ANALYSIS OF FLEXURAL VIBRATION OF VISCOELASTICALLY DAMPED SANDWICH PLATES

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The simplified governing equations and corresponding boundary conditions of flexural vibration of viscoelastically damped unsymmetrical sandwich plates are given. The asymptotic solution to the equations is then discussed. If only the first terms of the asymptotic solution of all variables are taken as an approximate solution, the result is identical with that obtained from the Modal Strain Energy (MSE) Method. As more terms of the asymptotic solution are taken, the successive calculations show improved accuracy. With the natural frequencies and the modal loss factors of a damped sandwich plate known, one can calculate the response of the plate to various loads providing a reliable basis for engineering design.

1. INTRODUCTION

It is known that structural vibration can be reduced by utilizing layers of viscoelastic damping material [1, 2]. One effective approach is the constrained layer damping treatment. A constrained damped plate is generally an unsymmetrical sandwich plate, in which a layer of viscoelastic damping material is sandwiched between a primary structural layer and a constraining layer. The structural layer and the constraining layer may be made of different material. In general they are of unequal thicknesses. Such a sandwich plate will provide both strength and rigidity and yet have a low response to vibration over a wide frequency range.

The governing equations of flexural vibration of symmetrical sandwich plates have been given in reference [3]. Successively, a set of governing equations for the vibratory bending of unsymmetrical sandwich plates was given in references [4, 5]. It is a set of twelfth-order partial differential equations including bending–extension coupling. Solving the equations is difficult. In recently published papers [6–8] the flexural vibrations of constrained damped plates with complex shapes and various boundary conditions have been computed by finite element methods, the mechanical plate model presented in reference [4] having been used. The viscoelastic core was modelled with solid elements and the face layers were modelled with plate elements. The number of degrees of freedom of all the elements was so large that the work was expensive. In reference [8] it was suggested that available design charts be used insofar as possible. However, the number of design charts is limited. Because of this, the analysis and design of unsymmetrical sandwich plates are restricted. It is the aim of this paper to present a set of simplified governing equations for unsymmetrical sandwich plates, not including bending–extension coupling, and to discuss its asymptotic solution in terms of real values.

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2. THE GOVERNING EQUATIONS OF FLEXURAL VIBRATION

To derive the governing equations of flexural vibration of unsymmetrical sandwich plates, the following assumptions are made: (1) the face-layers are elastic and isotropic and suffer no transverse shear deformation; (2) the core carries transverse shear, but no in-plane stresses; it is linearly viscoelastic and has a complex shear modulus; (3) no slip occurs at the interfaces of the core and face-layers; all points on a normal to the plate move with the same transverse displacement; (4) though the metallic material of the structural layer and the constraining layer may be different, the values of their Poisson ratios may be approximately equal; (5) when the sandwich plate is in flexural vibration, the in-plane inertia effects of the plate are ignored and only the transverse inertia effects are considered. Assumptions (1)–(3) are generally made. Both assumptions (4) and (5) are necessary to establish a set of simplified governing equations of unsymmetrical sandwich plates not including bending–extension coupling. The influence of the in-plane inertia forces upon dynamic characteristics of a plate was discussed in references [4, 5, 9]. In most cases assumption (5) is acceptable.

The unsymmetrical sandwich plate configuration is shown in Figure 1. The thicknesses of the constraining layer (face 1), the viscoelastic layer (face 2) and the structural layer (face 3) are t_1, t_2 and t_3 respectively. The transverse displacement of the plate in flexural vibration is $W(x, y, t)$. The in-plane displacements of the points in the middle planes of face 1 and face 3 are U_1, V_1 and U_3, V_3 respectively. The in-plane displacements of interfaces are U_{12}, V_{12} and U_{32}, V_{32} respectively. According to the assumptions, there are uniform transverse shear strain components γ_{xz} and γ_{yz} in the viscoelastic layer:

$$\gamma_{xz} = \frac{U_{12} - U_{32}}{t_2} + \frac{\partial W}{\partial x} = \frac{U_1 - U_3}{t_2} + \frac{c}{t_2} \frac{\partial W}{\partial x}, \qquad \gamma_{yz} = \frac{V_{12} - V_{32}}{t_2} + \frac{\partial W}{\partial y} = \frac{V_1 - V_3}{t_2} + \frac{c}{t_2} \frac{\partial W}{\partial y}.$$

$$(1)$$

Here $c = t_2 + \frac{1}{2}(t_1 + t_3)$. The expression for the strain energy of the plate U is

$$U = \frac{1}{2} \int \int \left\{ \gamma_1 \left[\left(\frac{\partial U_1}{\partial x} \right)^2 + 2\nu \frac{\partial U_1}{\partial x} \frac{\partial V_1}{\partial y} + \left(\frac{\partial V_1}{\partial y} \right)^2 + \frac{(1 - \nu)}{2} \left(\frac{\partial U_1}{\partial y} + \frac{\partial V_1}{\partial x} \right)^2 \right] \right.$$
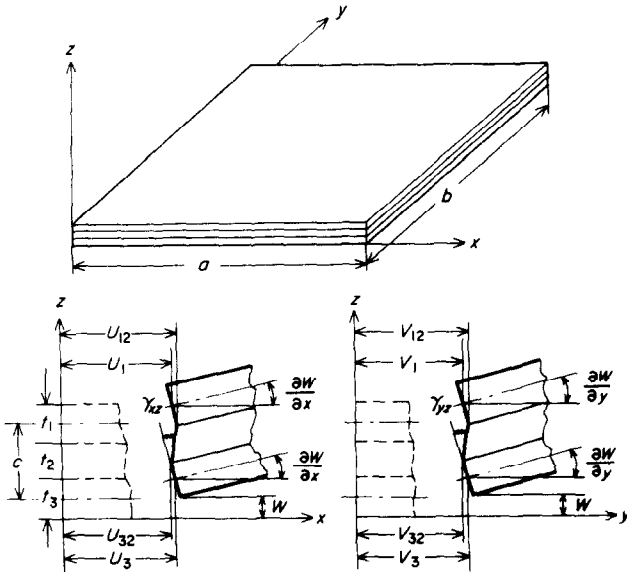

Figure 1. The dimensions and the displacements of a damped sandwich plate.

$$+ \gamma_3 \left[\left(\frac{\partial U_3}{\partial x} \right)^2 + 2\nu \frac{\partial U_3}{\partial x} \frac{\partial V_3}{\partial y} + \left(\frac{\partial V_3}{\partial y} \right)^2 + \frac{(1-\nu)}{2} \left(\frac{\partial U_3}{\partial y} + \frac{\partial V_3}{\partial x} \right)^2 \right]$$

$$+ (D_1 + D_3) \left[\left(\frac{\partial^2 W}{\partial x^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 2(1-\nu) \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right]$$

$$+ \frac{\gamma_2}{t_2^2} \left[\left(U_1 - U_3 + c \frac{\partial W}{\partial x} \right)^2 + \left(V_1 - V_3 + c \frac{\partial W}{\partial y} \right)^2 \right] \Bigg\} \, dx \, dy. \qquad (2)$$

Here

$$\gamma_1 = \frac{E_1 t_1}{1 - \nu^2}, \qquad \gamma_2 = G_2 t_2, \qquad \gamma_3 = \frac{E_3 t_3}{1 - \nu^2}, \qquad D_1 = \frac{E_1 t_1^3}{12(1 - \nu^2)}, \qquad D_3 = \frac{E_3 t_3^3}{12(1 - \nu^2)},$$
$$(3)$$

where E_1 and E_3 are the elastic moduli of the faces 1 and 3, ν is their common Poisson ratio, G_2 is the shear modulus of the core 2, which is taken as a real quantity temporarily. One can introduce

$$U_m(x, y, t) = [1/(\gamma_1 + \gamma_3)][\gamma_1 U_1(x, y, t) + \gamma_3 U_3(x, y, t)],$$

$$V_m(x, y, t) = [1/(\gamma_1 + \gamma_3)][\gamma_1 V_1(x, y, t) + \gamma_3 V_3(x, y, t)],$$

$$\psi_x(x, y, t) = (1/c)[U_1(x, y, t) - U_3(x, y, t)], \qquad \psi_y(x, y, t) = (1/c)[V_1(x, y, t) - V_3(x, y, t)].$$
$$(4, 5)$$

Here U_m and V_m may be regarded as the weighted mean in-plane displacements of the unsymmetrical sandwich plate, ψ_x and ψ_y are the rotary angles of a line connecting the two corresponding points at the middle planes of the faces 1 and 3 after bending. For the sake of convenience later, the following dimensionless variables are introduced:

$$\xi = x/a, \qquad \eta = y/a, \qquad u_m = U_m/a, \qquad v_m = V_m/a, \qquad w = W/a, \qquad \psi_\xi = \psi_x,$$

$$\psi_\eta = \psi_y, \qquad g = \gamma_2(\gamma_1 + \gamma_3)a^2/\gamma_1 \gamma_3 t_2^2, \qquad Y = \gamma_1 \gamma_3 c^2/(\gamma_1 + \gamma_3)(D_1 + D_3). \qquad (6)$$

Here a is a characteristic dimension of the plate, e.g., the length of an edge of a rectangular plate; g and Y are called the "shear parameter" and the "geometric parameter" respectively.

The strain energy U can now be rewritten as

$$U = \tfrac{1}{2}(\gamma_1 + \gamma_3)a^2 \int \int \left[\left(\frac{\partial u_m}{\partial \xi} \right)^2 + 2\nu \frac{\partial u_m}{\partial \xi} \frac{\partial v_m}{\partial \eta} + \left(\frac{\partial v_m}{\partial \eta} \right)^2 + \frac{(1-\nu)}{2} \left(\frac{\partial u_m}{\partial \eta} + \frac{\partial v_m}{\partial \xi} \right)^2 \right] d\xi \, d\eta$$

$$+ \tfrac{1}{2}(D_1 + D_3) \int \int \Bigg\{ \left[\left(\frac{\partial^2 w}{\partial \xi^2} \right)^2 + 2\nu \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} + \left(\frac{\partial^2 w}{\partial \eta^2} \right)^2 + 2(1-\nu) \left(\frac{\partial^2 w}{\partial \xi \partial \eta} \right)^2 \right]$$

$$+ Y \left[\left(\frac{\partial \psi_\xi}{\partial \xi} \right)^2 + 2\nu \frac{\partial \psi_\xi}{\partial \xi} \frac{\partial \psi_\eta}{\partial \eta} + \left(\frac{\partial \psi_\eta}{\partial \eta} \right)^2 + \frac{(1-\nu)}{2} \left(\frac{\partial \psi_\xi}{\partial \eta} + \frac{\partial \psi_\eta}{\partial \xi} \right)^2 \right]$$

$$+ Yg \left[\left(\psi_\xi + \frac{\partial w}{\partial \xi} \right)^2 + \left(\psi_\eta + \frac{\partial w}{\partial \eta} \right)^2 \right] \Bigg\} d\xi \, d\eta. \qquad (7)$$

The kinetic energy of the plate T is

$$T = \tfrac{1}{2}\rho a^4 \int \int \left(\frac{\partial w}{\partial t} \right)^2 d\xi \, d\eta, \qquad (8)$$

where ρ is the mass per unit area of the plate.

According to Hamilton's principle, one can obtain a set of simplified governing equations of motion of unsymmetrical sandwich plates (in dimensionless form):

$$L_1(u_m, v_m) \equiv \frac{\partial^2 u_m}{\partial \xi^2} + \frac{(1+\nu)}{2} \frac{\partial^2 v_m}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 u_m}{\partial \eta^2} = 0,$$

$$L_2(u_m, v_m) \equiv \frac{\partial^2 v_m}{\partial \eta^2} + \frac{(1+\nu)}{2} \frac{\partial^2 u_m}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 v_m}{\partial \xi^2} = 0, \qquad (9)$$

$$L_3(w, \psi_\xi, \psi_\eta) \equiv \frac{\partial^2 \psi_\xi}{\partial \xi^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \psi_\eta}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 \psi_\xi}{\partial \eta^2} - g\left(\psi_\xi + \frac{\partial w}{\partial \xi}\right) = 0,$$

$$L_4(w, \psi_\xi, \psi_\eta) \equiv \frac{\partial^2 \psi_\eta}{\partial \eta^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \psi_\xi}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 \psi_\eta}{\partial \xi^2} - g\left(\psi_\eta + \frac{\partial w}{\partial \eta}\right) = 0,$$

$$L_5(w, \psi_\xi, \psi_\eta) \equiv \nabla^4 w - Yg\left(\frac{\partial \psi_\xi}{\partial \xi} + \frac{\partial \psi_\eta}{\partial \eta} + \nabla^2 w\right) + \frac{\rho a^4}{(D_1 + D_3)} \frac{\partial^2 w}{\partial t^2} = 0. \qquad (10)$$

Here $\nabla^2 = \partial^2/\partial \xi^2 + \partial^2/\partial \eta^2$. Equations (9) and (10) are uncoupled. The homogeneous equation (9) constitutes a plane stress problem in elasticity, in which u_m and v_m are unknown functions. The corresponding homogeneous boundary conditions are

$$u_m = 0, \quad \text{or} \quad \left(\frac{\partial u_m}{\partial \xi} + \nu \frac{\partial v_m}{\partial \eta}\right) l + \frac{(1-\nu)}{2} \left(\frac{\partial u_m}{\partial \eta} + \frac{\partial v_m}{\partial \xi}\right) m = 0,$$

$$v_m = 0, \quad \text{or} \quad \frac{(1-\nu)}{2} \left(\frac{\partial u_m}{\partial \eta} + \frac{\partial v_m}{\partial \xi}\right) l + \left(\nu \frac{\partial u_m}{\partial \xi} + \frac{\partial v_m}{\partial \eta}\right) m = 0. \qquad (11)$$

In most cases any in-plane rigid body motions of the plate are suppressed, so one has

$$u_m = v_m = 0. \qquad (12)$$

Therefore, only the set of eighth-order partial differential equations (10) has to be solved.

When a plate is in simple harmonic vibration, the forms

$$w(\xi, \eta, t) = w(\xi, \eta)\, e^{i\omega t}, \qquad \psi_\xi(\xi, \eta, t) = \psi_\xi(\xi, \eta)\, e^{i\omega t}, \qquad \psi_\eta(\xi, \eta, t) = \psi_\eta(\xi, \eta)\, e^{i\omega t}, \qquad (13)$$

are introduced. The circular frequency ω can be expressed in dimensionless form as

$$\Omega = \omega \sqrt{\rho a^4/(D_1 + D_3)}. \qquad (14)$$

Equations (13) can now be substituted into equations (10). However, as the plate is assumed to vibrate harmonically, the shear modulus G_2 of the core has to be changed into the complex modulus $G_2(1+i\beta)$; here β is the loss factor of the viscoelastic material. The inertia term in equations (10) $[\rho a^4/(D_1+D_3)] \partial^2 w/\partial t^2 = -\Omega^2 w$ must also be changed correspondingly into $-\Omega^2(1+i\eta^*)w$; here the quantity η^* is called the modal loss factor of the plate. The physical significance of the complex frequency $\Omega^2(1+i\eta^*)$ has been discussed in reference [10]. The amplitudes $w(\xi, \eta)$, $\psi_\xi(\xi, \eta)$ and $\psi_\eta(\xi, \eta)$ are all complex quantities. They must satisfy the following equations:

$$\frac{\partial^2 \psi_\xi}{\partial \xi^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \psi_\eta}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 \psi_\xi}{\partial \eta^2} - g(1+i\beta)\left(\psi_\xi + \frac{\partial w}{\partial \xi}\right) = 0,$$

$$\frac{\partial^2 \psi_\eta}{\partial \eta^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \psi_\xi}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 \psi_\eta}{\partial \xi^2} - g(1+i\beta)\left(\psi_\eta + \frac{\partial w}{\partial \eta}\right) = 0, \qquad (15)$$

$$\nabla^4 w - Yg(1+i\beta)\left(\frac{\partial \psi_\xi}{\partial \xi} + \frac{\partial \psi_\eta}{\partial \eta} + \nabla^2 w\right) - \Omega^2(1+i\eta^*)w = 0.$$

In the particular case of a symmetrical sandwich plate equations (15) are identical with equations (5-7) of reference [3]. Typical homogeneous boundary conditions of unsymmetrical sandwich plates are as follows: (1) simply supported edges (I),

$$\partial\psi_\xi/\partial\xi(l^2+\nu m^2)+\partial\psi_\eta/\partial\eta(m^2+\nu l^2)+(1-\nu)(\partial\psi_\xi/\partial\eta+\partial\psi_\eta/\partial\xi)lm=0,$$

$$\psi_s=0, \qquad w=0, \qquad \partial^2 w/\partial n^2+(\nu/\rho_s)\partial w/\partial n=0; \tag{16}$$

(2) simply supported edges (II),

$$\partial\psi_\xi/\partial\xi(l^2+\nu m^2)+\partial\psi_\eta/\partial\eta(m^2+\nu l^2)+(1-\nu)(\partial\psi_\xi/\partial\eta+\partial\psi_\eta/\partial\xi)lm=0,$$

$$-(\partial\psi_\xi/\partial\xi-\partial\psi_\eta/\partial\eta)lm+\tfrac{1}{2}(\partial\psi_\xi/\partial\eta+\partial\psi_\eta/\partial\xi)(l^2-m^2)=0,$$

$$w=0, \qquad \partial^2 w/\partial n^2+(\nu/\rho_s)\partial w/\partial n=0; \tag{17}$$

(3) clamped edges,

$$\psi_n=0, \qquad \psi_s=0, \qquad w=0, \qquad \partial w/\partial n=0; \tag{18}$$

(4) free edges,

$$\partial\psi_\xi/\partial\xi(l^2+\nu m^2)+\partial\psi_\eta/\partial\eta(m^2+\nu l^2)+(1-\nu)(\partial\psi_\xi/\partial\eta+\partial\psi_\eta/\partial\xi)lm=0,$$

$$-(\partial\psi_\xi/\partial\xi-\partial\psi_\eta/\partial\eta)lm+\tfrac{1}{2}(\partial\psi_\xi/\partial\eta+\partial\psi_\eta/\partial\xi)(l^2-m^2)=0.$$

$$-\frac{\partial}{\partial n}(\nabla^2 w)-(1-\nu)\frac{\partial}{\partial s}\left[\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right]+Yg(1+i\beta)\left[\left(\psi_\xi+\frac{\partial w}{\partial\xi}\right)l+\left(\psi_\eta+\frac{\partial w}{\partial\eta}\right)m\right]=0,$$

$$\frac{\partial^2 w}{\partial n^2}+\nu\left(\frac{\partial^2 w}{\partial s^2}+\frac{1}{\rho_s}\frac{\partial w}{\partial n}\right)=0; \tag{19}$$

(5) at a corner,

$$w=0, \qquad \text{or} \qquad \left[\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right)\right]\Bigg|_{s_i+0}^{s_i-0}=0. \tag{20}$$

Here $l=\cos(n,\xi)$ and $m=\cos(n,\eta)$ are the direction cosines of the normal n drawn outwards on the boundary of the plate, where n (and s) are dimensionless. The ρ_s is the local radius of curvature of the periphery. Thus to investigate flexural vibration of a viscoelastically damped unsymmetrical sandwich plate, one must first solve equations (15) for a given set of boundary conditions to find the natural frequencies Ω, the modal loss factors η^*, and the corresponding complex modes w, ψ_ξ and ψ_η.

3. THE ASYMPTOTIC SOLUTION OF THE GOVERNING EQUATIONS

Solving equations (15) is difficult. The exact solution can be obtained only in some particular cases. To obtain an approximate and practical solution and to avoid calculation with complex values, an asymptotic solution with $\mu=i\beta$ as a complex parameter can be introduced. The same procedure has been used in reference [11] to find the loss factors of a sandwich cantilever. One first expands the solution in power series:

$$w(\xi,\eta)=w_0(\xi,\eta)+\mu w_1(\xi,\eta)+\mu^2 w_2(\xi,\eta)+\mu^3 w_3(\xi,\eta)+\cdots,$$

$$\psi_\xi(\xi,\eta)=\psi_{\xi 0}(\xi,\eta)+\mu\psi_{\xi 1}(\xi,\eta)+\mu^2\psi_{\xi 2}(\xi,\eta)+\mu^3\psi_{\xi 3}(\xi,\eta)+\cdots,$$

$$\psi_\eta(\xi,\eta)=\psi_{\eta 0}(\xi,\eta)+\mu\psi_{\eta 1}(\xi,\eta)+\mu^2\psi_{\eta 2}(\xi,\eta)+\mu^3\psi_{\eta 3}(\xi,\eta)+\cdots,$$

$$\Omega^2=\Omega_0^2+\mu^2\Omega_2^2+\cdots,$$

$$i\eta^*=\mu\eta_1^*+\mu^3\eta_3^*+\cdots. \tag{21}$$

Substituting equations (21) into equations (15) gives the successive equations that the asymptotic solution must satisfy:

$$L_3^*(w_0, \psi_{\xi 0}, \psi_{\eta 0}) \equiv \frac{\partial^2 \psi_{\xi 0}}{\partial \xi^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \psi_{\eta 0}}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 \psi_{\xi 0}}{\partial \eta^2} - g\left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi}\right) = 0,$$

$$L_4^*(w_0, \psi_{\xi 0}, \psi_{\eta 0}) \equiv \frac{\partial^2 \psi_{\eta 0}}{\partial \eta^2} + \frac{(1+\nu)}{2} \frac{\partial^2 \psi_{\xi 0}}{\partial \xi \partial \eta} + \frac{(1-\nu)}{2} \frac{\partial^2 \psi_{\eta 0}}{\partial \xi^2} - g\left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta}\right) = 0,$$

$$L_5^*(w_0, \psi_{\xi 0}, \psi_{\eta 0}) \equiv \nabla^4 w_0 - Yg\left(\frac{\partial \psi_{\xi 0}}{\partial \xi} + \frac{\partial \psi_{\eta 0}}{\partial \eta} + \nabla^2 w_0\right) - \Omega_0^2 w_0 = 0, \tag{22}$$

$$L_3^*(w_1, \psi_{\xi 1}, \psi_{\eta 1}) = g\left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi}\right), \qquad L_4^*(w_1, \psi_{\xi 1}, \psi_{\eta 1}) = g\left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta}\right),$$

$$L_5^*(w_1, \psi_{\xi 1}, \psi_{\eta 1}) = Yg\left(\frac{\partial \psi_{\xi 0}}{\partial \xi} + \frac{\partial \psi_{\eta 0}}{\partial \eta} + \nabla^2 w_0\right) + \Omega_0^2 \eta_1^* w_0, \tag{23}$$

$$L_3^*(w_2, \psi_{\xi 2}, \psi_{\eta 2}) = g\left(\psi_{\xi 1} + \frac{\partial w_1}{\partial \xi}\right), \qquad L_4^*(w_2, \psi_{\xi 2}, \psi_{\eta 2}) = g\left(\psi_{\eta 1} + \frac{\partial w_1}{\partial \eta}\right),$$

$$L_5^*(w_2, \psi_{\xi 2}, \psi_{\eta 2}) = Yg\left(\frac{\partial \psi_{\xi 1}}{\partial \xi} + \frac{\partial \psi_{\eta 1}}{\partial \eta} + \nabla^2 w_1\right) + \Omega_0^2 \eta_1^* w_1 + \Omega_2^2 w_0, \tag{24}$$

$$L_3^*(w_3, \psi_{\xi 3}, \psi_{\eta 3}) = g\left(\psi_{\xi 2} + \frac{\partial w_2}{\partial \xi}\right), \qquad L_4^*(w_3, \psi_{\xi 3}, \psi_{\eta 3}) = g\left(\psi_{\eta 2} + \frac{\partial w_2}{\partial \eta}\right),$$

$$L_5^*(w_3, \psi_{\xi 3}, \psi_{\eta 3}) = Yg\left(\frac{\partial \psi_{\xi 2}}{\partial \xi} + \frac{\partial \psi_{\eta 2}}{\partial \eta} + \nabla^2 w_2\right) + \Omega_0^2(\eta_1^* w_2 + \eta_3^* w_0) + \Omega_2^2(\eta_1^* w_0 + w_1),$$

$$\cdots. \tag{25}$$

Here L_3^*, L_4^* and L_5^* are linear partial differential operators defined in equations (22). Substituting equations (21) into the boundary conditions (16)-(20), one finds that, except for the third of equations (19) for free edges which must be expanded in power series of μ, the expressions for all other types of boundary conditions which the successive terms w_i, $\psi_{\xi i}$, $\psi_{\eta i}$ ($i = 0, 1, 2, \ldots$) must satisfy are the same in form as before. Therefore, from equations (22) and the boundary conditions with respect to w_0, $\psi_{\xi 0}$, $\psi_{\eta 0}$ one can solve first a real eigenvalue problem and obtain all the eigenvalues Ω_0^2 and corresponding modes w_0, $\psi_{\xi 0}$ and $\psi_{\eta 0}$. The modes can be normalized as follows.

If all the boundary conditions of the plate are homogeneous, one can prove that orthogonality exists between two complex modes. This means that if there are two different complex eigenvalues $\Omega_I^2(1 + i\eta_I^*)$ and $\Omega_{II}^2(1 + i\eta_{II}^*)$ and their corresponding complex modes w_I, $\psi_{\xi I}$, $\psi_{\eta I}$ and w_{II}, $\psi_{\xi II}$, $\psi_{\eta II}$, then

$$\int\int w_I w_{II} \, d\xi \, d\eta = 0. \tag{26}$$

Equation (26) is analogous to the orthogonality condition in the case of sandwich beams which was given in reference [12]. Therefore, the normalization of the complex modes can be stated as

$$\int\int w^2 \, d\xi \, d\eta = 1. \tag{27}$$

Substituting the first of equations (21) into equation (27), one can obtain

$$\int \int w_0^2 \, d\xi \, d\eta = 1, \qquad \int \int 2w_0 w_1 \, d\xi \, d\eta = 0, \qquad (28, 29)$$

$$\int \int (2w_0 w_2 + w_1^2) \, d\xi \, d\eta = 0, \qquad \int \int (2w_0 w_3 + 2w_1 w_2) \, d\xi \, d\eta = 0, \dots, \quad (30, 31)$$

successively. In accordance with equation (28) one can normalize the real mode w_0, $\psi_{\xi 0}$ and $\psi_{\eta 0}$. From equations (22),

$$\int \int \left[Y\psi_{\xi 0} L_3^*(w_0, \psi_{\xi 0}, \psi_{\eta 0}) + Y\psi_{\eta 0} L_4^*(w_0, \psi_{\xi 0}, \psi_{\eta 0}) - w_0 L_5^*(w_0, \psi_{\xi 0}, \psi_{\eta 0}) \right] d\xi \, d\eta = 0.$$
$$(32)$$

Through integrations by parts and other calculations one obtains

$$\Omega_0^2 = \int \int \left\{ \left[\left(\frac{\partial^2 w_0}{\partial \xi^2} \right)^2 + 2\nu \frac{\partial^2 w_0}{\partial \xi^2} \frac{\partial^2 w_0}{\partial \eta^2} + \left(\frac{\partial^2 w_0}{\partial \eta^2} \right)^2 + 2(1-\nu) \left(\frac{\partial^2 w_0}{\partial \xi \, \partial \eta} \right)^2 \right] \right.$$
$$+ Y \left[\left(\frac{\partial \psi_{\xi 0}}{\partial \xi} \right)^2 + 2\nu \frac{\partial \psi_{\xi 0}}{\partial \xi} \frac{\partial \psi_{\eta 0}}{\partial \eta} + \left(\frac{\partial \psi_{\eta 0}}{\partial \eta} \right)^2 + \frac{(1-\nu)}{2} \left(\frac{\partial \psi_{\xi 0}}{\partial \eta} + \frac{\partial \psi_{\eta 0}}{\partial \xi} \right)^2 \right]$$
$$\left. + Yg \left[\left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi} \right)^2 + \left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta} \right)^2 \right] \right\} d\xi \, d\eta. \qquad (33)$$

The sum of the first two integrals in equation (33) is proportional to the strain energy in the faces 1 and 3 due to the normalized real mode w_0, $\psi_{\xi 0}$ and $\psi_{\eta 0}$, while the third integral is proportional to the strain energy in the core. Both proportionality coefficients are $2/(D_1 + D_3)$. This is obvious from comparison of equations (33) and (7).

Next one can solve equations (23). First the value of η_1^* must be determined. By using a similar procedure as before, when equation (32) was obtained, one gets a relation from equation (23):

$$\int \int \left[Y\psi_{\xi 0} L_3^*(w_1, \psi_{\xi 1}, \psi_{\eta 1}) + Y\psi_{\eta 0} L_4^*(w_1, \psi_{\xi 1}, \psi_{\eta 1}) - w_0 L_5^*(w_1, \psi_{\xi 1}, \psi_{\eta 1}) \right] d\xi \, d\eta$$

$$= \int \int \left\{ Yg \left[\psi_{\xi 0} \left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi} \right) + \psi_{\eta 0} \left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta} \right) - w_0 \left(\frac{\partial \psi_{\xi 0}}{\partial \xi} + \frac{\partial \psi_{\eta 0}}{\partial \eta} + \nabla^2 w_0 \right) \right] \right.$$

$$\left. - \Omega_0^2 \eta_1^* w_0^2 \right\} d\xi \, d\eta. \qquad (34)$$

From equation (34) one obtains

$$\eta_1^* = \frac{1}{\Omega_0^2} \int \int Yg \left[\left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi} \right)^2 + \left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta} \right)^2 \right] d\xi \, d\eta, \qquad (35)$$

η_1^* is evidently a fraction of elastic strain energy attributable to the viscoelastic core when the damped plate deforms in the mode w_0, $\psi_{\xi 0}$ and $\psi_{\eta 0}$. Up to now all the first terms of each expression in equations (21) have been obtained. If less accuracy is acceptable, the approximate solution can be expressed as

$$w(\xi, \eta) = w_0(\xi, \eta), \qquad \psi_\xi(\xi, \eta) = \psi_{\xi 0}(\xi, \eta), \qquad \psi_\eta(\xi, \eta) = \psi_{\eta 0}(\xi, \eta),$$
$$\Omega^2 = \Omega_0^2, \qquad \eta^* = \eta_1^* \beta. \qquad (36)$$

In references [7] and [8] the Modal Strain Energy (MSE) Method is suggested for analysis of viscoelastically damped structures. For viscoelastically damped sandwich plates, adopting the simplified scheme of this paper, one obtains equations (33), (35) and (36), which are the same as the results obtained by means of MSE. Therefore, MSE can be considered to be the first-order approximation of the asymptotic solution presented in this paper.

To improve accuracy, one must calculate successive terms of the asymptotic solution. Having determined the value of η_1^*, one can obtain the unique solution for w_1, $\psi_{\xi 1}$ and $\psi_{\eta 1}$ of all modes in accordance with equations (23) and corresponding boundary conditions. The general solution for w_1, $\psi_{\xi 1}$ and $\psi_{\eta 1}$ may be written in the following form:

$$w_1(\xi, \eta) = w_{1s}(\xi, \eta) + C_1 w_0(\xi, \eta), \qquad \psi_{\xi 1}(\xi, \eta) = \psi_{\xi 1 s}(\xi, \eta) + C_1 \psi_{\xi 0}(\xi, \eta),$$

$$\psi_{\eta 1}(\xi, \eta) = \psi_{\eta 1 s}(\xi, \eta) + C_1 \psi_{\eta 0}(\xi, \eta). \tag{37}$$

Here w_{1s}, $\psi_{\xi 1 s}$ and $\psi_{\eta 1 s}$ are a set of particular solutions, where C_1 is an undetermined constant. The value of C_1 can be given by means of the normalization condition, equation (29). Then the expressions for w_1, $\psi_{\xi 1}$ and $\psi_{\eta 1}$ can be determined completely.

Thereafter one solves equation (24). Through calculations similar to those previously, the expression for Ω_2^2 is first obtained.

$$\Omega_2^2 = Yg \int \int \left[\left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi} \right) \left(\psi_{\xi 1} + \frac{\partial w_1}{\partial \xi} \right) + \left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta} \right) \left(\psi_{\eta 1} + \frac{\partial w_1}{\partial \eta} \right) \right] d\xi \, d\eta. \tag{38}$$

Using the normalization condition (30), one can obtain the unique solution for w_2, $\psi_{\xi 2}$ and $\psi_{\eta 2}$ in accordance with equations (24) and corresponding boundary conditions.

Before solving equations (25) one determines the value of η_3^*;

$$\eta_3^* = -\eta_1^* \left(\int \int w_0 w_2 \, d\xi \, d\eta + \frac{\Omega_2^2}{\Omega_0^2} \right)$$

$$+ \frac{Yg}{\Omega_0^2} \int \int \left[\left(\psi_{\xi 0} + \frac{\partial w_0}{\partial \xi} \right) \left(\psi_{\xi 2} + \frac{\partial w_2}{\partial \xi} \right) + \left(\psi_{\eta 0} + \frac{\partial w_0}{\partial \eta} \right) \left(\psi_{\eta 2} + \frac{\partial w_2}{\partial \eta} \right) \right] d\xi \, d\eta. \tag{39}$$

Then, using equation (31), one can also obtain the unique solution for w_3, $\psi_{\xi 3}$ and $\psi_{\eta 3}$. If the calculation is not to be continued, one is left with the expressions for the asymptotic solution given explicitly in equations (21), without the residual terms indicated by $(\cdots)$ on their right sides. The last of equations (21) may be rewritten as

$$\eta^* = \eta_1^* \beta / [1 + (\eta_3^*/\eta_1^*)\beta^2], \tag{40}$$

By calculating in accordance with equation (40), one can often obtain better results.

4. AN EXAMPLE

Consider a rectangular viscoelastically damped sandwich plate with simple support (I) along all edges. Its length and width are a and b respectively. Define $\theta = b/a$. Introduce a dimensionless co-ordinate system $0\xi\eta$ and let the edges of the plate to be $\xi = 0, 1$ and $\eta = 0, \theta$. One can obtain the asymptotic solution for the natural frequencies, modes and modal loss factors of this plate as follows:

$$\Omega_0^2 = \alpha_{pq}^2 \left(1 + \frac{Yg}{\alpha_{pq} + g} \right), \qquad w_0(\xi, \eta) = \frac{2}{\sqrt{\theta}} \sin p\pi\xi \sin \frac{q\pi\eta}{\theta},$$

$$\psi_{\xi 0}(\xi, \eta) = -\frac{2p\pi g}{\sqrt{\theta}(\alpha_{pq} + g)} \cos p\pi\xi \sin \frac{q\pi\eta}{\theta},$$

$$\psi_{\eta 0}(\xi, \eta) = -\frac{2q\pi g}{\theta\sqrt{\theta}(\alpha_{pq} + g)} \sin p\pi\xi \cos \frac{q\pi\eta}{\theta}; \tag{41}$$
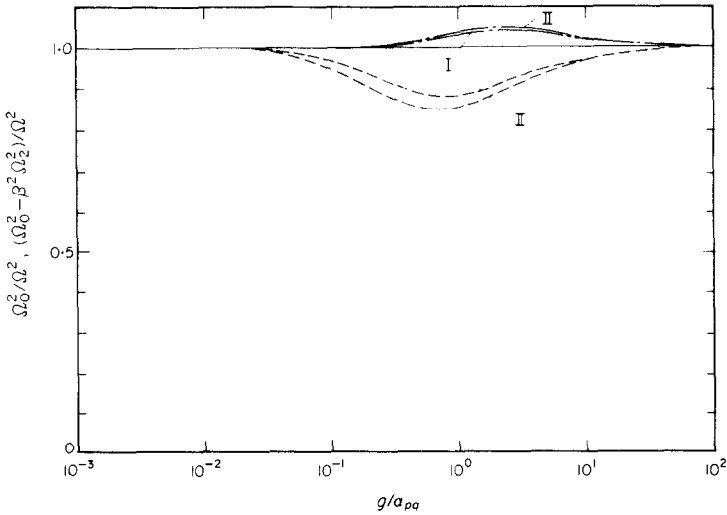
Figure 2. Ratio of the asymptotic solutions of natural frequencies to the exact ones vs. the variable g/α_{pq}. - - -, Ω_0^2/Ω^2; - · -, $(\Omega_0^2 - \beta^2 \Omega_2^2)/\Omega^2$. I, $Y = 3 \cdot 5$; II, $Y = 10$ ($\beta = 1 \cdot 0$).

here $\alpha_{pq} = p^2 \pi^2 + q^2 \pi^2/\theta^2$, and p and q are positive integers;

$$\eta_1^* = \alpha_{pq} Y g/[(\alpha_{pq} + g)^2 + Y g(\alpha_{pq} + g)], \qquad w_1(\xi, \eta) = 0.$$

$$\psi_{\xi 1}(\xi, \eta) = -\frac{2 p \pi g \alpha_{pq}}{\sqrt{\theta}(\alpha_{pq} + g)^2} \cos p \pi \xi \sin \frac{q \pi \eta}{\theta},$$

$$\psi_{\eta 1}(\xi, \eta) = -\frac{2 q \pi g \alpha_{pq}}{\theta \sqrt{\theta}(\alpha_{pq} + g)^2} \sin p \pi \xi \cos \frac{q \pi \eta}{\theta}; \qquad (42)$$

$$\Omega_2^2 = - Y g^2 \alpha_{pq}^3/(\alpha_{pq} + g)^3, \qquad w_2(\xi, \eta) = 0,$$

$$\psi_{\xi 2}(\xi, \eta) = \frac{2 p \pi g^2 \alpha_{pq}}{\sqrt{\theta}(\alpha_{pq} + g)^3} \cos p \pi \xi \sin q \pi \eta/\theta,$$

$$\psi_{\eta 2}(\xi, \eta) = \frac{2 q \pi g^2 \alpha_{pq}}{\theta \sqrt{\theta}(\alpha_{pq} + g)^3} \sin p \pi \xi \cos \frac{q \pi \eta}{\theta}; \qquad (43)$$

$$\eta_3^* = \eta_1^*(1 + Y)g^2/[(\alpha_{pq} + g)^2 + Y g(\alpha_{pq} + g)], \qquad w_3(\xi, \eta) = 0,$$

$$\psi_{\xi 3}(\xi, \eta) = -\frac{2 p \pi g^3 \alpha_{pq}}{\sqrt{\theta}(\alpha_{pq} + g)^4} \cos p \pi \xi \sin \frac{q \pi \eta}{\theta},$$

$$\psi_{\eta 3}(\xi, \eta) = -\frac{2 q \pi g^3 \alpha_{pq}}{\theta \sqrt{\theta}(\alpha_{pq} + g)^4} \sin p \pi \xi \cos \frac{q \pi \eta}{\theta}, \ldots \qquad (44)$$

As this problem is simple, the exact solution of it can be obtained directly from equations (15)

$$\Omega^2 = \alpha_{pq}^2 \left\{ 1 + \frac{Y g[\alpha_{pq} + g(1 + \beta^2)]}{(\alpha_{pq} + g)^2 + g^2 \beta^2} \right\}, \qquad w(\xi, \eta) = \frac{2}{\sqrt{\theta}} \sin p \pi \xi \sin \frac{q \pi \eta}{\theta},$$

$$\psi_\xi(\xi, \eta) = -\frac{2 p \pi g(1 + i\beta)}{\sqrt{\theta}[\alpha_{pq} + g(1 + i\beta)]} \cos p \pi \xi \sin \frac{q \pi \eta}{\theta},$$

$$\psi_\eta(\xi, \eta) = -\frac{2 q \pi g(1 + i\beta)}{\theta \sqrt{\theta}[\alpha_{pq} + g(1 + i\beta)]} \sin p \pi \xi \cos \frac{q \pi \eta}{\theta}.$$

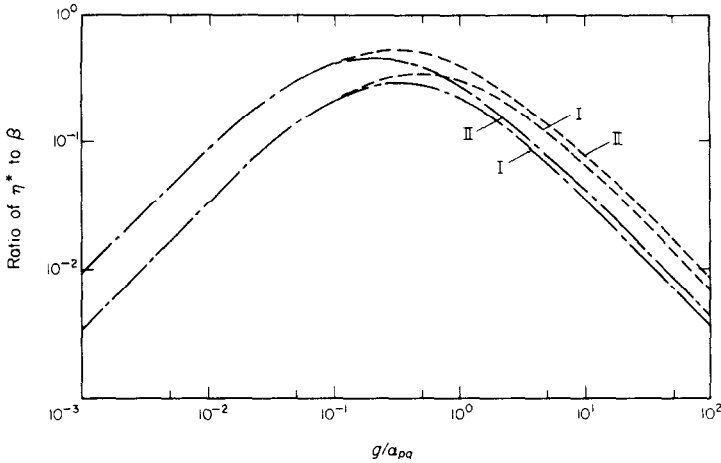$$\eta^* = \alpha_{pq} Y g \beta/[(\alpha_{pq} + g)^2 + Y g(\alpha_{pq} + g) + (1 + Y)g^2 \beta^2]. \qquad (45)$$

Figure 3. Ratio of η^* to β vs. the variable g/α_{pq}. $\beta = 1.0$. - - -, η_1^*; — · —, $\eta_1^*/[1+(\eta_3^*/\eta_1^*)\beta^2]$. I, $Y = 3.5$; II, $Y = 10$.

To examine the accuracy of the asymptotic solution obtained, numerical calculation was done for two plates, both with $\beta = 1.0$, but with $Y = 3.5$ or 10 respectively. In Figure 2 the curves representing the variation of Ω_0^2/Ω^2 and $(\Omega_0^2 - \beta^2\Omega_2^2)/\Omega^2$ with the variable g/α_{pq} are given; here Ω^2 is the exact solution as shown in equations (45). Obviously the values $\Omega_0^2 - \beta^2\Omega_2^2$ approximate to the exact values Ω^2 better than the values Ω_0^2, which are equivalent to the result obtained by the MSE Method. Figure 3 shows η^*/β as a function of the variable g/α_{pq}, where the values of η^* are calculated according to $\eta_1^*\beta$, or equation (40). The latter in this example is just the exact expression of η^* as shown in the last of equations (45). Figure 2 and Figure 3 are applicable to all modes of simply supported rectangular plates with various values of θ.

5. CONCLUSIONS

In this paper a set of simplified governing equations and corresponding boundary conditions of viscoelastically damped unsymmetrical sandwich plates in flexural vibration are given. To avoid calculation with complex values, an asymptotic solution of the simplified governing equations has been introduced, with the loss factor of the viscoelastic material of the core used as a parameter. In simpler cases the asymptotic solution in analytical form can be obtained. In other cases one can use approximate methods to obtain numerical solutions. If in the asymptotic solution only the first terms of all quantities are adopted, then the result is identical in principle with that as given in accordance with the Modal Strain Energy (MSE) Method. However, when the value of the loss factor β of the viscoelastic material is larger, then the errors in the values of the natural frequencies and modal loss factors are somewhat appreciable in a certain range of the value of g. By taking more terms of the asymptotic solution, with successive calculations, accuracy can be improved. With the character of the free flexural vibrations of a viscoelastically damped sandwich plate thus established, one can further analyze the response of the plate to various types of dynamic loads to provide a reliable basis for design.

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