ii) the interphase drag force due to the relative motion between phases; (iii) the interphase force produced by the concentration gradient of the dispersed phase in a pressure field. And $E_k$ is composed of the following four terms, that is, the energy carried by the interphase mass transfer, the work produced by the interphase forces of the second and third parts above, and the heat transfer between phases.

It is concluded from the results that (i) the term, $(-\alpha_k \nabla p)$, which is related to the pressure gradient in the momentum equation, can be derived from the basic conservation laws without introducing the "shared-pressure presumption"; (ii) the mean velocity of the action point of the interphase drag is the mean velocity of the interface displacement, $\bar{\boldsymbol{v}}_i$. It is approximately equal to the mean velocity of the dispersed phase, $\bar{\boldsymbol{v}}_d$. Hence the work terms produced by the drag forces are $\boldsymbol{f}_{dc} \cdot \bar{\boldsymbol{v}}_d$, and $\boldsymbol{f}_{cd} \cdot \bar{\boldsymbol{v}}_d$, respectively, with $\bar{\boldsymbol{v}}_i$ not being replaced by the mean velocity of the continuous phase, $\bar{\boldsymbol{v}}_c$; (iii) by analogy, the terms of the momentum transfer due to phase change are $\bar{\boldsymbol{v}}_d \Gamma_c$ and $\bar{\boldsymbol{v}}_d \Gamma_d$, respectively; (iv) since the transformation between explicit heat and latent heat occurs in the process of phase change, the algebraic sum of the heat transfer between phases is not equal to zero. $Q_{ic}$ and $Q_{id}$ are composed of the explicit heat and latent heat, so that the sum $(Q_{ic} + Q_{id})$ is equal to zero.

**Keywords: interactions between phases, suspension flows, shared-pressure presumption.**

## I. INTRODUCTION

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Some of these differences arose from different physical models of two-phase flows. while others were produced by confusion in concepts.

Many authors[8,9] started from the equations of fluid mechanics, extended them and obtained equations for describing two-phase flows. Although the equations of fluid mechanics were reasonably established and proved by many experimental data, the validity of this extension has not been demonstrated. In fact, different authors used different approaches and achieved different results. At least, it is certainly impossible that all the extension approaches are correct. Because of the difficulties encountered in the measurements of two-phase flows, we cannot prove the validity of these equations by experiments.

Many other authors[1-3] have built up the equations for two-phase flows based on a quite rigorous mathematical derivation, starting from the conservation laws. These equations are the most reliable at present.

Bouré and Delhaye[1] have chosen a control volume element $\delta v$, which contains the media of two phases, as the system to be investigated. The conservation equations (of mass, momentum, energy and other quantities) for $\delta v$ were built up in integral form with the effects of the discontinuous surfaces (interfaces) on the conservative quantities included. They obtained, by using Leibnitz's rule and Gauss's law, the instantaneous local conservation equations of each phase in differential form and the conservation relation suitable to these interfaces. Only within each phase, can these conservation equations be applied and are physical quantities continuous and differentiable. However, discontinuity of the quantities will arise when crossing these interfaces. There are many discontinuous surfaces in motion. Therefore, each quantity in the fluid field comprises fluctuating components which vary rapidly with time and position. By averaging, the high-frequency fluctuations can be smoothed out in order to make the physical quantities of each phase continuous and differentiable. So each phase can be regarded as a pseudofluid which exists in the whole fluid field. Bouré and Delhaye's approach is essentially similar to Ishii's[2] but with surface tension and phase change neglected. It greatly simplifies the derivation procedures. Some other authors, such as Drew et al.[3], used similar methods in their researches.

Every mean conservation equation (of mass, momentum and energy) for two-phase flows obtained by the above methods contains an interaction term between phases, which is expressed as a sum of infinite discontinuities. It is not convenient to use them and to compare them directly with other types of equations for two-phase flows.

In this paper, we start from Ishii's equations and the general expressions for the interaction between phases and derive the formulae for the rates of interphase momentum transfer $\boldsymbol{M}_k$ and that of interphase energy transfer $E_k$ which have concise physical significance. We obtain a set of equations for the suspension flows, which is convenient to use and easily compared with other types of equations. Finally, we analyze and discuss in detail the physical significance of each interaction term between phases.

## II. Equations for General Two-Phase Flows

Using the approach of time averaging, Ishii presented a set of equations for two-phase flows with the effects of phase change and surface tension included. In this paper, the surface tension is neglected in order to simplify the problem.

In Ishii's model, the equation for general two-phase flows is[2]

$$\frac{\partial}{\partial t}(\alpha_k\bar{\bar{\rho}}_k\bar{\psi}_k)+\nabla\cdot(\alpha_k\bar{\bar{\rho}}_k\bar{v}_k\bar{\psi}_k)=-\nabla\cdot[\alpha_k(\bar{\bar{J}}_k+J_k^T)]$$
$$+\alpha_k\bar{\bar{\rho}}_k\bar{\phi}_k+I_k,\ (k=c,d),\tag{2.1}$$

where the interaction term between phases $I_k$ is

$$I_k\equiv-\sum_j\{[\dot{m}_k\psi_{ki}+n_k\cdot J_{ki}]/L_j\},\quad (k=c,\ d)\tag{2.2}$$

and the mass flux across interfaces $\dot{m}_k$ is

$$\dot{m}_k\equiv n_k\cdot\rho_{ki}(v_{ki}-v_i),\quad (k=c,d).\tag{2.3}$$

The subscripts $k=c$ and $k=d$ denote the continuous and dispersed phases, respectively. Eq. (2.1) is the generalized form of the conservation equations of mass, momentum and energy. For various kinds of the conservation equations, $\psi_k$, $J_k$ and $\phi_k$ have their own expressions (see Table 1). $\psi_k$ represents the mass, momentum and energy (including kinetic energy $v_k^2/2$ and internal energy $u_k$) per unit mass in phase $k$, $J_k$ the flux of $\psi_k$ flowing out of phase $k$. $J_k^T$ is the flux of turbulent transportation corresponding to $J_k$. $\rho_k\phi_k$ is the source term of $\psi_k$, $\alpha_k$ is the time fraction occupied by phase $k$ at a point. It can be proved that $\alpha_k$ is also equal to the probability occupied by phase $k$ of a volume element, an area element, or a line element containing the point. $v_k$ and $\rho_k$ are the velocity and density of phase $k$, respectively. A double line "=" over a letter $F_k$ (such as $F=\rho$, $p$, $q$). denotes the mean value of a physical quantity $F_k$ within phase $k$. If a physical quantity has an even distribution in phase $k$, then $\bar{\bar{F}}_k=F_k$. The time-averaged value of the physical quantity $F_k$ can be expressed as $\alpha_k\bar{\bar{F}}_k$. $\alpha_k\bar{\bar{\rho}}_k,\alpha_k\bar{\bar{p}}_k\cdots$ are defined as partial density, partial pressure $\cdots$ of phase $k$. A single line" — "over a letter $\varphi_k$ (such as $\varphi=\psi$, $v$, $u$, etc.) represents the mass-weighted mean value of a physical quantity $\varphi_k$ of phase $k$,

$$\bar{\varphi}_k\equiv\overline{\overline{\rho_k\varphi_k}}/\bar{\bar{\rho}}_k,\tag{2.4}$$

$n_k$ is a unit vector drawn normal out of phase $k$. Obviously, we have

$$n_c=-n_d.\tag{2.5}$$

$v_i$ is the velocity vector of interface displacement and $v_{in}$ is its normal component.

$L_j \equiv v_{in}\Delta t$, where $\Delta t$ is the averaging time interval. $\sum_j$ denotes the summation of all the discontinuities through the observation point within $\Delta t$. The subscript $ki$ (i. e. $v_{ci}$, $\rho_{di}$, etc.) represents the value of phase $k$ at the interface. $\boldsymbol{b}_k$ is the external body force acting on phase $k$. $\boldsymbol{T}_k$ is the stress tensor in phase $k$, including the hydrostatic pressure $(-p_k\boldsymbol{I})$ and the viscous stress $\boldsymbol{\tau}_k$. Here $\boldsymbol{I}$ is the unit tensor. The body heating $Q_k$ arises from an external energy source. $\boldsymbol{q}_k$ is the heat flux in phase $k$. $\Gamma_k$, $\boldsymbol{M}_k$ and $E_k$ are the rates of the interphase transfer of mass momentum and energy, respectively.

**Table 1**

| Conservation Equation of the Following | $\psi_k$ | $J_k$ | $\phi_k$ | $I_k$ | $\bar{\psi}_k$ |
|---|---|---|---|---|---|
| Mass | 1 | 0 | 0 | $\Gamma_k$ | 1 |
| Momentum | $\boldsymbol{v}_k$ | $-\boldsymbol{T}_k$ | $\boldsymbol{b}_k$ | $\boldsymbol{M}_k$ | $\bar{\boldsymbol{v}}_k$ |
| Energy | $\frac{1}{2}v_k^2 + u_k$ | $-\boldsymbol{T}_k\cdot\boldsymbol{v}_k + \boldsymbol{q}_k$ | $\boldsymbol{b}_k\cdot\boldsymbol{v}_k + Q_k/\rho_k$ | $E_k$ | $\frac{1}{2}\bar{v}_k^2 + \frac{1}{2}\overline{v_k'^2} + \bar{u}_k$ |

The conservation relation for every interface element is given by

$$\sum_{k=c,d}(\dot{m}_k\psi_{ki} + \boldsymbol{n}_k\cdot\boldsymbol{J}_{ki}) = 0. \tag{2.6}$$

From Eqs. (2.2) and (2.6), we obtain

$$I_c + I_d = 0. \tag{2.7}$$

Using the definition of the averaging and that of the time fraction, we have[2]

$$\partial\alpha_k/\partial t = \sum_j(\boldsymbol{n}_k\cdot\boldsymbol{v}_i/L_j), \quad (k = c,\ d), \tag{2.8}$$

$$\nabla\alpha_k = -\sum_j(\boldsymbol{n}_k/L_j), \quad (k = c,\ d). \tag{2.9}$$

## III. Expression for Interaction Terms $\boldsymbol{M}_k$ and $E_k$

In this section, we will discuss the discontinuities of physical quantities on interfaces. For this purpose, we introduce the velocity $\boldsymbol{v}_i$, stress tensor $\boldsymbol{T}_i$ and heat flux $\boldsymbol{q}_i$ of the interfaces, utilize them to express the discontinuities of velocity, stress tensor and heat flux at the interfaces and obtain some relations at the interfaces which are simple in form and concise in physical significance. Then the averaging rule of physical quantities along the interfaces will be introduced. Finally, we obtain the quite simple expressions for the interaction terms between phases, $\boldsymbol{M}_k$ and $E_k$.

If there is no slip[1)] between the tangential velocity of fluid $\boldsymbol{v}_{ci,t}$ and that of a particle $\boldsymbol{v}_{di,t}$ at an interface element (it can be satisfied in general two-phase flows), then the two tangential velocities are equal to each other and the tangential velocity of the interface displacement can be defined as

$$\boldsymbol{v}_{ci,t} = \boldsymbol{v}_{di,t} \equiv \boldsymbol{v}_{i,t}. \tag{3.1}$$

Using Eq. (2.3), the velocity discontinuity on the interface can be obtained

$$\boldsymbol{v}_{ki} - \boldsymbol{v}_i = (\dot{m}_k/\rho_{ki})\boldsymbol{n}_k, \quad (k = c, d). \tag{3.2}$$

It is assumed that all the interfaces are geometric surfaces which have neither mass nor surface tension. Hence the tangential stresses $\boldsymbol{T}_{di,nt}$ and $\boldsymbol{T}_{ci,nt}$ on the two sides of an interface are equal to each other, and the tangential stress of the interface $\boldsymbol{T}_{i,nt}$ can be defined as

$$\boldsymbol{T}_{ci,nt} = \boldsymbol{T}_{di,nt} \equiv \boldsymbol{T}_{i,nt}. \tag{3.3}$$

From Eq. (2.6), the normal stress $T_{i,nn}$ of the interface can be defined as

$$\dot{m}_k\boldsymbol{n}_k \cdot \boldsymbol{v}_{ki} - T_{ki,nn} \equiv \dot{m}_k\boldsymbol{n}_k \cdot \boldsymbol{v}_i - T_{i,nn}, \quad (k = c,\ d). \tag{3.4}$$

Using Eqs. (3.2), (3.3) and (3.4) we can obtain the discontinuity of the stress tensor on the interface

$$\boldsymbol{T}_{ki} - \boldsymbol{T}_i = B_k\boldsymbol{I}, \quad (k = c,\ d), \tag{3.5}$$

$$B_k \equiv \dot{m}_k^2/\rho_{ki}, \quad (k = c,\ d). \tag{3.6}$$

By analogy, the heat flux of an interface $\boldsymbol{q}_i$ can be introduced. From Eq. (2.6), the normal component of $\boldsymbol{q}_i$ can be defined as

$$\begin{aligned} &\dot{m}_k\left(\frac{1}{2}v_{ki}^2 + u_{ki}\right) - \boldsymbol{n}_k \cdot \boldsymbol{T}_{ki} \cdot \boldsymbol{v}_{ki} + \boldsymbol{n}_k \cdot \boldsymbol{q}_{ki} \\ &\equiv \dot{m}_k\left[\frac{1}{2}v_i^2 + (\boldsymbol{v}_{ki} - \boldsymbol{v}_i)\cdot\boldsymbol{v}_i\right] - \boldsymbol{n}_k \cdot \boldsymbol{T}_{ki} \cdot \boldsymbol{v}_i + \boldsymbol{n}_k \cdot \boldsymbol{q}_i \\ &= \frac{1}{2}v_i^2\dot{m}_k - \boldsymbol{n}_k \cdot \boldsymbol{T}_i \cdot \boldsymbol{v}_i + \boldsymbol{n}_k \cdot \boldsymbol{q}_i, \quad (k = c, d). \end{aligned} \tag{3.7}$$

The second equal sign in the above formula is obtained by using Eqs. (3.2) and (3.5). From the definition of $\boldsymbol{q}_i$, we can obtain the normal discontinuity of the heat flux on the interface,

$$\boldsymbol{n}_k \cdot (\boldsymbol{q}_{ki} - \boldsymbol{q}_i) = -\dot{m}_k\left[\frac{1}{2}|\boldsymbol{v}_{ki} - \boldsymbol{v}_i|^2 + u_{ki} - T_{ki,nn}/\rho_{ki}\right]$$

1) In gasdynamics, the difference between the tangential velocity of a wall and that of gas adjacent to the wall is defined as slip velocity. It is directly proportional to a molecular free path and the gas velocity gradient there. However, in two-phase flows, the difference of mean velocities of the two phases is defined as slip velocity too. These are two completely different concepts. The slip used in this paper is the former. In order to avoid confusion with the slip in gasdynamics, we suggest that "slip velocity" and "slip (phenomenon)" in two-phase flows can be replaced by "difference of the velocities between phases" and "non-equilibrium (phenomenon) of the velocities between phases", respectively.

$$= -\dot{m}_k h_{ki} + \delta_k, \quad (k = c, d), \tag{3.8}$$

where

$$\delta_k \equiv \dot{m}_k (T_{ki,nn} + p_{ki})/\rho_{ki} - \frac{1}{2} \dot{m}_k^3/\rho_{ki}^2. \tag{3.9}$$

In general, $\delta_k$ can be neglected because it is very small. $h_k (\equiv u_k + p_k/\rho_k)$ is the enthalpy of phase $k$.

Eq. (3.7) represents the total energy flux (related to the laboratory coordinates) flowing out of phase $k$ and into the interface. The left side of Eq. (3.7) shows that the total energy flux is composed of the following three parts: (i) the internal energy $u_{ki}$ and kinetic energy $\frac{1}{2}v_{ki}^2$ carried by the mass flux $\dot{m}_k$, (ii) the work produced by the stress of phase $k$, $(-\boldsymbol{n}_k \cdot \boldsymbol{T}_{ki} \cdot \boldsymbol{v}_{ki})$, (iii) the heat flux $(\boldsymbol{n}_k \cdot \boldsymbol{q}_{ki})$. The right side of Eq. (3.7) indicates that the total energy flux can also be regarded as the sum of the following three terms: (i) the kinetic energy (of the motion of following) carried by the mass flux $\left(\frac{1}{2} v_i^2 m_i\right)$, (ii) the work produced by the stress of the interface, $(-\boldsymbol{n}_k \cdot \boldsymbol{T}_i \cdot \boldsymbol{v}_i)$, (iii) the total energy flux in the coordinates relative to the observer moving along with the interface, $(\boldsymbol{n}_k \cdot \boldsymbol{q}_i)$. From Eq. (3.8), we can see that the total energy flux in the relative coordinates $(\boldsymbol{n}_k \cdot \boldsymbol{q}_i)$ is composed of the explicit heat flux $(\boldsymbol{n}_k \cdot \boldsymbol{q}_{ki})$, the internal energy flux $(u_{ki}\dot{m}_k)$, the kinetic energy flux $\left(\frac{1}{2}|\boldsymbol{v}_{ki} - \boldsymbol{v}_i|^2 \dot{m}_k\right)$ and the work produced by the stress $[-\boldsymbol{n}_k \cdot \boldsymbol{T}_{ki} \cdot (\boldsymbol{v}_{ki} - \boldsymbol{v}_i)]$. In fact, the sum of the latter three parts is the total enthalpy in the relative coordinates $\left(h_{ki} + \frac{1}{2}|\boldsymbol{v}_{ki} - \boldsymbol{v}_i|^2\right)$ multiplied by $\dot{m}_k$ and can be named the latent heat flux.

The advantage of using the second expression of the total energy flux related to the laboratory coordinates (the right side of Eq. (3.7)) is that every sum of the corresponding terms of phase $c$ and phase $d$ is zero. That is

$$\begin{cases} \frac{1}{2} v_i^2 \dot{m}_c + \frac{1}{2} v_i^2 \dot{m}_d = 0, \\ -\boldsymbol{n}_c \cdot \boldsymbol{T}_i \cdot \boldsymbol{v}_i - \boldsymbol{n}_d \cdot \boldsymbol{T}_i \cdot \boldsymbol{v}_i = 0, \\ \boldsymbol{n}_c \cdot \boldsymbol{q}_i + \boldsymbol{n}_d \cdot \boldsymbol{q}_i = 0. \end{cases} \tag{3.10}$$

Such an advantage no lonaer exists when using the first expression. For some of these terms, the algebraic sum of the corresponding terms of phase $c$ and phase $d$ is not equal to zero. For example, from Eq. (3.8) we have

$$\boldsymbol{n}_c \cdot \boldsymbol{q}_{ci} + \boldsymbol{n}_d \cdot \boldsymbol{q}_{di} \approx -\dot{m}_d (h_{di} - h_{ci}). \tag{3.11}$$

This formula indicates that the transformation between the explicit heat and the latent heat occurs in the process of phase change.

When there is no phase change $(\dot{m}_c = \dot{m}_d = 0)$, we can obtain from Eqs. (3.2)

(3.5) and (3.8)

$$\begin{cases} \boldsymbol{v}_{ci} = \boldsymbol{v}_{di} = \boldsymbol{v}_i, \\ \boldsymbol{T}_{ci} = \boldsymbol{T}_{di} = \boldsymbol{T}_i, \\ \boldsymbol{q}_{ci} = \boldsymbol{q}_{di} = \boldsymbol{q}_i, \end{cases} \tag{3.12}$$

because $\boldsymbol{q}_{ci,t}, \boldsymbol{q}_{di,t}$ and $\boldsymbol{q}_{i,t}$ are obviously equal to each other.

The mean value of a quantity $F_k$ along all the interfaces in $\delta v$ is the same as the mean value for all the discontinuities in $\Delta t$. It is defined as

$$\bar{\bar{F}}_{ki} \equiv \sum_j (F_{ki}/L_j) \Big/ \sum_j L_j^{-1} = L_s \sum_j (F_{ki}/L_j), \tag{3.13}$$

where the total area of the interfaces in unit volume is

$$L_s^{-1} \equiv \sum_j L_j^{-1}. \tag{3.14}$$

For suspension flows with particle radius $r$, it is not difficult to prove that $L_s^{-1} = 3\alpha_d/r$.

Let $\bar{\bar{\boldsymbol{T}}}_i$ be the average value of the stress tensor of the interfaces, $\boldsymbol{T}_i'$ the part deviating from the average value. Using Eq. (3.5), we can obtain

$$\begin{aligned} &\boldsymbol{T}_i = \bar{\bar{\boldsymbol{T}}}_i + \boldsymbol{T}_i', \\ &\boldsymbol{T}_{ki} = B_k\boldsymbol{I} + \bar{\bar{\boldsymbol{T}}}_i + \boldsymbol{T}_i', \\ &\bar{\bar{\boldsymbol{T}}}_i' = 0, \\ &\bar{\bar{\boldsymbol{T}}}_{ki} = \bar{\bar{B}}_k I + \bar{\bar{\boldsymbol{T}}}_i. \end{aligned} \tag{3.15}$$

Let $(-p_\beta \boldsymbol{I})$ be the part of the hydrostatic pressure of tensor $\boldsymbol{T}_\beta$, and $\boldsymbol{\tau}_\beta$ the part of viscous stress, we have

$$\boldsymbol{T}_\beta = -p_\beta \boldsymbol{I} + \boldsymbol{\tau}_\beta, \quad (\beta = c, d, i, ci, di). \tag{3.16}$$

Using Eqs. (3.15), (2.2), (3.2), (3.6), (2.9) and (2.2), we have

$$\begin{aligned} \boldsymbol{M}_k &= -\sum_j \{[\dot{m}_k \boldsymbol{v}_i + \bar{\bar{p}}_i \boldsymbol{n}_k - \boldsymbol{n}_k \cdot (\boldsymbol{T}_i' + \bar{\bar{\boldsymbol{\tau}}}_i)]/L_j\} \\ &= \bar{\bar{\boldsymbol{v}}}_i \Gamma_k + \bar{\bar{p}}_i \nabla \alpha_k + \boldsymbol{f}_{k'k}, \quad (k = c \text{ or } d,\ k' = d \text{ or } c), \end{aligned} \tag{3.17}$$

where the drag force between phases $\boldsymbol{f}_{k'k}$ is defined as

$$\begin{aligned} \boldsymbol{f}_{cd} &\equiv \sum_j \{[\boldsymbol{n}_d \cdot (\boldsymbol{T}_i' + \bar{\bar{\boldsymbol{\tau}}}_i) - \dot{m}_d \boldsymbol{v}_i']/L_j\} \\ &= -\sum_j \{[\boldsymbol{n}_c \cdot (\boldsymbol{T}_i' + \bar{\bar{\boldsymbol{\tau}}}_i) - \dot{m}_c \boldsymbol{v}_i']/L_j\} \equiv -\boldsymbol{f}_{dc}, \end{aligned} \tag{3.18}$$

$$\boldsymbol{v}_i' \equiv \boldsymbol{v}_i - \bar{\bar{\boldsymbol{v}}}_i. \tag{3.19}$$

Obviously, $\boldsymbol{M}_c$ and $\boldsymbol{M}_d$ expressed by Eq. (3.17) still satisfy $\boldsymbol{M}_c + \boldsymbol{M}_d = 0$.

Using Eqs. (3.7), (2.2), (3.15), (3.16), (3.18) and (2.8), we obtain

$$E_k = -\sum_j \left\{\left[\frac{1}{2}\dot{m}_k v_i^2 + \boldsymbol{n}_k \cdot \boldsymbol{q}_i - \boldsymbol{n}_k \cdot (\boldsymbol{T}'_i + \bar{\bar{\boldsymbol{\tau}}}_i) \cdot \boldsymbol{v}_i + \boldsymbol{n}_k \cdot (\bar{\bar{p}}_i \boldsymbol{I}) \cdot \boldsymbol{v}_i \right] / L_j\right\}$$

$$= \frac{1}{2}\overline{\overline{v_i^2}}\Gamma_k + Q_{ik} + \boldsymbol{f}_{k'k} \cdot \bar{\bar{\boldsymbol{v}}}_i - \bar{\bar{p}}_i \frac{\partial \alpha_k}{\partial t}$$

$$= \left(\frac{1}{2}\bar{\bar{v}}_i^2 + \frac{1}{2}\overline{\overline{v_i'^2}} + \bar{\bar{h}}_{ki}\right)\Gamma_k + Q_{k'k} + \boldsymbol{f}_{k'k} \cdot \bar{\bar{\boldsymbol{v}}}_i - \bar{\bar{p}}_i \frac{\partial \alpha_k}{\partial t},$$

$$\begin{pmatrix} k = c \text{ or } d, \\ k' = d \text{ or } c, \end{pmatrix} \tag{3.20}$$

where the heat flux between phases $Q_{ik}$ (with the explicit heat and latent heat included) and $Q_{k'k}$ (with the explicit heat only) are, respectively, defined as

$$Q_{ik} \equiv \sum_j \left\{\left[-\boldsymbol{n}_k \cdot \boldsymbol{q}_i + \boldsymbol{n}_k \cdot (\boldsymbol{T}'_k + \bar{\bar{\boldsymbol{\tau}}}_i) \cdot \boldsymbol{v}'_i - \frac{1}{2}\dot{m}_k(v_i'^2 - \overline{\overline{v_i'^2}})\right] / L_i \right\} = Q_{k'k} + \bar{\bar{h}}_{ki}\Gamma_k, \quad (k = c, d); \tag{3.21}$$

$$Q_{k'k} \equiv \sum_j \left\{\left[-\boldsymbol{n}_k \cdot \boldsymbol{q}_{ki} + \boldsymbol{n}_k \cdot (\boldsymbol{T}'_k + \bar{\bar{\boldsymbol{\tau}}}_i) \cdot \boldsymbol{v}'_i - \frac{1}{2}\dot{m}_k(v_i'^2 - \overline{\overline{v_i'^2}}) - \dot{m}_k(h_{ki} - \bar{\bar{h}}_{ki}) + \tau_{ki,nn}\dot{m}_k/\rho_{ki} - \frac{1}{2}\dot{m}_k^3/\rho_{ki}^2\right] / L_j \right\}$$

$$\approx \sum_j (-\boldsymbol{n}_k \cdot \boldsymbol{q}_{ki}/L_j), \qquad \begin{pmatrix} k = c \text{ or } d \\ k' = d \text{ or } c \end{pmatrix}. \tag{3.22}$$

Obviously,

$$\begin{cases} Q_{ic} + Q_{id} = 0, \\ Q_{cd} + Q_{dc} = (\bar{\bar{h}}_{ci} - \bar{\bar{h}}_{di})\Gamma_d, \\ E_c + E_d = 0. \end{cases} \tag{3.23}$$

In general, Eq. $(3.23)_2$ is not equal to zero except $\Gamma_k = 0$. In analogy to Eq. (3.11), Eq. $(3.23)_2$ represents the transformation between the explicit heat and latent heat in the process of phase change.

Apart from the assumptions included in Ishii's Eq. (2.11), some additional assumptions have been introduced in this section. They are mainly as follows. There is no slip between the tangential velocities of a particle and of fluid adjacent to the interface. The interface is a geometric surface with neither mass nor surface tension. Most of the preceding results are correct for various two-phase flows where these assumptions are valid. However, these results are still too complicated to solve practical problems. In the next section we will deal with the suspension flows. In most cases of these flows, some of the terms are small and can be neglected. As a result, we can obtain a set of equations for suspension flows which is suitable for solving practical problems.

## IV. Equations for Suspension Flows

Substituting Eqs. (3.17) and (3.20) into Eq. (2.1), we can obtain equations for suspension flows with phase change ($k=c$ or $d$, $k'=d$ or $c$):

$$\frac{\partial}{\partial t}(\alpha_k\bar{\bar{\rho}}_k)+\nabla\cdot(\alpha_k\bar{\bar{\rho}}_k\bar{\boldsymbol{v}}_k)=\Gamma_k, \tag{4.1}$$

$$\frac{\partial}{\partial t}(\alpha_k\bar{\bar{\rho}}_k\bar{\boldsymbol{v}}_k)+\nabla\cdot(\alpha_k\bar{\bar{\rho}}_k\bar{\boldsymbol{v}}_k\bar{\boldsymbol{v}}_k)=-\alpha_k\nabla p_k+\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{\tau}}}_k+\boldsymbol{T}_k^T)]$$
$$+(\bar{\bar{p}}_i-\bar{\bar{p}}_k)\nabla\alpha_k+\alpha_k\bar{\bar{\rho}}_k\boldsymbol{b}_k+\boldsymbol{f}_{k'k}+\bar{\bar{v}}_i\Gamma_k, \tag{4.2}$$

$$\frac{\partial}{\partial t}\left[\alpha_k\bar{\bar{\rho}}_k\left(\bar{u}_k+\frac{1}{2}\overline{v_k'^2}+\frac{1}{2}\bar{v}_k^2\right)\right]+\nabla$$
$$\cdot\left[\alpha_k\bar{\bar{\rho}}_k\bar{\boldsymbol{v}}_k\left(u_k+\frac{1}{2}\overline{v_k'^2}+\frac{1}{2}\bar{v}_k^2\right)\right]=-\nabla\cdot(\alpha_k\bar{\bar{p}}_k\bar{\boldsymbol{v}}_k)$$
$$+\nabla\cdot[\alpha_k(\bar{\bar{\rho}}_k+\boldsymbol{T}_k^T)\cdot\bar{\boldsymbol{v}}_k]-\nabla\cdot[\alpha_k(\bar{\bar{q}}_k+\boldsymbol{q}_k^T)]$$
$$+\alpha_k\bar{\bar{\rho}}_k\boldsymbol{b}_k\cdot\bar{\boldsymbol{v}}_k+\alpha_k\bar{\bar{Q}}_k+\boldsymbol{f}_{k'k}\cdot\bar{\bar{\boldsymbol{v}}}_i+Q_{k'k}-\bar{\bar{p}}_i\frac{\partial\alpha_k}{\partial t}$$
$$+\left(\bar{\bar{h}}_{ki}+\frac{1}{2}\overline{\overline{v_i'^2}}+\frac{1}{2}\bar{\bar{v}}_i^2\right)\Gamma_k. \tag{4.3}$$

For the phase of constant density, using the continuity equation (4.1), we can reduce some of the terms in the energy equation (4.3) as

$$-\nabla\cdot(\alpha_k\bar{\bar{p}}_k\bar{\boldsymbol{v}}_k)-\bar{\bar{p}}_i\frac{\partial\alpha_k}{\partial t}+\bar{\bar{h}}_{ki}\Gamma_k=-\alpha_k\bar{\boldsymbol{v}}_k\cdot\nabla\bar{\bar{p}}_k$$
$$-(\bar{\bar{p}}_i-\bar{\bar{p}}_k)\frac{\partial\alpha_k}{\partial t}+(\bar{\bar{h}}_{ki}-\bar{h}_k)\Gamma_k+\bar{u}_k\Gamma_k. \tag{4.4}$$

If the density $\rho_k$ is constant, the corresponding three terms in the other phase $k'$ can be reduced as

$$-\nabla\cdot(\alpha_{k'}\bar{\bar{p}}_{k'}\bar{\boldsymbol{v}}_{k'})-\bar{\bar{p}}_i\frac{\partial\alpha_{k'}}{\partial t}+\bar{\bar{h}}_{k'i}\Gamma_{k'}=-\alpha_{k'}\boldsymbol{v}_{k'}\cdot\nabla\bar{\bar{p}}_{k'}$$
$$-(\bar{\bar{p}}_i-\bar{\bar{p}}_{k'})\frac{\partial\alpha_{k'}}{\partial t}+(\bar{\bar{h}}_{k'i}-\bar{\bar{h}}_{k'})\Gamma_{k'}+\bar{u}_{k'}\Gamma_{k'}+\Delta, \tag{4.5}$$

where

$$\Delta\equiv p_{k'}\left[\frac{\Gamma_k}{\bar{\bar{\rho}}_{k'i}}-\nabla\cdot(\alpha_{k'}\bar{\boldsymbol{v}}_{k'})+\frac{\Gamma_k}{\rho_k}-\nabla\cdot(\alpha_k\bar{\boldsymbol{v}}_k)\right]$$
$$+[\overline{\overline{p_{k'}/\rho_{k'}}}-\bar{\bar{p}}_{k'}/\bar{\bar{\rho}}_{k'}]\Gamma_{k'}. \tag{4.6}$$

If both the phases $c$ and $d$ are incompressible, then $\Delta=0$.

For the suspension flows, it is valid in general that

$$\bar{\bar{\boldsymbol{v}}}_{di} \approx \bar{\boldsymbol{v}}_d, \tag{4.7}$$

$$\bar{\bar{\vartheta}}_{di} \approx \bar{\vartheta}_d, \tag{4.8}$$

(see Appendix). This shows that the mean value of the velocity and temperature of the particles averaged along the interfaces, $\bar{\bar{\boldsymbol{v}}}_{di}$ and $\bar{\bar{\vartheta}}_{di}$, are approximately equal to those averaged in the control volume (it is equal to the value of time averaging), $\bar{\boldsymbol{v}}_d$ and $\bar{\vartheta}_d$. For phase $c$, there are no such relations as Eqs. (4.7) and (4.8). If there is no phase change ($\dot{m}_c = \dot{m}_d = 0$), we can obtain $\boldsymbol{v}_i = \boldsymbol{v}_{di} = \boldsymbol{v}_{ci}$ from Eq. (3.2). We can also obtain $\bar{\bar{\boldsymbol{v}}}_i = \bar{\bar{\boldsymbol{v}}}_{di} = \bar{\bar{\boldsymbol{v}}}_{ci}$ after averaging them along the interfaces. If $\dot{m}_c = -\dot{m}_d \neq 0$, we still have

$$\bar{\bar{\boldsymbol{v}}}_i \approx \bar{\bar{\boldsymbol{v}}}_{di} \approx \bar{\bar{\boldsymbol{v}}}_{ci}, \tag{4.9}$$

because $\overline{\overline{(\dot{m}_k/\rho_{ki})\boldsymbol{n}_k}}$ ($k = c, d$) is not a large term in most cases.

From Eqs. (4.7) and (4.9), we have

$$\bar{\bar{\boldsymbol{v}}}_i \approx \bar{\boldsymbol{v}}_d. \tag{4.10}$$

From Eq. (4.8), we can also obtain

$$\bar{\bar{h}}_{di} = \bar{h}_d. \tag{4.11}$$

The temperature jump between the phases must be small provided that the slip between the tangential velocities of the two phases is small. That is,

$$\vartheta_{ci} \approx \vartheta_{di}, \ \bar{\bar{\vartheta}}_{ci} \approx \bar{\bar{\vartheta}}_{di} \approx \bar{\vartheta}_d. \tag{4.12}$$

Thus, the enthalpy of the fluid phase averaged along the interfaces can be expressed as

$$\bar{\bar{h}}_{ci} \approx h_c(\bar{\vartheta}_d), \tag{4.13}$$

where $h_c(\bar{\vartheta}_d)$ is the enthalpy of the fluid phase corresponding to the temperature $\bar{\vartheta}_d$.

When phase change does not occur very rapidly ($\dot{m}_k$ and $\Gamma_k$ are somewhat small), we can assume approximately

$$\delta_k \approx 0, \ B_k \approx 0, \ (k = c, d), \tag{4.14}$$

(see Eqs. (3.9) and (3.6)) so that there are

$$\boldsymbol{q}_{ki} - \boldsymbol{q}_i \approx -\dot{m}_k h_{ki} \boldsymbol{n}_k, \tag{4.15}$$

$$\boldsymbol{T}_{ki} - \boldsymbol{T}_i \approx 0, \tag{4.16}$$

$$\boldsymbol{f}_{k'k} \approx \sum_j [\boldsymbol{n}_k \cdot (\boldsymbol{T}'_i + \bar{\bar{\boldsymbol{\tau}}}_i)/L_j], \tag{4.17}$$

$$Q_{k'k} \approx \sum_j [-\boldsymbol{n}_k \cdot \boldsymbol{q}_{ki}/L_j], \tag{4.18}$$

(see Eqs. (3.8), (3.5), (3.18) and (3.22)). It is not a bad assumption that $\bar{\bar{p}}_{ci} \approx \bar{\bar{p}}_c$, $\bar{\bar{p}}_{di} \approx \bar{\bar{p}}_d$. Combining it with Eqs. (4.16) and (3.15), we obtain

$$\bar{\bar{p}}_c \approx \bar{\bar{p}}_{ci} \approx \bar{\bar{p}}_i \approx \bar{\bar{p}}_{di} \approx \bar{\bar{p}}_d \equiv p, \tag{4.19}$$

Assuming that both $\rho_c$ and $\rho_d$ are constant and using Eqs. (4.4), (4.10), (4.11), (4.13) and (4.19), we can reduce the right hand sides of Eqs. (4.2) and (4.3) as

$$-\alpha_k\nabla p+\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{\tau}}}_k+\boldsymbol{T}_k^T)]+\alpha_k\bar{\bar{\rho}}_k\boldsymbol{b}_k+\boldsymbol{f}_{k'k}+\bar{\boldsymbol{v}}_d\Gamma_k, \qquad (k=c,d) \tag{4.2$'$}$$

and

$$\begin{aligned}&-\alpha_k\bar{\boldsymbol{v}}_k\cdot\nabla p+\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{\tau}}}+\boldsymbol{T}_k^T)\cdot\bar{\boldsymbol{v}}_k]-\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{q}}}_k+\boldsymbol{q}_k^t)]\\&+\alpha_k\bar{\bar{\rho}}_k\boldsymbol{b}_k\cdot\bar{\boldsymbol{v}}_k+\alpha_k\bar{\bar{Q}}_k+\boldsymbol{f}_{k'k}\cdot\bar{\boldsymbol{v}}_d+Q_{k'k}\\&+\left(\bar{u}_k+\frac{1}{2}\overline{\overline{v_i'^2}}+\frac{1}{2}\bar{v}_d^2\right)\Gamma_k+[h_k(\vartheta_d)-h_k(\vartheta_k)]\Gamma_k,\\&\begin{pmatrix}k=c\ \text{or}\ d\\k'=d\ \text{or}\ c\end{pmatrix}.\end{aligned}\tag{4.3$'$}$$

In order to make it easy to discuss later, we write down the total enthalpy equation, kinetic energy equation and the equation of heat enthalpy plus turbulent kinetic energy by using Eqs. (4.1), (4.2) and (4.3) as follows ($k=c$ or $d$, $k'=d$ or $c$):

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\alpha_k\bar{\bar{\rho}}_k\left(\bar{h}_k+\frac{1}{2}\overline{v_k'^2}+\frac{1}{2}\bar{v}_k^2\right)\right]+\nabla\cdot\left[\alpha_k\bar{\bar{\rho}}_k\bar{\boldsymbol{v}}_k\left(\bar{h}_k+\frac{1}{2}\overline{v_k'^2}\right.\right.\\&\left.\left.+\frac{1}{2}\bar{v}_k^2\right)\right]=\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{\tau}}}_k+\boldsymbol{T}_k^T)\cdot\bar{\boldsymbol{v}}_k]-\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{q}}}_k+\boldsymbol{q}_k^T)]\\&+\alpha_k\bar{\bar{\rho}}_k\boldsymbol{b}_k\cdot\bar{\boldsymbol{v}}_k+\alpha_k\bar{\bar{Q}}_k+\boldsymbol{f}_{k'k}\cdot\bar{\boldsymbol{v}}_d+Q_{k'k}+\alpha_k\frac{\partial p}{\partial t}\\&+\left[h_k(\vartheta_d)+\frac{1}{2}\overline{\overline{v_i'^2}}+\frac{1}{2}\bar{v}_d^2\right]\Gamma_k,\end{aligned}\tag{4.20}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{1}{2}\alpha_k\bar{\bar{\rho}}_k\bar{v}_k^2\right)+\nabla\cdot\left(\frac{1}{2}\alpha_k\bar{\bar{\rho}}_k\bar{v}_k^2\bar{\boldsymbol{v}}_k\right)=-\alpha_k\bar{\boldsymbol{v}}_k\cdot\nabla p+\bar{\boldsymbol{v}}_k\\&\cdot\{\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{\tau}}}_k+\boldsymbol{T}_k^T)]\}+\alpha_k\bar{\bar{\boldsymbol{\tau}}}_k\boldsymbol{b}_k\cdot\bar{\boldsymbol{v}}_k\\&+\boldsymbol{f}_{k'k}\cdot\bar{\boldsymbol{v}}_k+\left(\bar{\boldsymbol{v}}_d\cdot\bar{\boldsymbol{v}}_k-\frac{1}{2}\bar{\boldsymbol{v}}_k^2\right)\Gamma_k,\end{aligned}\tag{4.21}$$

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\alpha_k\bar{\bar{\rho}}_k\left(\bar{h}_k+\frac{1}{2}\overline{v_k'^2}\right)\right]+\nabla\cdot\left[\alpha_k\bar{\bar{\rho}}_k\bar{\boldsymbol{v}}_k\left(\bar{h}_k+\frac{1}{2}\overline{v_k'^2}\right)\right]\\&=\alpha_k\left(\frac{\partial}{\partial t}+\bar{\boldsymbol{v}}_k\cdot\nabla\right)p+\alpha_k(\bar{\bar{\boldsymbol{\tau}}}_k+\boldsymbol{T}_k^T):\nabla\bar{\boldsymbol{v}}_k-\nabla\cdot[\alpha_k(\bar{\bar{\boldsymbol{q}}}_k+\boldsymbol{q}_k^T)]\\&+\alpha_k\bar{\bar{Q}}_k+\boldsymbol{f}_{k'k}\cdot(\bar{\boldsymbol{v}}_d-\bar{\boldsymbol{v}}_k)+Q_{k'k}\\&+\left[h_k(\vartheta_d)+\frac{1}{2}\overline{v_i'^2}+\frac{1}{2}|\bar{\boldsymbol{v}}_d-\bar{\boldsymbol{v}}_k|^2\right]\Gamma_k.\end{aligned}\tag{4.22}$$

Using the generalized Gibbs' relation

$$\vartheta_k d\bar{s}_k=d\left(\bar{h}_k+\frac{1}{2}\overline{v_k'^2}\right)-dp/\bar{\bar{\rho}}_k,$$

the energy equation can be obtained as

$$\alpha_k\bar{\bar{\rho}}_k\bar{\vartheta}_k\left(\frac{\partial}{\partial t}+\bar{\boldsymbol{v}}_k\cdot\boldsymbol{\nabla}\right)\bar{s}_k=\alpha_k(\bar{\bar{\boldsymbol{\tau}}}_k+\boldsymbol{T}_k^T):\boldsymbol{\nabla}\bar{\boldsymbol{v}}_k-\boldsymbol{\nabla}\cdot[\alpha_k(\bar{\bar{\boldsymbol{q}}}_k$$

$$+\boldsymbol{q}_k^T)]+\alpha_k\bar{\bar{Q}}_k+\boldsymbol{f}_{k'k}\cdot(\bar{\boldsymbol{v}}_d-\bar{\boldsymbol{v}}_k)+Q_{k'k}$$

$$+\left[h_k(\bar{\vartheta}_d)-h_k(\bar{\vartheta}_k)+\frac{1}{2}\overline{\overline{v_i'^2}}-\frac{1}{2}\overline{v_k'^2}+\frac{1}{2}|\bar{\boldsymbol{v}}_d-\bar{\boldsymbol{v}}_k|^2\right]\Gamma_k. \qquad (4.23)$$

## V. Discussion

1. The interphase force $\boldsymbol{M}_k$ (see Eq. (3.17)) is composed of the following three terms: $\bar{\bar{\boldsymbol{v}}}_i\Gamma_k$, $\boldsymbol{f}_{k'k}$ and $\bar{\bar{p}}_i\boldsymbol{\nabla}\alpha_k$. The first term is due to the mass transfer between phases. When some of mass is transported from one phase to the other, the momentum possessed by the mass will be carried into the latter phase. The second term is caused by the relative motion between the two phases and depends on the difference of the velocities and that of the accelerations between the two phases. The third term has nothing to do with the relative motion and the mass transfer. It remains even under the conditions without phase change and relative motion. The buoyance acting on the dispersed phase is related to this term.

The rate of the interphase energy transfer $E_k$ (see Eq. (3.20)) includes the similar three terms: $\left(\frac{1}{2}\overline{\overline{v_i^2}}+\bar{\bar{u}}_{ki}\right)\Gamma_k$, $\boldsymbol{f}_{k'k}\cdot\bar{\bar{\boldsymbol{v}}}_i$ and $[\bar{\bar{p}}_{ki}\Gamma_k/\bar{\bar{\rho}}_{ki}-\bar{\bar{p}}_i\partial\alpha_k/\partial t]$. Besides, there is the fourth term: the heat transfer between the phases, $Q_{k'k}$. For the phase $\rho_k\neq$ constant, it seems that the third term of $E_k$ and that of $\boldsymbol{M}_k$ have no relation to each other in form. Only for the phase $\rho_k=$ constant, can the third term of $E_k$ be approximately expressed as $[\bar{\bar{p}}_i\boldsymbol{\nabla}\cdot(\alpha_k\bar{\boldsymbol{v}}_k)]$ (but not $\bar{\bar{p}}_i\bar{\boldsymbol{v}}_k\cdot\boldsymbol{\nabla}\alpha_k$!), so that the relation between the third term of $E_k$ and that of $\boldsymbol{M}_k$ becomes obvious and can be easily understood.

2. Since the process of phase change is accompanied with the release or absorption of latent heat as well as the expansion or contraction of volume, the algebraic sum of heat transfer between phases, $(Q_{cd}+Q_{dc})$, is not equal to zero, and the algebraic sum of the work produced by the third term of the interphase force,

$$\sum_{k=c,d}\left[\bar{\bar{p}}_{ki}\Gamma_k/\bar{\bar{\rho}}_{ki}-\bar{\bar{p}}_i\frac{\partial\alpha_k}{\partial t}\right],$$

is not equal to zero too.

If the enthalpy (with the enthalpy of formation included) carried by the mass, which is transported in the process of phase change, is added to the heat transfer between phases, $Q_{k'k}$, we can obtain the total energy flux out of phase $k$ in the relative coordinates, $Q_{ik}$ (see Eq. (3.21)). Then the algebraic sum $(Q_{ic}+Q_{id})$ is always equal to zero.

3. Since it is on the interfaces that the drag forces $\boldsymbol{f}_{cd}$ and $\boldsymbol{f}_{dc}$ act, their action points are moving at the velocity of the interfaces, $\bar{\bar{\boldsymbol{v}}}_i$. Therefore, the expressions for the work produced by $\boldsymbol{f}_{cd}$ and $\boldsymbol{f}_{dc}$ are $\boldsymbol{f}_{cd}\cdot\bar{\bar{\boldsymbol{v}}}_i$ and $\boldsymbol{f}_{dc}\cdot\bar{\bar{\boldsymbol{v}}}_i$, respectively.

They are approximately equal to $\boldsymbol{f}_{cd}\cdot\bar{\boldsymbol{v}}_d$ and $\boldsymbol{f}_{dc}\cdot\bar{\boldsymbol{v}}_d$ with $\bar{\bar{\boldsymbol{v}}}_i$ not being replaced by $\bar{\boldsymbol{v}}_c$.

4. The term $(-\alpha_k\nabla p)$ in Eq. (4.2) results from the combination of the surface force $[-\nabla(\alpha_k p)]$ and the third part of the interphase force $(p\nabla\alpha_k)$ (Eq. (4.19) is assumed to be valid). In mathematics, they can be combined in such a way, but in physics they are two different quantities. The former is a kind of surface force (it can be expressed as a body force only after using Gauss's law) while the latter is a kind of body force. For the chosen control volume, the former is an external force which remains in the equation for the mixture while the latter is a kind of interphase force which will be offset by the corresponding term of the other phase in the equation for the mixture. The expression of the work produced by the former is $[-\nabla\cdot(\alpha_k p\bar{\boldsymbol{v}}_k)]$ while that produced by the latter is $\left[p\left(\Gamma_k/\bar{\bar{\rho}}_k-\frac{\partial\alpha_k}{\partial t}\right)\right]$. In general, the combination of these two kinds of work is not equal to $(-\alpha_k\bar{\boldsymbol{v}}_k\cdot\nabla p)$ unless $\rho_k$ is constant (see Eq. (4.4)). If we ignore the fact that $(-\alpha_k\nabla p)$ is composed of the two different kinds of forces in concepts and this term is introduced into the momentum equation as the so-called "shared-pressure presumption"[8], we would get something wrong with the corresponding term in the energy equation, or cannot understand why that is $\left[-\nabla\cdot(\alpha_k p\bar{\boldsymbol{v}}_k)+p\left(\Gamma_k/\bar{\bar{\rho}}_k-\frac{\partial\alpha_k}{\partial t}\right)\right]$.

5. In the enthalpy equation (4.22) and the entropy equation (4.23), the term $\boldsymbol{f}_{k'k}\cdot(\bar{\boldsymbol{v}}_d-\bar{\boldsymbol{v}}_k)$ is approximately equal to zero for the phase $k=d$. For the phase $k=c$, this term is always greater than zero except $\bar{\boldsymbol{v}}_c=\bar{\boldsymbol{v}}_d$. The velocity of the fluid surrounding a particle is gradually changed, from $\bar{\bar{\boldsymbol{v}}}_i(\approx\bar{\boldsymbol{v}}_d)$ at the interface to $\bar{\boldsymbol{v}}_c$ far from the particle in the average sense. There exists the process in which the mechanical energy is transformed into heat in this layer (boundary layer or other kinds of viscous layer). It is this term that represents the irreversible energy transformation.

## Appendix

Choose a control volume $V$ in a suspension as the system to be investigated. Let $V_d$ be the mean volume of a particle. Thus, the particle number in $V$ is

$$N_V=\alpha_d V/V_d.$$

Let $A$ be the surface area of the control volume $V$. In statistics, a part of it, $\alpha_d A$, is within the particles at the control surface. The area crossed by $A$ and a particle is proportional to $V_d^{2/3}$ while $A$ is proportional to $V^{2/3}$. Hence, the particle number crossed with $A$ is $N_A$

$$N_A\propto\alpha_d(V/V_d)^{2/3},\ N_A/N_V\propto(V_d/V)^{1/3}=(\alpha_d/N_V)^{1/3}.$$

If $N_V$ is very large, then $N_A\ll N_V$. The number of the integral particle $N_I$ in the control volume $V$ is approximately equal to $\left(N_V-\frac{1}{2}N_A\right)$.

Let $\bar{\boldsymbol{v}}_d^V$ be the mass weighted mean value of the particle's velocities in the control volume. $\bar{\bar{\boldsymbol{v}}}^V_{di}$ is the mean value of the particle velocities at the interfaces. The average goes over all the interfaces in the control volume $V$. From the ergodic hypothesis, $\bar{\boldsymbol{v}}^V_d$ and $\bar{\bar{\boldsymbol{v}}}^V_{di}$ are equal to the time averaging value $\bar{\boldsymbol{v}}_d$ and $\bar{\bar{\boldsymbol{v}}}_{di}$, respectively. Let $\tilde{\boldsymbol{v}}^V_d$ be the mass weighted mean value of the velocities over the $N_I$ integral particles, $\tilde{\bar{\boldsymbol{v}}}^V_{di}$ is the mean value of the particle velocities at the interfaces. The average goes over all the interfaces of the $N_I$ integral particles. Les $m_s$ and $a_s$ be the particle mass and surface area of particls $s$, $\boldsymbol{v}_{sp}$ and $\boldsymbol{v}_{sa}$ the mass-center velocity and the mean surface velocity of this particle, $\Delta \boldsymbol{v}_s \equiv \boldsymbol{v}_{sa} - \boldsymbol{v}_{sp}$ the difference of these two velocities. Hence,

$$\tilde{\boldsymbol{v}}^V_d = \sum_{s=1}^{N_I} (m_s \boldsymbol{v}_{sp}) \Big/ \sum_{s=1}^{N_I} m_s,$$

$$\tilde{\bar{\boldsymbol{v}}}^V_{di} = \sum_{s=1}^{N_I} (v_s \boldsymbol{v}_{sa}) \Big/ \sum_{s=1}^{N_I} a_s = \tilde{\boldsymbol{v}}^V_d + \boldsymbol{\delta}_3 + \boldsymbol{\delta}_4,$$

where

$$\boldsymbol{\delta}_3 \equiv \sum_{s=1}^{N_I} (a_s \boldsymbol{v}_{sp}) \Big/ \sum_{s=1}^{N_I} a_s - \sum_{s=1}^{N_I} (m_s \boldsymbol{v}_{sp}) \Big/ \sum_{s=1}^{N_I} m_s,$$

$$\boldsymbol{\delta}_4 \equiv \sum_{s=1}^{N_I} (a_s \nabla \boldsymbol{v}_s) \Big/ \sum_{s=1}^{N_I} a_s.$$

Furthermore, we assume $\boldsymbol{\delta}_1 \equiv \bar{\boldsymbol{v}}^V_d - \tilde{\boldsymbol{v}}^V_d$, $\boldsymbol{\delta}_2 \equiv \bar{\bar{\boldsymbol{v}}}^V_{di} - \tilde{\bar{\boldsymbol{v}}}^V_{di}$, then we have

$$\bar{\boldsymbol{v}}_d \approx \bar{\boldsymbol{v}}^V_d = \tilde{\boldsymbol{v}}^V_d + \boldsymbol{\delta}_1,$$

$$\bar{\bar{\boldsymbol{v}}}_{di} \approx \bar{\bar{\boldsymbol{v}}}^V_{di} = \tilde{\boldsymbol{v}}^V_d + \boldsymbol{\delta}_2 + \boldsymbol{\delta}_3 + \boldsymbol{\delta}_4.$$

If the particle number in the control volume is very large, there is not much difference in size and shape of all particles, and there is no sharp mass distribution inside every particle, then $\boldsymbol{\delta}_1 - \boldsymbol{\delta}_4$ are all very small. Consequently, the mean velocity of the particle phase $\bar{\boldsymbol{v}}_d$ is approximately equal to the mean value of the particle velocities at the interfaces $\bar{\bar{\boldsymbol{v}}}_{di}$. By analogy, it can be shown that the mean temperature of the particle phase $\bar{\vartheta}_d$ is approximately equal to the mean value of the particle temperature at the interface $\bar{\bar{\vartheta}}_{di}$. That is,

$$\bar{\bar{\boldsymbol{v}}}_{di} \approx \bar{\boldsymbol{v}}_d, \quad \bar{\bar{\vartheta}}_{di} \approx \bar{\vartheta}_d.$$

## References


[1] Bouré, J. A. & Delhaye, J. M., General equations and two-phase flow modeling, in *Handbook of Multiphase Systems* (Ed. Hetsroni, G.), Hemisphere Publishing Co., Washington, 1982.

[2] Ishii, M., *Thermo-Fluid Dynamic Theory of Two-Phase Flow*, Eyrolles, Paris, 1975.

[3] Drew, D. A. & Segel, L., *Studies in Applied Math.*, 50(1971), 205—231.

[4] Soo, S. L., *Fluid Dynamics of Multiphase Systems*, Blaisdell Publishing Co., Waltham, 1967.

[5] Pai, S. I., *Two-Phase Flow*, Vieweg-Verlag, Braunschweig, 1977.

[6] Rudinger, G. *Fundamentals of Gas-Particle Flow*, Elsevier Scientific Publishing Co., Amsterdam, 1980.
[7] Schmitt-v. Schubert, B., *ZAMP*, **20**(1969), 922—935.
[8] 蔡树棠等,应用数学和力学, **7**(1986), 477—485.
[9] 凌志光,工程热物理学报, **5**(1984), 51—54.
[10] 刘大有,力学学报, **19**(1987), 213—221.