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Theoretical analysis of the relationships between hardness, elastic modulus, and the work of indentation for work-hardening materials

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(Received 17 May 2010; accepted 11 August 2010)

In our previous paper, the expanding cavity model (ECM) and Lamé solution were used to obtain an analytical expression for the scale ratio between hardness (H) to reduced modulus (E_r) and unloading work (W_u) to total work (W_t) of indentation for elastic-perfectly plastic materials. In this paper, the more general work-hardening (linear and power-law) materials are studied. Our previous conclusions that this ratio depends mainly on the conical angle of indenter, holds not only for elastic perfectly-plastic materials, but also for work-hardening materials. These results were also verified by numerical simulations.

Over the past two decades, instrumented indentation has gradually evolved into a conventional testing method for measuring mechanical properties of materials at small-scale and in the process has advanced our understanding of mechanical behavior of materials.[1] The most frequently used analytic method, developed by Oliver and Pharr,[2] is based on the solutions of elastic contact; thus its estimation of contact area has been found to be incapable of accounting for pileup behavior of material. Another method based on dimensional analysis and finite element calculations, proposed by Cheng and Cheng[3] in 1998, uses an approximate linear relation of $(H/E_r)/(W_u/W_t)$ to overcome the dependence on the uncertain contact area. Recently, many questions have been raised in regard to this scale ratio relation. Alkorta et al.[4] and Malzbender[5] indicated that this method can incur significant error for soft materials, while Chen and Bull[6] found that when H/E_r is larger than 0.1, significant deviation occurs in this relation. In our previous paper,[7] analytical approaches were adopted to uncover the physical nature of this scale ratio, and it was found that for elastic perfectly-plastic case, the scale ratio has the form $[2(1-\nu)\cot\alpha]/3$, where ν is Poisson's ratio and α is the half-included angle of the conical indenter; in the linear elastic regime, this ratio reduces to the form $(\cot\alpha)/2$.

In this paper, the same approach is taken with certain relaxed constraints to investigate work-hardened materials. The leading order of the derived scale ratio $(H/E_r)/(W_u/W_t)$ is found to be identical to that derived in our previous paper[7] for both linear-hardening and power-law hardened materials. Again, this ratio mainly depends on the half-included angle of conical indenter and slightly on Poisson's ratio.

In deriving the scale ratio, we consider a three-dimensional, rigid, conical indenter of half-included angle α, indenting normally into the surface of a homogeneous work-hardening solid. A general stress–strain relation for the material can be written as

$$\tilde{\sigma} = \begin{cases} E\tilde{\varepsilon}, & \text{for } \tilde{\varepsilon} \le Y/E \\ f(\tilde{\varepsilon}), & \text{for } \tilde{\varepsilon} > Y/E \end{cases} \quad , \tag{1}$$

where $\tilde{\sigma}$ and $\tilde{\varepsilon}$ are the equivalent stress and strain, respectively, E is the elastic modulus, and Y is the yield stress. The function $f(\tilde{\varepsilon})$ is the constitutive equation of material, which for linear-hardening materials takes the form,

$$f(\tilde{\varepsilon}) = Y + E_P\left(\tilde{\varepsilon} - \frac{Y}{E}\right) \quad , \tag{2}$$

while for power-law hardening material, it can be written as

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DOI: 10.1557/JMR.2010.0267

$$f(\tilde{\varepsilon}) = \frac{E^n}{Y^{n-1}}\tilde{\varepsilon}^n \equiv K\tilde{\varepsilon}^n \quad , \tag{3}$$

where E_P is the tangent modulus and n is the work-hardening exponent.

The problem is simplified by adopting the assumptions given by Johnson's ECM[8] as shown in Fig. 1: (i) the displacement field produced by the indenter is approximately spherically symmetric; (ii) the material beneath the indenter can be divided into a core region ($r < a$), a plastic region ($a < r < c$) and an elastic region ($r > c$); (iii) the material of core region is assumed to be incompressible; and (iv) the geometrical similarity of this problem leads to $a/c = da/dc$ during penetration. To obtain a better approximation of the total work, Johnson's assumption of an incompressible fluid is replaced by one of an incompressible solid for the core region. Meanwhile, the state of the material of core region is considered uniform with the same value as that at $r = a$, i.e., on the inner boundary of the plastic region.

By solving this problem, the seven equations that govern the plastic region can be integrated to give two equations, the equivalent strain equation and the constitutive equation, which can be expressed as:

$$\tilde{\varepsilon} + \frac{4(1-2\nu)}{3E}f(\tilde{\varepsilon}) = \frac{(7-8\nu)Y}{3E}\frac{c^3}{r^3} \quad , \tag{4}$$

$$\tilde{\sigma} = f(\tilde{\varepsilon}) \quad . \tag{5}$$

These expressions greatly simplify the problem. Given any specific stress–strain relation, the equivalent strain and stress can be obtained from Eq. (4). Moreover, taken together with volume conservation of the core region and geometrical similarity of this problem leads to

$$\tilde{\varepsilon}\big|_{r=a} = \frac{1}{3}\cot\alpha \quad , \tag{6}$$

and the relative volumetric size of plastic region as

$$\frac{c^3}{a^3} = \frac{3E}{(7-8\nu)Y}\left[\tilde{\varepsilon}\big|_{r=a} + \frac{4(1-2\nu)}{3E}\tilde{\sigma}\big|_{r=a}\right] \quad . \tag{7}$$

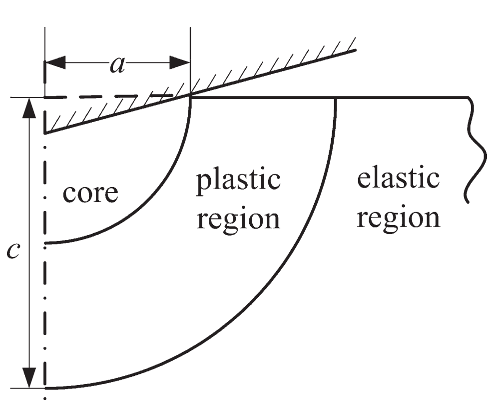

FIG. 1. Stress field of expanding cavity model for conical indentation.

If the material of elastic and plastic regions is incompressible, in which $\nu = 0.5$, Eq. (4) leads to

$$\tilde{\varepsilon} = \frac{Y}{E}\frac{c^3}{r^3} \quad , \tag{8}$$

which is consistent with the solutions given by Gao et al.[9,10] For more general materials, the results are as discussed next.

Beginning with linear-hardening materials, the equivalent strain and stress can be solved by inserting Eq. (2) into Eq. (4), yielding

$$\tilde{\varepsilon} = \frac{(7-8\nu)Y}{4(1-2\nu)E_P + 3E}\frac{c^3}{r^3} - \frac{4(1-2\nu)(E-E_P)Y}{[4(1-2\nu)E_P + 3E]E} \equiv C_1\frac{c^3}{r^3} - C_2 \quad , \tag{9}$$

$$\tilde{\sigma} = \frac{E_P(7-8\nu)Y}{4(1-2\nu)E_P + 3E}\frac{c^3}{r^3} + \frac{3(E-E_P)Y}{4(1-2\nu)E_P + 3E} \equiv D_1\frac{c^3}{r^3} + D_2 \quad , \tag{10}$$

here notations of C_1, C_2, D_1, and D_2 are taken to represent the coefficients in equations. If the material is incompressible, the components of displacement, strain, and stress for the plastic region reduce, respectively, to the simplified forms

$$u = \frac{Y}{2E}\frac{c^3}{r^2} \quad , \tag{11}$$

$$\varepsilon_r = -\frac{Y}{E}\frac{c^3}{r^3} \quad , \tag{12}$$

$$\varepsilon_\theta = \varepsilon_\varphi = \frac{Y}{2E}\frac{c^3}{r^3} \quad , \tag{13}$$

$$\sigma_r = -\frac{2E_PY}{3E}\frac{c^3}{r^3} + \frac{2E_PY}{3E} + \frac{2(E-E_P)Y}{E}\ln\frac{r}{c} - \frac{2Y}{3} \quad , \tag{14}$$

$$\sigma_\theta = \sigma_\varphi = \frac{E_PY}{3E}\frac{c^3}{r^3} - \frac{E_PY}{3E} + \frac{2(E-E_P)Y}{E}\ln\frac{r}{c} + \frac{Y}{3} \quad . \tag{15}$$

The relative volumetric size of the plastic region is determined as

$$\frac{c^3}{a^3} = \frac{E\cot\alpha}{3Y} \quad . \tag{16}$$

The total work is then given as

$$W_{\mathrm{t}} \approx \frac{\pi Y^2 a^3}{E}\left[\frac{2(E-E_{\mathrm{P}})}{E}\frac{E\cot\alpha}{3Y}\ln\frac{c}{a}+\frac{2E_{\mathrm{P}}}{3E}\left(\frac{E\cot\alpha}{3Y}\right)^2\right] \quad , \tag{17}$$

and the hardness as

$$H = -\sigma_{\mathrm{r}}|_{r=a} = \frac{2Y}{3}\left[\frac{\cot\alpha}{3Y}E_{\mathrm{P}}-\frac{E_{\mathrm{P}}}{E}+\frac{3(E-E_{\mathrm{P}})}{E}\ln\frac{c}{a}+1\right] \quad . \tag{18}$$

By the Lamé solution,[7,11] the unloading work is expressible as

$$W_{\mathrm{u}} = \frac{\pi(1+\nu)}{2E}H^2a^3 = \frac{3\pi}{4E}H^2a^3 \quad . \tag{19}$$

Thus the scale ratio can be obtained as

$$\frac{H/E_{\mathrm{r}}}{W_{\mathrm{u}}/W_{\mathrm{t}}} \approx \frac{1}{3}\cot\alpha \quad . \tag{20}$$

If the material is compressible, the components of displacement, strain, and stress for the plastic region can be solved and written, respectively, as

$$u = \frac{C_1c^3}{2r^2}+\frac{3}{2}C_2r\ln\frac{r}{c}+C_3r \quad , \tag{21}$$

$$\varepsilon_{\mathrm{r}} = -\frac{C_1c^3}{r^3}+\frac{3}{2}C_2\ln\frac{r}{c}+C_3+\frac{3}{2}C_2 \quad , \tag{22}$$

$$\varepsilon_\theta = \varepsilon_\varphi = \frac{C_1c^3}{2r^3}+\frac{3}{2}C_2\ln\frac{r}{c}+C_3 \quad , \tag{23}$$

$$\sigma_{\mathrm{r}} = -\frac{2D_1}{3}\frac{c^3}{r^3}-\frac{2D_2}{3}+2D_2\ln\frac{r}{c}-\frac{Y}{6} \quad , \tag{24}$$

$$\sigma_\theta = \sigma_\varphi = \frac{D_1}{3}\frac{c^3}{r^3}+\frac{D_2}{3}+2D_2\ln\frac{r}{c}-\frac{Y}{6} \quad , \tag{25}$$

where C_1, C_2, D_1, and D_2 are the same as above in Eqs. (9) and (10), while an additional coefficient is deduced by using the displacement consistency condition at elastic-plastic boundary ($r = c$), which is given as

$$C_3 = \frac{(1-2\nu)[8(1+\nu)E_{\mathrm{P}}-15E]}{6E[4(1-2\nu)E_{\mathrm{P}}+3E]}Y \quad . \tag{26}$$

The relative volumetric size of plastic region is determined from

$$\frac{c^3}{a^3} = \frac{E\cot\alpha}{(7-8\nu)Y}+\frac{4(1-2\nu)}{3(7-8\nu)}\left[\frac{3(E-E_{\mathrm{P}})}{E}+\frac{E_{\mathrm{P}}\cot\alpha}{Y}\right] \quad . \tag{27}$$

The total work is given as

$$W_{\mathrm{t}} \approx \frac{2\pi D_1^2c^6}{3E_{\mathrm{P}}a^3}+\frac{2\pi D_1D_2c^3}{E_{\mathrm{P}}}\ln\frac{c}{a} \quad , \tag{28}$$

the hardness as

$$H \approx \frac{2D_1}{3}\frac{c^3}{a^3}+2D_2\ln\frac{c}{a} \quad , \tag{29}$$

and the unloading work can be expressed as

$$W_{\mathrm{u}} = \frac{\pi(1+\nu)}{2E}H^2a^3 \quad . \tag{30}$$

Thus, the scale ratio can then be evaluated yielding the result

$$\frac{H/E_{\mathrm{r}}}{W_{\mathrm{u}}/W_{\mathrm{t}}} \approx \frac{2(1-\nu)}{3}\cot\alpha \quad . \tag{31}$$

For power-law hardening materials, we find by inserting Eq. (3) into Eq. (4) that the equivalent strain in the plastic region can be solved from

$$\tilde{\varepsilon}+\frac{4(1-2\nu)}{3E}\frac{E^n}{Y^{n-1}}\tilde{\varepsilon}^n = \frac{(7-8\nu)Y}{3E}\frac{c^3}{r^3} \quad . \tag{32}$$

If the material is incompressible, the expressions for components of displacement and strain are the same as these in Eqs. (11) to (13); meanwhile the stress components are

$$\sigma_{\mathrm{r}} = -\frac{2Y}{3}\left[\frac{1}{n}\frac{c^{3n}}{r^{3n}}+1-\frac{1}{n}\right] \quad , \tag{33}$$

$$\sigma_\theta = \sigma_\varphi = -\frac{2Y}{3}\left[\left(\frac{1}{n}-\frac{3}{2}\right)\frac{c^{3n}}{r^{3n}}+1-\frac{1}{n}\right] \quad . \tag{34}$$

The total work can be deduced as

$$W_{\mathrm{t}} = \frac{\pi Y^2a^3}{E}\left[\frac{2}{3n}\left(\frac{E\cot\alpha}{3Y}\right)^{n+1}+\frac{2(n-1)}{3n}\frac{E\cot\alpha}{3Y} -\frac{\cot\alpha}{3(n+1)}\left(\frac{E\cot\alpha}{3Y}\right)^{n+1}-\frac{(n-1)\cot\alpha}{6(n+1)}\right] \quad , \tag{35}$$

with hardness becoming

$$H = -\sigma_r|_{r=a} = \frac{2Y}{3}\left[\frac{1}{n}\left(\frac{E\cot\alpha}{3Y}\right)^n+1-\frac{1}{n}\right] \quad . \tag{36}$$

The unloading work has the same form as in Eq. (19), while the scale ratio is the same as Eq. (20). If the material is compressible, there is no explicit solution for Eq. (32), except when $n = ½$ and $⅓$ ($n = ¼$ is also solvable, but the expressions are too complex to interpret). By solving this problem implicitly, the stress components are given as

$$\sigma_{\mathrm{r}} \approx -\frac{2}{3n}\tilde{\sigma} \quad , \tag{37}$$

$$\sigma_\theta = \sigma_\varphi \approx -\frac{2-3n}{3n}\tilde{\sigma} \quad . \tag{38}$$

The relative size of plastic region is determined as

$$\frac{c^3}{a^3} = \frac{E\cot\alpha}{(7-8\nu)Y} + \frac{4(1-2\nu)}{7-8\nu}\left(\frac{E\cot\alpha}{3Y}\right)^n \quad , \tag{39}$$

the total work as

$$W_t \approx \frac{2\pi K^{-1/n}}{n+1}\frac{1}{3n}a^3\tilde{\sigma}_a{}^{(n+1)/n} + \frac{2}{3}\pi a^3\frac{1}{1+n}K\left(\frac{\cot\alpha}{3}\right)^{n+1}$$
$$= \frac{2}{3n}\pi a^3 K\left(\frac{\cot\alpha}{3}\right)^{n+1} \quad , \tag{40}$$

and the hardness becomes

$$H \approx \frac{2}{3n}\tilde{\sigma}_a = \frac{2}{3n}K\left(\frac{\cot\alpha}{3}\right)^n \quad . \tag{41}$$

The unloading work is in the same form of Eq. (30), and thus the ratio remains the same as Eq. (31).

The results for work-hardening materials can reduce to those for elastic-perfectly plastic case. Linear-hardening material becomes elastic-perfectly plastic when E_P approaches zero, which leads to

$$W_t \approx 2\pi C_1 Y c^3 \ln\frac{c}{a} \quad , \tag{42}$$

$$H \approx 2Y\ln\frac{c}{a} \quad . \tag{43}$$

Meanwhile, taking $n = 0$, the power-law hardening results reduce as well to those for elastic-perfectly plastic material. After noting that

$$\lim_{x\to 0}\frac{1}{x}(y^x - 1) = \ln y \quad \text{for } y > 0 \quad . \tag{44}$$

We specifically find

$$W_t \approx \lim_{n\to 0}\frac{2}{3n}\pi a^3 K\left(\frac{\cot\alpha}{3}\right)^{n+1} \approx \frac{2\cot\alpha}{3}\pi a^3 Y\ln\frac{c}{a} \quad , \tag{45}$$

$$H \approx \lim_{n\to 0}\frac{2}{3n}K\left(\frac{\cot\alpha}{3}\right)^n \approx 2Y\ln\frac{c}{a} \quad . \tag{46}$$

Thus these results obtained under various limiting processes are consistent with those published previously.[7]

The scale ratio for both linear and power-law hardening presented in this paper can be expressed equivalently as $[2(1-\nu)\cot\alpha]/3$, and it is apparent that this ratio is independent of work hardening. Indeed, this ratio mainly depends on the half-included angle of conical indenter and slightly on Poisson's ratio.

Extensive finite element method (FEM) calculations are performed using ABAQUS[12] to verify our analytical results given in this paper. Situations involving different conical angles, $\alpha = 42.3°$, 50°, 60°, 70.3°, and 80° were studied, with $2.5 \times 10^{-4} \leq Y/E \leq 2.5 \times 10^{-3}$ for most cases except for $\alpha = 70.3°$ where we extended the upper range to $Y/E \leq 1 \times 10^{-1}$. The FEM results are presented in Figs. 2 to 4, in which H/E_r is normalized by cot α.

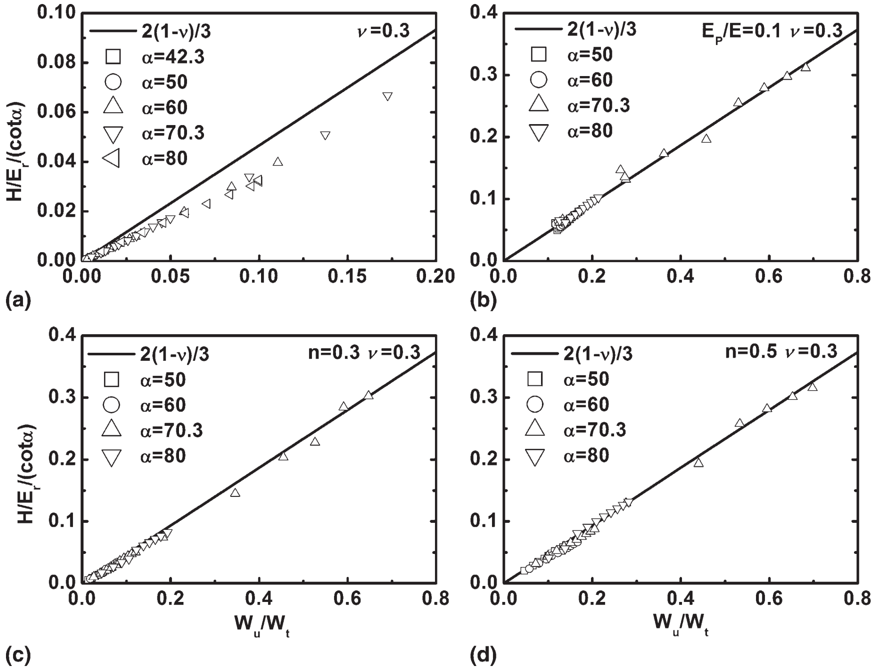

FIG. 2. Analytical result (solid curve) and FEM results of the scale relation for various conical angles when $\nu = 0.3$, the stress–strain relations are: (a) elastic-perfectly plastic, (b) linear hardening, and (c, d) power-law hardening.

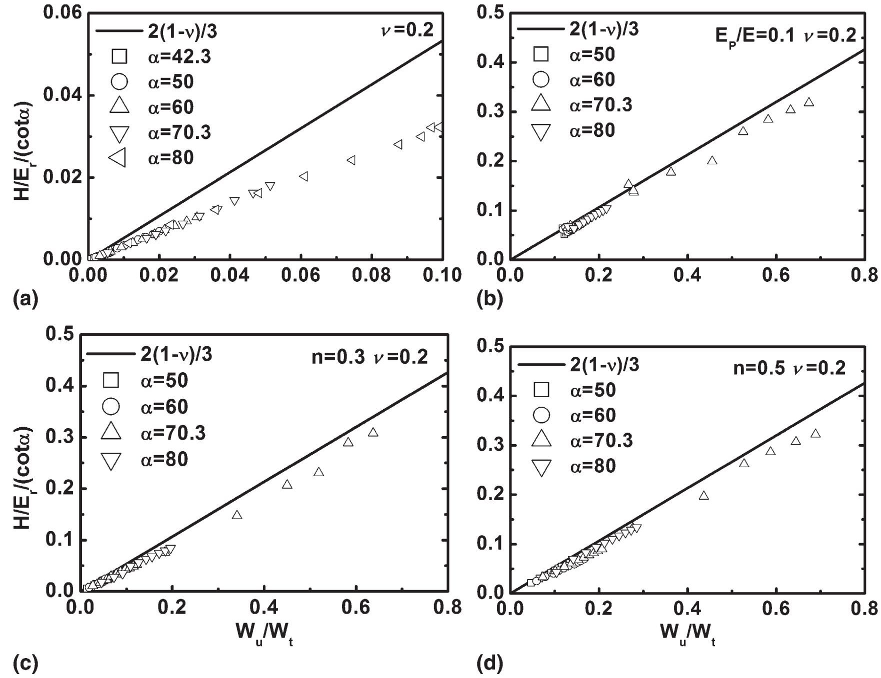

FIG. 3. Analytical result (solid curve) and FEM results of the scale relation for various conical angles when $\nu = 0.2$, the stress–strain relations are: (a) elastic-perfectly plastic, (b) linear hardening, and (c, d) power-law hardening.

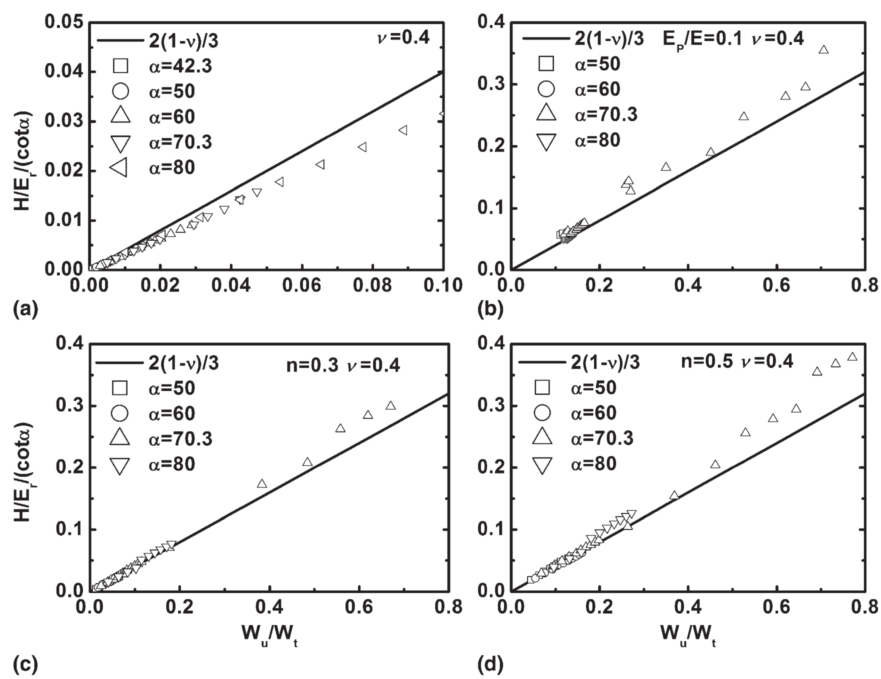

FIG. 4. Analytical result (solid curve) and FEM results of the scale relation for various conical angles when $\nu = 0.4$, the stress–strain relations are: (a) elastic-perfectly plastic, (b) linear hardening, and (c, d) power-law hardening.

The analytical form of the scale ratio $[2(1 - \nu) \cot \alpha]/3$ agrees with these FEM results. The normalized relation in these figures is found to be independent of conical angle; meanwhile, deviations still exist for different values of Poisson's ratio and elastic-perfectly plastic materials.

By the assumptions of ECM, the validity range of analytic expressions derived in this paper are as follows: (i) The half-included angle of indenter should be $45° < \alpha < 90°$. The lower limit of indenter angle stands for the minimum angle for ensuring all material within the radius

a is in the core region. However, the FEM results on $\alpha = 42.3°$ shows that ECM still works for that angle. (ii) Plasticity should predominate in the deformation of the sample material, which has limited our expression for metals. Even so, the FEM results shows that our analytic expression still holds for materials with larger W_u/W_t ratio according to Figs. 3(b) and 3(c), etc. Beside those limits mentioned previously, there are other aspects that should be focused on in our model, such as the effect of sink-in and pileup to the contact radius and the nonlinear relation of ratio $(H/E_r)/(W_u/W_t)$ addressed in the work of Alkorta et al.[4] and Chen and Bull,[6] as well as the effect of relative modulus between indenter and sample material toward the scale ratio that Ma and Ong[13] took into account.

In summary, the results derived in this paper were based on more general stress–strain relations than our previous paper.[7] The scale ratio of $(H/E_r)/(W_u/W_t)$ for work-hardening materials was found to depend mainly on the geometry of indenter and thus hints at a more fundamental role in this energy approach to indentation for more general materials. Improvements are still needed in the theoretical analysis of determining the effects of Poisson's ratio but left for future work.

ACKNOWLEDGMENTS

The authors would like to thank Prof. Yilong Bai for helpful discussions. The support from NSF of China (Nos. 10872200 and 10432050) is also gratefully acknowledged.

REFERENCES

1. G.M. Pharr, Y.T. Cheng, I.M. Hutchings, M. Sakai, N.R. Moody, G. Sundararajan, and M.V. Swain: Indentation methods in advanced materials research introduction. *J. Mater. Res.* **24,** 579 (2009).
2. W.C. Oliver and G.M. Pharr: An improved technique for determining hardness and elastic modulus using load and displacement sensing indentation experiments. *J. Mater. Res.* **7,** 1564 (1992).
3. Y.T. Cheng and C.M. Cheng: Relationships between hardness, elastic modulus, and the work of indentation. *Appl. Phys. Lett.* **73,** 614 (1998).
4. J. Alkorta, J.M. Martinez-Esnaola, and J.G. Sevillano: Comments on "Comment on the determination of mechanical properties from the energy dissipated during indentation" by J. Malzbender [*J. Mater. Res.* 20, 1090 (2005)]. *J. Mater. Res.* **21,** 302 (2006).
5. J. Malzbender: Reply to the comments on "Comment on the determination of mechanical properties from the energy dissipated during indentation" by J. Malzbender [*J. Mater. Res.* 20, 1090 (2005)]. *J. Mater. Res.* **21,** 306 (2006).
6. J. Chen and S.J. Bull: Relation between the ratio of elastic work to the total work of indentation and the ratio of hardness to Young's modulus for a perfect conical tip. *J. Mater. Res.* **24,** 590 (2009).
7. R. Yang, T.H. Zhang, P. Jiang, and Y.L. Bai: Experimental verification and theoretical analysis of the relationships between hardness, elastic modulus, and the work of indentation. *Appl. Phys. Lett.* **92,** 231906 (2008).
8. K.L. Johnson: *Contact Mechanics* (Cambridge University Press, Cambridge, 1985).
9. X.L. Gao, X.N. Jing, and G. Subhash: Two new expanding cavity models for indentation deformations of elastic strain-hardening materials. *Int. J. Solids Struct.* **43,** 2193 (2006).
10. X.L. Gao: Elasto-plastic analysis of an internally pressurized thick-walled cylinder using a strain gradient plasticity theory. *Int. J. Solids Struct.* **40,** 6445 (2003).
11. R. Hill: *The Mathematical Theory of Plasticity* (Oxford University Press, New York, 1998).
12. ABAQUS, HKS Inc., Pawtucket, RI.
13. D.J. Ma and C.W. Ong: Further analysis of energy-based indentation relationship among Young's modulus, nominal hardness, and indentation work. *J. Mater. Res.* **25,** 1131 (2010).