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On contraction factors of Hermitian and skew-Hermitian splitting iteration method for generalized saddle point problems

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Abstract The Hermitian and skew-Hermitian splitting (HSS) iteration method was presented and studied by Bai, et al. for solving non-Hermitian positive definite linear systems (Bai Z Z, Golub G H, Ng M K. Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. *SIAM J. Matrix Anal. Appl.*, 2003, **24**: 603–626). In this paper, contraction factors of the HSS iteration method in terms of the weighted 2-norm and the 2-norm are given, respectively, for the generalized saddle point problems. These contraction factors rather than the spectral radius of the iteration matrix essentially control the actual convergent speed of the HSS iteration method in practical computations. According to the analyses, the contraction factor of the HSS iteration method for the generalized saddle point problem is one in the weighted 2-norm. However, it may be greater than or equal to one in the 2-norm and less than one in other suitable norms. Finally, numerical examples are used to examine the correctness of the theoretical results.

Key words contraction factor; weighted 2-norm; 2-norm; generalized saddle point problem; Hermitian and skew-Hermitian splitting (HSS) iteration method

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关于广义鞍点问题的 HSS 迭代方法的收缩因子

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摘要 白中治等提出了解非埃尔米特正定线性方程组的埃尔米特和反埃尔米特分裂 (HSS) 迭代方法 (Bai Z Z, Golub G H, Ng M K. Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems. *SIAM J. Matrix Anal. Appl.*, 2003, **24**: 603–626). 本文精确地估计了用 HSS 迭代方法求解广义鞍点问题时在加权 2-范数和 2-范数下的收缩因子. 在实际的计算中, 正是这些收缩因子而不是迭代矩阵的谱半径, 本质上控制着 HSS 迭代方法的实际收敛速度. 根据文中的分析, 求解广义鞍点问题的 HSS 迭代方法的收缩因子在加权 2-范数下等于 1, 在 2-范数下它会大于等于 1, 而在某种适当选取的范数之下, 它则会小于 1. 最后, 用数值算例说明了理论结果的正确性.

关键词 收缩因子; 加权 2-范数; 2-范数; 广义鞍点问题; HSS 迭代方法

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0 Introduction

We consider the large and spare system of linear equations

$$\begin{pmatrix} B & E \\ -E^* & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \quad \text{or} \quad Az = f, \tag{1}$$

where $B \in \mathbb{C}^{m\times m}$ is Hermitian positive definite, $C \in \mathbb{C}^{n\times n}$ is Hermitian positive semidefinite, and $\mathrm{null}(C) \cap \mathrm{null}(E) = \{0\}$ with $m \geqslant n$, $b \in \mathbb{C}^m$, and $q \in \mathbb{C}^n$. Here, E^* is used to denote the conjugate transpose of the matrix E, and $\mathrm{null}(C)$ and $\mathrm{null}(E)$ denote the null spaces of the matrices C and E, respectively. Usually, linear systems of the form (1) are called generalized saddle point problems.

The generalized saddle point problem (1) frequently arises in many areas of scientific computing and engineering applications, including mixed finite element approximations of elliptic partial differential equations[1], optimal control[2], electronic networks[3] and others[4–6]; see also [7].

Many researchers have discussed iteration methods and preconditioning techniques for the generalized saddle point problems[8–13]. When the (2, 2) block C of the matrix A is zero, the linear system (1) reduces to the standard saddle point problem. Recently, various kinds of iteration methods have been proposed and studied for the generalized saddle point problem (1), and among them we just mention the Hermitian and skew-Hermitian splitting (HSS) iteration method[14], which directly applies the HSS iteration method to the generalized saddle point problem (1) (see [11, 13, 15–16]).

In this paper, we discuss the contraction factors of the HSS iteration method used to solve the generalized saddle point problem (1). To this end, we first review the HSS iteration method for a general non-Hermitian positive definite linear system $Ax = b$ with $A \in \mathbb{C}^{\tilde{n}\times\tilde{n}}$, $b \in \mathbb{C}^{\tilde{n}}$, and $\tilde{n}$ being a positive integer. It is easy to know that the coefficient matrix A has the Hermitian and skew-Hermitian splitting

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*), \quad S = \frac{1}{2}(A - A^*).$$

Based on this splitting, the HSS iteration method[14] can be described as follows:

The HSS iteration method Given an initial guess $x^{(0)} \in \mathbb{C}^{\tilde{n}}$ for $k = 0, 1, 2, \cdots$ until the iteration sequence $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ using the following procedure:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases}$$

where α is a given positive constant, and I is the identity matrix.

When the HSS iteration method is specified to the generalized saddle point problem (1), we can obtain the following iteration method (see [13, 16]):

$$\begin{cases} \begin{pmatrix} \alpha I + B & 0 \\ 0 & \alpha I + C \end{pmatrix} \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} = \begin{pmatrix} \alpha I & -E \\ E^* & \alpha I \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} + \begin{pmatrix} b \\ q \end{pmatrix}, \\ \begin{pmatrix} \alpha I & E \\ -E^* & \alpha I \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha I - B & 0 \\ 0 & \alpha I - C \end{pmatrix} \begin{pmatrix} x^{(k+\frac{1}{2})} \\ y^{(k+\frac{1}{2})} \end{pmatrix} + \begin{pmatrix} b \\ q \end{pmatrix}. \end{cases}$$

Note that now

$$H = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

is only Hermitian positive semidefinite. We know that the convergence of the HSS iteration method is, however, still guaranteed for the generalized saddle point problem (1) (see [11, 13, 15–16]).

In general, an iteration method is convergent when the spectral radius of the corresponding iteration matrix is less than one. In fact, the contraction factor of an iteration method controls the convergent speed in actual computations. We remark that the contraction factor of the GSOR iteration method[17] was discussed in [18]. In this paper, we will estimate the contraction factors of the HSS iteration method in the weighted 2-norm and the 2-norm, respectively, for the generalized saddle point problem (1).

1 Formulas of contraction factors

In this section, we discuss the contraction factors of the HSS iteration method in the weighted 2-norm and the 2-norm, respectively. The iteration matrix $T(\alpha)$ of the HSS iteration method is given by $T(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)$.

Define a weighted vector norm $|||x||| = \|(\alpha I + S)x\|_2$ (for all $x \in \mathbb{C}^{\tilde{n}}$). Then, the correspondingly induced matrix norm is $|||X||| = \|(\alpha I+S)X(\alpha I+S)^{-1}\|_2$ (for all $X \in \mathbb{C}^{\tilde{n}\times\tilde{n}}$) (see [14]). Define the contraction factor of the HSS iteration method to be $|||T(\alpha)|||$. Then, for the generalized saddle point problem (1), it holds that

$$\begin{aligned} |||T(\alpha)||| &= \|(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S)^{-1}\|_2 \\ &= \|(\alpha I - H)(\alpha I + H)^{-1}\|_2 \\ &= \max\{\|(\alpha I - B)(\alpha I + B)^{-1}\|_2, \|(\alpha I - C)(\alpha I + C)^{-1}\|_2\}. \end{aligned}$$

Here, we have used the fact that $Q(\alpha) = (\alpha I - S)(\alpha I + S)^{-1}$ is a Cayley transform and is, thus, unitary, and the $\|\cdot\|_2$ norm is unitarily invariant. Because C is Hermitian positive semidefinite, we have $\|(\alpha I - C)(\alpha I + C)^{-1}\|_2 = 1$. In addition, that B is Hermitian positive definite leads to $\|(\alpha I - B)(\alpha I + B)^{-1}\|_2 < 1$. Therefore, we obtain $|||T(\alpha)||| = 1$.

Next, we discuss the contraction factor $\|T(\alpha)\|_2$ of the HSS iteration method. It follows from straightforward computations that

$$\begin{aligned} \|T(\alpha)\|_2^2 &= \|(\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)\|_2^2 \\ &= \rho((\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S) \\ &\quad \cdot(\alpha I + S)(\alpha I + H)^{-1}(\alpha I - H)(\alpha I - S)^{-1}) \\ &= \rho((\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S)(\alpha I + S) \\ &\quad \cdot(\alpha I + H)^{-1}(\alpha I - H)(\alpha I - S)^{-1}(\alpha I + S)^{-1}) \\ &:= \rho(W(\alpha)V(\alpha)W(\alpha)V(\alpha)^{-1}) \\ &= \rho(V(\alpha)^{-\frac{1}{2}}W(\alpha)V(\alpha)^{\frac{1}{2}}V(\alpha)^{\frac{1}{2}}W(\alpha)V(\alpha)^{-\frac{1}{2}}) \\ &= \|V(\alpha)^{-\frac{1}{2}}W(\alpha)V(\alpha)^{\frac{1}{2}}\|_2^2, \end{aligned}$$

where

$$\begin{aligned} W(\alpha) &= (\alpha I - H)(\alpha I + H)^{-1} = \begin{pmatrix} (\alpha I - B)(\alpha I + B)^{-1} & 0 \\ 0 & (\alpha I - C)(\alpha I + C)^{-1} \end{pmatrix} \\ &:= \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \end{aligned}$$

with $W_1(\alpha) = (\alpha I - B)(\alpha I + B)^{-1}$ and $W_2(\alpha) = (\alpha I - C)(\alpha I + C)^{-1}$, and

$$\begin{aligned} V(\alpha) &= (\alpha I - S)(\alpha I + S) = \begin{pmatrix} \alpha^2 I + EE^* & 0 \\ 0 & \alpha^2 I + E^*E \end{pmatrix} \\ &:= \begin{pmatrix} V_1(\alpha) & 0 \\ 0 & V_2(\alpha) \end{pmatrix} \end{aligned}$$

with $V_1(\alpha) = \alpha^2 I + EE^*$ and $V_2(\alpha) = \alpha^2 I + E^*E$. Hence, we have

$$\|T(\alpha)\|_2 = \max\{\|V_1(\alpha)^{-\frac{1}{2}} W_1(\alpha) V_1(\alpha)^{\frac{1}{2}}\|_2,\ \|V_2(\alpha)^{-\frac{1}{2}} W_2(\alpha) V_2(\alpha)^{\frac{1}{2}}\|_2\}.$$

Note that

$$\begin{aligned}\|V_2(\alpha)^{-\frac{1}{2}} W_2(\alpha) V_2(\alpha)^{\frac{1}{2}}\|_2 &= \sqrt{\rho(V_2(\alpha)^{-\frac{1}{2}} W_2(\alpha) V_2(\alpha) W_2(\alpha) V_2(\alpha)^{-\frac{1}{2}})} \\ &= \sqrt{\rho(W_2(\alpha) V_2(\alpha) W_2(\alpha) V_2(\alpha)^{-1})} \\ &= \max_{x \neq 0} \frac{x^* W_2(\alpha) V_2(\alpha) W_2(\alpha) x}{x^* V_2(\alpha) x}.\end{aligned}$$

Because C is Hermitian positive semidefinite, there exists a unity matrix Q such that

$$C = Q^* \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} Q$$

with Λ a diagonal matrix of positive diagonal entries. Let

$$\tilde{x} = Q^* \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

Then, it holds that

$$\frac{\tilde{x}^* W_2(\alpha) V_2(\alpha) W_2(\alpha) \tilde{x}}{\tilde{x}^* V_2(\alpha) \tilde{x}} = \frac{\tilde{x}^* V_2(\alpha) \tilde{x}}{\tilde{x}^* V_2(\alpha) \tilde{x}} = 1.$$

Based on this fact, we easily know that $\|V_2(\alpha)^{-\frac{1}{2}} W_2(\alpha) V_2(\alpha)^{\frac{1}{2}}\|_2 \geqslant 1$. Therefore, $\|T(\alpha)\|_2 \geqslant 1$.

2 Numerical examples

When $C = 0$, for the saddle point problem defined through (1), we can obtain

$$\|T(\alpha)\|_2 = \max\{\|V_1(\alpha)^{\frac{1}{2}} W_1(\alpha) V_1(\alpha)^{-\frac{1}{2}}\|_2, 1\} \geqslant 1.$$

In particular, when $B = I$ and $C = 0$, it holds that

$$\|T(\alpha)\|_2 = \max\left\{\left\|\frac{\alpha - 1}{\alpha + 1} I\right\|_2, 1\right\} = \max\left\{\frac{|\alpha - 1|}{\alpha + 1}, 1\right\} = 1.$$

This shows that the lower bound 1 of the contraction factor in the 2-norm can be exactly achieved.

We use two examples to examine the theoretical results in this paper. Our results are run in MATLAB with machine precision 10^{-16}.

Example 1 Consider the generalized saddle point problem (1) with the blocks in the coefficient matrix being given by

$$B = \begin{pmatrix} 1.3111 & 0.4889 & 0.5333 & 0.4444 \\ 0.4889 & 2.7111 & 0.6667 & 0.3556 \\ 0.5333 & 0.6667 & 3.4000 & -0.2667 \\ 0.4444 & 0.3556 & -0.2667 & 2.5778 \end{pmatrix},$$

$$E = \begin{pmatrix} 1.0667 & -0.5333 \\ 6.1333 & -13.0667 \\ -4.0000 & 8.0000 \\ -5.3333 & 10.6667 \end{pmatrix}, \quad C = \begin{pmatrix} 1.0800 & -1.4400 \\ -1.4400 & 1.9200 \end{pmatrix}.$$

When $\alpha = 1.50$, we can obtain

$$\|T(\alpha)\|_2 = 3.21, \quad \rho(T(\alpha)) = 0.45.$$

Obviously, it holds that $\rho(T(\alpha)) < 1 < \|T(\alpha)\|_2$.

Example 2 Consider the generalized saddle point problem (1) with the blocks in the coefficient matrix being given by

$$B = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2l^2 \times 2l^2},$$

$$E = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2l^2 \times l^2},$$

and

$$C = \begin{pmatrix} 1^2 & & & & \\ 1 & 2^2 & & & \\ & \ddots & \ddots & & \\ & & 1 & (l^2-1)^2 & \\ & & & 1 & 0 \end{pmatrix},$$

where

$$T = \frac{1}{h^2} \cdot \mathrm{tridiag}\,(-1, 2, -1) \in \mathbb{R}^{l \times l}, \quad F = \frac{1}{h} \cdot \mathrm{tridiag}(-1, 1, 0) \in \mathbb{R}^{l \times l}$$

with $\otimes$ being the Kronecker product symbol and $h = \frac{1}{l+1}$ the discretization mesh size.

We choose $l = 30$, then $B \in \mathbb{R}^{1\,800 \times 1\,800}$, $C \in \mathbb{R}^{900 \times 900}$. When $\alpha = 40$, we can get

$$\|T(\alpha)\|_2 = 1.47, \quad \rho(T(\alpha)) = 0.994.$$

Again, it holds that $\rho(T(\alpha)) < 1 < \|T(\alpha)\|_2$.

These examples show that the contraction factors in the 2-norm of the HSS iteration method are greater than or equal to one. However, the spectral radii of the iteration matrices are less than one. These results agree with our theoretical conclusion.

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