Nonlinear theory of a positive column in a magnetic field

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A nonlinear theory of an intermediate pressure discharge column in a magnetic field is presented. Motion of the neutral gas is considered. The continuity and momentum transfer equations for charged particles and neutral particles are solved by numerical methods. The main result obtained is that the rotating velocities of ionic gas and neutral gas are approximately equal. Bohm's criterion and potential inversion in the presence of neutral gas motion are also discussed.

I. INTRODUCTION

The theory of an intermediate-pressure discharge column in a magnetic field has been described in Refs. 1 and 2. "Intermediate pressure" implies that both the electron-neutral and ion-neutral collision mean free paths are of the same order of magnitude as the radius of the tube. This theory coincides with the "free-fall theory" in a magnetic field at low pressure, and also with the ambipolar diffusion theory at high pressure. The same range of pressure is considered in this paper.

In Refs. 1 and 2, it was assumed that the neutral gas is at rest. In fact, the neutral gas is in motion. In some applications,[3,4] the motion of neutral gas is of great importance. In this paper, therefore, we shall discuss not only the variation of electric field and the density and velocity of charged particles, but also the motion of neutral gas, especially the rotating velocity of neutral gas.

The discharge chamber is cylindrical, the wall of the chamber is dielectric. We assume that both the discharge and the applied magnetic field are longitudinal. The radial and azimuthal velocities of charged particles have been obtained in Ref. 1. The assumption of this nonlinear theory is that the ion motion close to the wall is inertia-controlled rather than collision-controlled. Persson[5,6] first showed that the nonlinear inertial term in the equation of motion of the ions gives rise, in the plasma approximation, to a plasma boundary where the density and potential are finite, and the ambipolar diffusion velocity equals the isothermal sound speed. Reference 1 indicated that this boundary condition is just the singular point of the equations, and the Bohm's criterion for formation of a monotonic sheath is satisfied. There are two singular points when the motion of neutral gas is taken into account. In this case, is Bohm's criterion still valid? This problem is well worth discussing.

We shall also investigate the rotational velocity of the neutral gas, the distribution of number densities and the potential inversion.

II. FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS

We assume that:

(1) the positive column is composed of three component gases: the electron gas, ion gas, and neutral gas;

(2) collisions between electrons and ions may be neglected;

(3) the collision frequency is independent of the relative velocity between particles;

(4) only single-stage ionization is taken into account;

(5) ion-electron recombination occurs at the dielectric wall only; both volume recombination and electron attachment are negligible;

(6) we use the first two moments of the Boltzmann equation; the ion temperature, the electron temperature and the temperature of neutral gas are assumed to be uniform in spatial distribution, and each of them has a Maxwellian distribution at its own temperature;

(7) the positive column is steady and the longitudinal gradient is negligible;

(8) the column extends very far so as to make end effects negligible;

(9) dissipative factors such as viscosity and heat conductivity are negligible.

The basic equations of a three-component gas are:

$$\nabla\cdot\mathbf{V}_e + (\mathbf{V}_e\cdot\nabla)\ln N_e = \nu^i, \tag{1}$$

$$\nabla\cdot\mathbf{V}_p + (\mathbf{V}_p\cdot\nabla)\ln N_p = (N_e/N_p)\nu^i, \tag{2}$$

$$\nabla\cdot\mathbf{V}_n + (\mathbf{V}_n\cdot\nabla)\ln N_n = -(N_e/N_n)\nu^i, \tag{3}$$

$$(\mathbf{V}_e\cdot\nabla)\mathbf{V}_e + \frac{KT_e}{M_e}\frac{\nabla N_e}{N_e} + \frac{e}{M_e}(\mathbf{E}+\mathbf{V}_e\times\mathbf{B}) = \nu_e(\mathbf{V}_n-\mathbf{V}_e) - \nu^i\mathbf{V}_e, \tag{4}$$

$$(\mathbf{V}_p\cdot\nabla)\mathbf{V}_p + \frac{KT_p}{M_p}\frac{\nabla N_p}{N_p} - \frac{e}{M_p}(\mathbf{E}+\mathbf{V}_p\times\mathbf{B}) = \nu_p(\mathbf{V}_n-\mathbf{V}_p) - \nu^i\mathbf{V}_p, \tag{5}$$

$$(\mathbf{V}_n\cdot\nabla)\mathbf{V}_n + \frac{KT_n}{M_n}\frac{\nabla N_n}{N_n} = -\frac{M_eN_e}{M_nN_n}\nu_e(\mathbf{V}_n-\mathbf{V}_e) - \frac{M_pN_p}{M_nN_n}\nu_p(\mathbf{V}_n-\mathbf{V}_p) + \frac{M_eN_e}{M_nN_n}\nu^i\mathbf{V}_e + \frac{M_pN_p}{M_nN_n}\nu^i\mathbf{V}_p, \tag{6}$$

where N_e, N_p, N_n are the number densities of electron gas, ion gas, and neutral gas; T_e, T_p, T_n are the temperatures of electron gas, ion gas, and neutral gas; $\mathbf{V}_e$, $\mathbf{V}_p$, $\mathbf{V}_n$ are the velocities of electron gas, ion gas, and neutral gas; M_e, M_p,

M_n are the mass of an electron, ion, and neutral particle; ν_e, ν_p are the electron-neutral and ion-neutral collision frequency; ν^i is the average ionizing collision frequency for an electron; **E** is the electric field; its component in the longitudinal direction is given, the potential in the radial direction is ϕ, and $\phi = 0$ at $r = 0$; **B** is the magnetic field, in the longitudinal direction; K is the Boltzmann constant; and e is the charge of an electron. As to the collision frequency we have $\nu^i = C_1 N_n$, $\nu_e = C_2 N_n$, $\nu_p = C_3 N_n$. The values of C_1, C_2, C_3 are different for different gases and can be found in appropriate refernces. In the case of argon, for example, C_1, C_2, C_3 are computed in Ref. 7 [expression (15)].

The cylindrical coordinate system (r,θ,z) is introduced and nondimensional substitution is made under the assumptions that all quantities are independent of θ and z; that the nonlinear inertial term of electrons may be neglected; and that $\nu_e \gg \nu^i$, $\nu_p \gg \nu^i$, $N_e = N_p$, $U_{er} = U_{pr}$, $M_p = M_n$, and $T_p = T_n$. Dimensionless variables are defined as

$$\bar\phi = \frac{e\phi}{KT_e}, \quad \mathbf{U} = \left(\frac{M_p}{K(T_e+T_p)}\right)^{1/2}\mathbf{V},$$

$$S = r\left(\frac{M_p}{K(T_e+T_p)}\right)^{1/2}\nu_p'(0), \quad \bar N_e = \frac{N_e}{N_n(0)},$$

$$\bar N_p = N_p/N_n(0), \quad \bar N_n = N_n/N_n(0), \tag{7}$$

where $-(d\phi/dr) = E_r$. The dimensionless parameters are

$$\tau = \frac{T_p}{T_e}, \quad A = \frac{\nu^i}{\nu_p'},$$

$$R' = \frac{M_p\nu_p'}{M_e\nu_e'}, \quad R = \frac{M_p\nu_p}{M_e\nu_e}, \quad M = \frac{\Omega_e}{\nu_e(0)},$$

where $\nu_p' = \nu_p + \nu^i$, $\nu_e' = \nu_e + \nu^i$, $\Omega_e = eB/M_e$ and the notation "0" in the parenthesis denotes $r = 0$.

Using the assumptions listed above, projecting Eqs. (4)–(6) in the directions of r and θ, and noting that $U_{e\theta}$ is uncoupled, we obtain seven nondimensional equations from (1)–(6) which form a closed set for seven unknown functions U_{er}, U_{nr}, $U_{n\theta}$, $U_{p\theta}$, $\ln\bar N_e$, $\ln\bar N_n$, $\bar\phi$. They can be rearranged in the following form:

$$\frac{dU_{er}}{dS} = \frac{1}{1-U_{er}^2}\left[A\bar N_n - U_{er}\frac{1+U_{p\theta}^2}{s} + \left(\frac{M^2}{R'\bar N_n} + \frac{R'+1}{R'}\bar N_n\right)U_{er}^2 - \frac{M}{R'}U_{er}U_{n\theta} - \left(\frac{1}{R'} + \frac{1}{1+A}\right)\bar N_n U_{er}U_{nr}\right], \tag{8}$$

$$\frac{d}{dS}\ln\bar N_e = \frac{1}{1-U_{er}^2}\Bigg\{-\left[A + \frac{1}{R'}\left(\frac{M^2}{\bar N_n^2}+1\right)+1\right]\bar N_n U_{er} + \frac{U_{er}^2 + U_{p\theta}^2}{S} + \frac{M}{R'}U_{p\theta} + \left(\frac{1}{R'} + \frac{1}{1+A}\right)\bar N_n U_{nr} - \frac{M}{R'}U_{n\theta}\Bigg\}, \tag{9}$$

$$\frac{d\bar\phi}{dS} = \frac{1}{1-U_{er}^2}\Bigg\{\left[-A-1+\frac{\tau}{R'}\left(\frac{M^2}{\bar N_n^2}+1\right)\right]\bar N_n U_{er} + \frac{U_{er}^2+U_{p\theta}^2}{S} + \frac{M}{R'}U_{p\theta} + \tau\frac{M}{R'}U_{n\theta} + \left(-\frac{\tau}{R'} + \frac{1}{1+A}\right)\bar N_n U_{nr} - (1+\tau)U_{er}^2\left[\frac{1}{R'}\left(\frac{M^2}{\bar N_n^2} + \bar N_n\right)U_{er} + \frac{M}{R'}U_{n\theta} - \frac{1}{R'}\bar N_n U_{nr}\right]\Bigg\}, \tag{10}$$

$$\frac{dU_{p\theta}}{dS} = -\frac{U_{p\theta}}{S} - \frac{M}{R'} - \bar N_n\frac{U_{p\theta}}{U_{er}} + \left(\frac{\bar N_n}{1+A}\right)\frac{U_{n\theta}}{U_{er}}, \tag{11}$$

$$\frac{d}{dS}\ln\bar N_n = \frac{1}{\dfrac{\tau}{1+\tau} - U_{nr}^2}\left[\frac{U_{n\theta}^2 + U_{nr}^2}{S} + \left(A - \frac{1}{R'} - \frac{1}{1+A}\right)\bar N_e U_{nr} + \left(1+\frac{1}{R'}\right)\bar N_e U_{er}\right], \tag{12}$$

$$\frac{dU_{nr}}{dS} = \frac{-1}{\dfrac{\tau}{1+\tau} - U_{nr}^2}\left[\frac{\tau}{1+\tau}A\bar N_e + \left(\frac{\tau}{1+\tau}\right)\frac{U_{nr}}{S} + U_{nr}\frac{U_{n\theta}^2}{S} + \left(1+\frac{1}{R'}\right)\bar N_e U_{er}U_{nr} - \left(\frac{1}{R'} + \frac{1}{1+A}\right)\bar N_e U_{nr}^2\right], \tag{13}$$

$$\frac{dU_{n\theta}}{dS} = -\frac{U_{n\theta}}{S} - \frac{1}{1+A}\bar N_e\frac{U_{n\theta}}{U_{nr}} + \left(\frac{1}{R'} + \frac{M_e}{M_n}A\right)M\left(\frac{\bar N_e}{\bar N_n}\right)\frac{U_{er}}{U_{nr}} + \bar N_e\frac{U_{p\theta}}{U_{nr}}. \tag{14}$$

The boundary conditions are as follows: at $S = 0$,

$$U_{er} = \bar\phi = U_{p\theta} = \ln\bar N_n = U_{nr} = U_{n\theta} = 0,$$
$$\ln\bar N_e = \ln\bar N_e(0), \tag{15}$$

and

$$S = S_b, \quad U_{er} = 1, \text{ or } U_{nr} = [\tau/(1+\tau)]^{1/2}, \tag{16}$$

where $\ln\bar N_e(0)$ is known, and S_b is to be determined.

Since some terms on the right-hand side of Eqs. (8)–(14) may become 0/0 at $S = 0$ under the condition of (15), i.e., they are indeterminate, other constraints are needed. We assume that, near $S = 0$, the quantities U_{er}, $U_{p\theta}$, U_{nr}, $U_{n\theta}$ are proportional to s. Substituting these relations into the equations, and taking the limit $S \to 0$, we have

$$\frac{dU_{er}}{dS} = \frac{A}{2}, \quad \frac{d\ln\bar N_e}{dS} = \frac{d\bar\phi}{dS} = 0,$$

TABLE I. Parameters identifying the two reference cases.

	τ	A	R'
Case A	0.0431	0.393	53.9
Case B	0.1	0.01	30.3

$$\frac{dU_{p\theta}}{dS}=\frac{-M}{2(1+A)(2+A)}\left(\frac{M_e}{M_n}+\frac{(1+A)^2}{R'}\right),$$
$$\frac{d\ln\overline{N}_n}{dS}=0,\quad \frac{dU_{nr}}{dS}=-\frac{A}{2}\overline{N}_e(0),$$
$$\frac{dU_{nr}}{dS}=-\frac{A}{2}\overline{N}_e(0),$$
$$\frac{dU_{n\theta}}{dS}=-\frac{M(1+A)}{2(2+A)}\left(\frac{M_e}{M_p}+\frac{1}{R'}\right). \quad (17)$$

The numerical results were obtained on a TQ-16 computer using the Runge–Kutta method with variable step sizes.

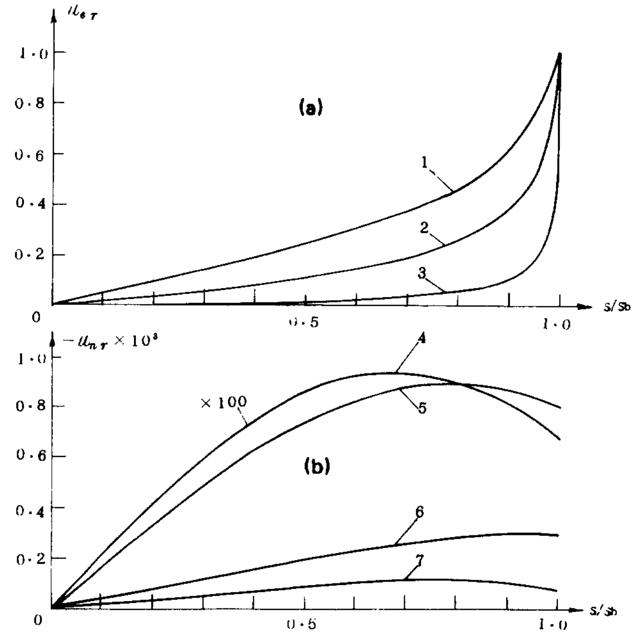

FIG. 1. Spatial variations of U_{er} and U_{nr} for case A.

III. RESULTS AND DISCUSSION

We have computed the case $U_{nr}=U_{n\theta}=\ln\overline{N}_n=0$ as a special one with $\tau=0.1, A=0.01, R'=30.3, M=172$ or 17.2. In Ref. 1, the computational results of the potential $\overline{\phi}$ were presented with $A=0.01$, $R'=30.3$, $c^2=1000$ ($M\sim172$) or $c^2=10$ ($M\sim17.2$). Comparison of the present values of $\overline{\phi}$ with the results in Ref. 1 shows very good agreement.

With motion of the neutral gas, the system of differential equations involves seven physical variables. Two cases were calculated: case A and case B (see Table I). For the sake of comparison, the values of the three parameters in case B are the same as Ref. 1. For brevity, the values of the other two parameters M and $\ln\overline{N}_e(0)$ are represented by the symbols M_k and N_j, respectively (see Table II). The special sets of the five parameters corresponding to the curves in Figs. 1–8 are listed in Table III. The computational results of the seven variables are presented and discussed in the following:

A. The radial velocities U_{er}, U_{nr} and Bohm's criterion

Bohm's criterion is an analytical result. It states that only when the kinetic energy of ions that arrive at the sheath is more than half the electron thermal energy, will a stable sheath then exist (see Ref. 8).

Reference 1 indicates that at the boundary the ambipolar diffusion velocity is equal to the ambipolar thermal velocity $U_{er}=1$, i.e., Bohm's criterion is valid here. However, $U_{er}=1$ is also a singular point of the equations, so that it is appropriate to integrate the equations until $U_{er}=1$ is reached. However, the motion of neutral gas introduces another singular point $U_{nr}=[\tau/(1+\tau)]^{1/2}$ in the equations. Mathematically, it may be possible that $U_{nr}=[\tau/(1+\tau)]^{1/2}$ is reached before the integration reaches $U_{er}=1$, so that further integration is meaningless. In this case, Bohm's criterion will not be satisfied. Physically, this corresponds to saying that a stable sheath may not exist in the presence of neutral gas motion and that the plasma sheath model must be reconsidered.

For the sets of parameters which we have computed, all integrations reach $U_{er}=1$ before $U_{nr}=[\tau/(1+\tau)]^{1/2}$, so Bohm's criterion is still valid for these sets of parameters.

The computational results for U_{er} and U_{nr} are shown in Figs. 1 and 2. It can be seen that the radial velocity of the neutral gas U_{nr} is one order smaller than that of the electron gas U_{er}.

TABLE II. Density and frequency ratios used in Figs. 1–8.

$\ln\overline{N}_e(0)$ \ M	$M_1=2.65$	$M_2=26.5$	$M_3=265$	$M_4=17.2$	$M_5=54.5$	$M_6=172$
$N_1=-4.605$	N_1+M_1	N_1+M_2	N_1+M_3	N_1+M_4	N_1+M_5	N_1+M_6
$N_2=-6.908$	N_2+M_1	N_2+M_2	N_2+M_3	N_2+M_4	N_2+M_5	N_2+M_6
$N_3=-9.211$	N_3+M_1	N_3+M_2	N_3+M_3	N_3+M_4	N_3+M_5	N_3+M_6

TABLE III. Parameters corresponding to the curves of Figs. 1–8.

	Case A				Case B			
	Fig. 1	Fig. 3	Fig. 6	Fig. 7	Fig. 2	Fig. 4	Fig. 5	Fig. 8
Curve 1	M_1+N_2	M_3+N_2	M_1+N_2	M_2+N_3	M_4+N_2	$M_6+\begin{cases}N_1\\N_2\\N_3\end{cases}$	$M_6+\begin{cases}N_1\\N_2\\N_3\end{cases}$	M_6+N_3
Curve 2	$M_2+\begin{cases}N_1\\N_2\\N_3\end{cases}$	$M_2+\begin{cases}N_1\\N_2\\N_3\end{cases}$	$M_2+\begin{cases}N_1\\N_2\\N_3\end{cases}$	M_3+N_2	M_5+N_2	M_5+N_2	M_5+N_2	$\left.\begin{matrix}M_4\\M_5\\M_6\end{matrix}\right\}+N_2$
Curve 3	M_3+N_2	M_1+N_2	M_3+N_2	M_2+N_2	$M_6+\begin{cases}N_1\\N_2\\N_3\end{cases}$	M_4+N_2	M_4+N_2	M_6+N_1
Curve 4	M_3+N_2			M_1+N_2	M_6+N_1			M_6+N_1
Curve 5	M_2+N_1			M_2+N_1	M_4+N_2			M_5+N_2
Curve 6	M_1+N_2			M_3+N_2	M_5+N_2			M_4+N_2
Curve 7	M_2+N_2			M_2+N_1	M_6+N_2			
Curve 8				$M_2+\begin{cases}N_2\\N_3\end{cases}$				
Curve 9				M_1+N_2				

B. The azimuthal velocities $U_{p\theta}$ and $U_{n\theta}$

For the sets of parameters computed here, $U_{p\theta}$ and $U_{n\theta}$ are found to have nearly the same value. In this case, the electromagnetic field gives momentum to charged particles, mainly to ions, and the ions give momentum to the neutral particles through collisions. It follows that it is not appropriate to neglect $U_{n\theta}$ and retain $U_{p\theta}$ in the equations such as was done in Refs. 1 and 2.

Since the steady state is considered, there is little influence of initial electron density on the magnitude of $U_{p\theta}$ and $U_{n\theta}$. However, if the accelerating process is considered, the greater the electron density, the shorter the transient process will be, i.e., the steady state should be reached more quickly.

The magnetic field has a larger effect on the neutral gas rotation. The higher the magnetic field, the greater $U_{p\theta}$ and $U_{n\theta}$, i.e., the faster the rotation. Since the driving force for the ion rotation is proportional to B, this result is reasonable.

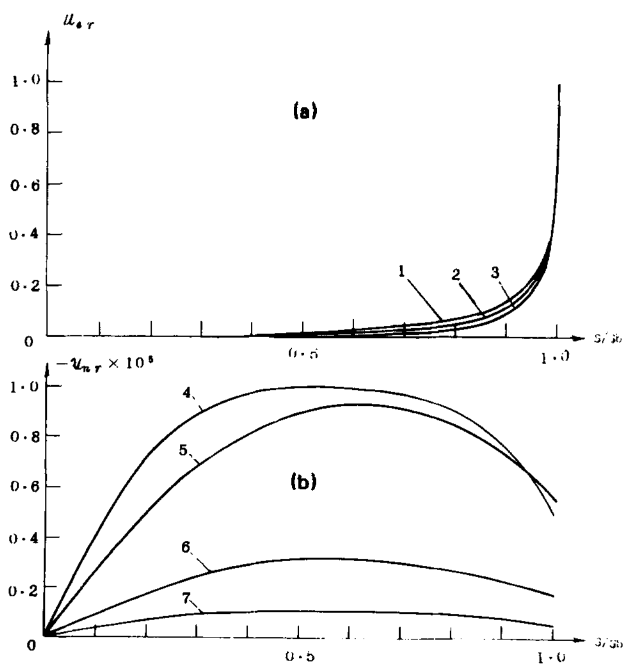

FIG. 2. Spatial variations of U_{er} and U_{nr} for case B.

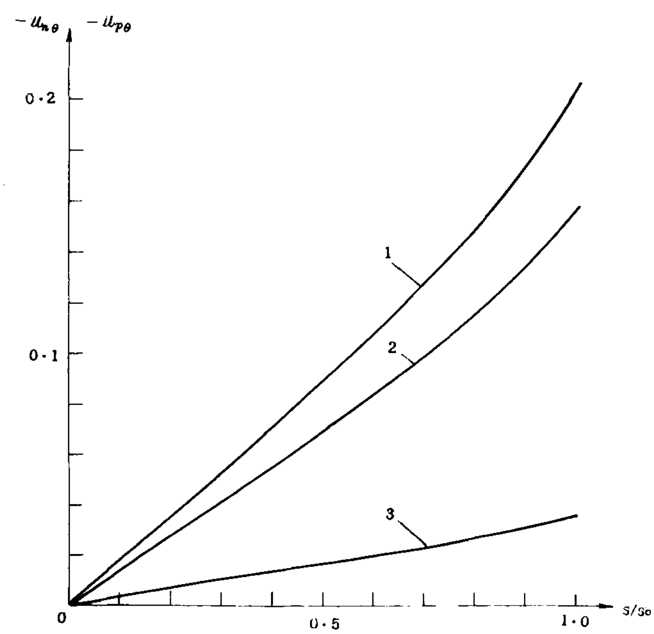

FIG. 3. Spatial variations of $U_{p\theta}$ and $U_{n\theta}$ (they are nearly the same) for case A.

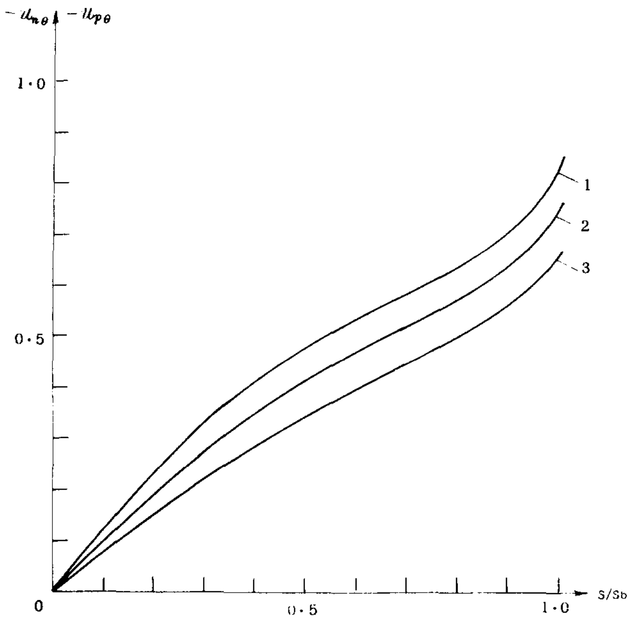

FIG. 4. Spatial variations of $U_{p\theta}$ and $U_{n\theta}$ (they are nearly the same) for case B.

The associated results are shown in Figs. 3 and 4.

When $\tau = 0.1$, $A = 0.01$, $R' = 30.3$, $M = 172$ (there is no effect from the initial electron density), we obtain $U_{n\theta} = -0.86$ at the wall. The magnitude of $U_{n\theta}|_{s=s_b}$ may be even greater for other chosen parameters.

C. The quantity $\bar{\phi}$ and potential inversion

If one considers the rotation of the neutral gas, the phenomenon of potential inversion becomes less frequent. It is seen from Fig. 5 that in the case of $M = 172$, all parameters are the same as Ref. 1, potential inversion appears when rotation of the neutral gas is not considered, but disappears

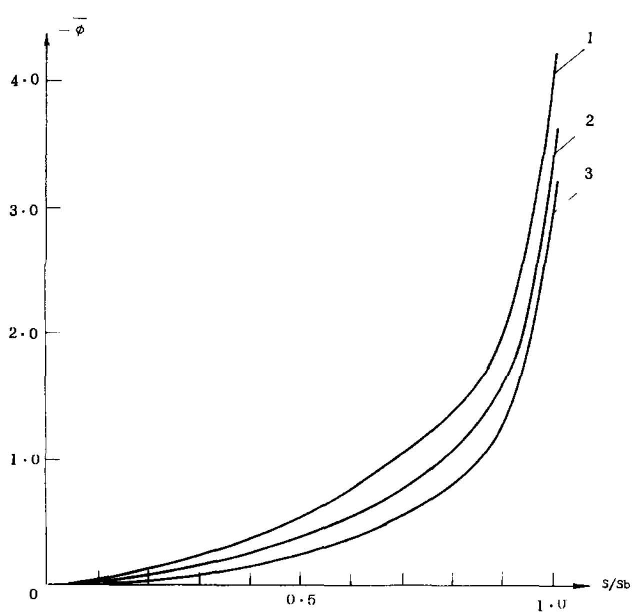

FIG. 5. Spatial variations of $\bar{\phi}$ for case B.

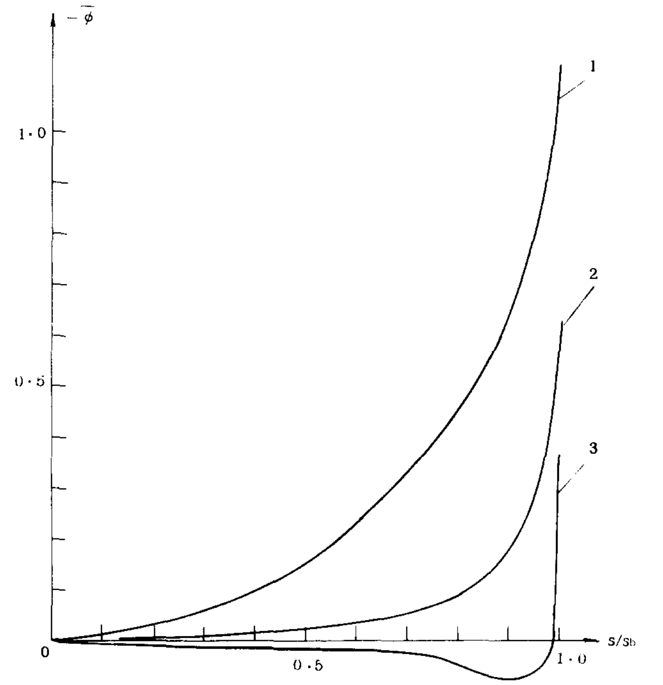

FIG. 6. Spatial variations of $\bar{\phi}$ for case A.

when rotation is considered. The potential inversion still appears with other sets of parameters, such as those given in Fig. 6, curve 3.

When the potential inversion is absent, the radial component of the electric field points in the direction of increasing radius. The associated charge distribution is such that there is more positive charge near the center and more negative near the wall.

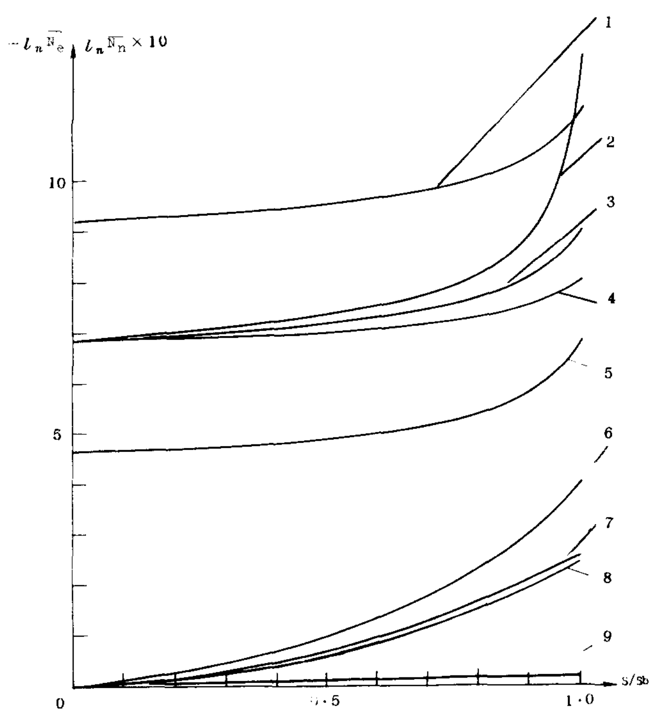

FIG. 7. Spatial variations of $\bar{N}_e$ (curves 1–5) and $\bar{N}_n$ (curves 6–9) for case A.

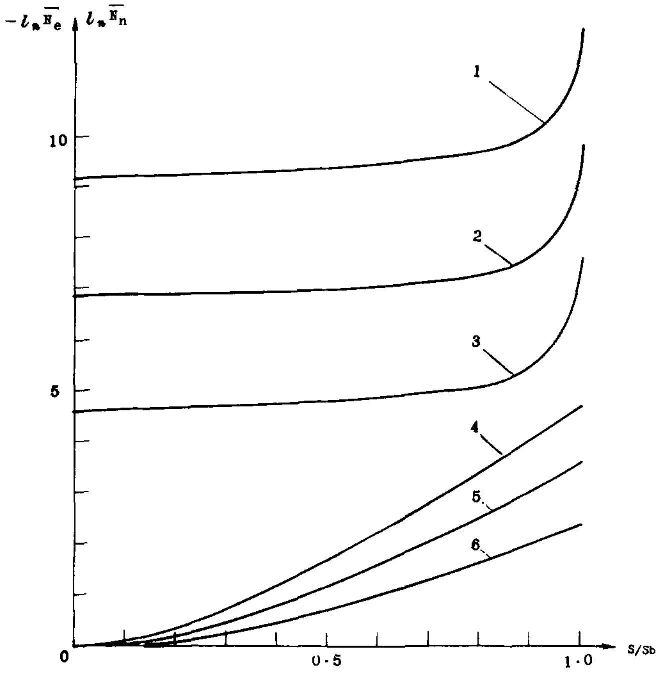

FIG. 8. Spatial variations of $\overline{N}_e$ (curves 1–3) and $\overline{N}_n$ (curves 4–6) for case B.

D. The density logarithms ln $\overline{N}_e$ and ln $\overline{N}_n$

The electron density profile shows that $\overline{N}_n$ is much larger at the center than at the wall, and the ion profile is the same since we have assumed a fully neutralized plasma. These profiles may be treated as the solutions of the zero-order iteration. When the electric field is known, from the electric field equation $\nabla \cdot \mathbf{E} = e(N_p - N_e)/\epsilon_0$, we may obtain the charge profile as the solution of the first-order iteration.

As the result of neutral gas rotation, a radial pressure gradient appears. For the isothermal case, this means that the radial gradient of neutral particle density is present. It can be seen from Figs. 7 and 8 that the greater the value of M (i.e., the stronger the magnetic field), the larger the pressure gradient. However, the $\overline{N}_n$ profile is little affected by the initial electron density.

[1] H. N. Ewald, F. W. Crawford, and S. A. Self, Appl. Phys. **38**, 2753 (1967).
[2] J. R. Forrest and R. N. Franklin, Brit. J. Appl. Phys. **17**, 1569 (1966).
[3] O. Kaneko, S. Sasaki, and N. Kawashima, Plasma Phys. **20**, 1167 (1978).
[4] B. W. James and S. W. Simpson, Plasma Phys. **18**, 289 (1976).
[5] K. B. Persson, Phys. Fluids **5**, 1625 (1962).
[6] E. R. Mosburg, Jr., and K.B. Persson, Phys. Fluids **7**, 1829 (1964).
[7] H. A. Hassan and C. C. Thompson, Plasma Phys. **12**, 727 (1970).
[8] A. Guthrie and R. K. Wakerling, *Characteristics of Electrical Discharges in Magnetic Fields* (McGraw-Hill, New York, 1949), p. 77.