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Some Remarks on Planar Boussinesq Equations

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Abstract The main purpose of this paper is to prove the well-posedness of the two-dimensional Boussinesq equations when the initial vorticity $\omega_0 \in L^1(R^2)$ (or the finite Radon measure space). Using the stream function form of the equations and the Schauder fixed-point theorem to get the new proof of these results, we get that when the initial vorticity is smooth, there exists a unique classical solutions for the Cauchy problem of the two dimensional Boussinesq equations.

Keywords Boussinesq equations, classical solutions, Schauder fixed-point theorem
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1 Introduction

In this paper, we consider the following two-dimensional Boussinesq equations

$$\begin{cases} u_t - \mu\triangle u + u\cdot\nabla u + \nabla p = \theta f, & (x,t)\in R^2\times[0,T), \\ \theta_t - \nu\Delta\theta + u\cdot\nabla\theta = 0, & (x,t)\in R^2\times[0,T), \\ \text{div } u = 0, & (x,t)\in R^2\times[0,T), \\ u\mid_{t=0}= u_0, \quad \theta\mid_{t=0}=\theta_0 & x\in R^2. \end{cases} \tag{1.1}$$

The unknown functions here are $u = u(x,t) = (u_1(x,t), u_2(x,t))$, $\theta = \theta(x,t)$ and $p = p(x,t)$, which stand for the velocity field, the temperature function and the pressure of the flow, respectively. The given functions $u_0 = u_0(x)$, $\theta_0 = \theta_0(x)$ are the initial velocity and the initial temperature, respectively. Moreover, $\mu > 0$ is the constant coefficient of fluid viscosity and $\nu > 0$ is the constant coefficient of heat conduction. For simplicity, we assume that $\mu = \nu = 1$.

Taking the curl on both sides of the first equation in (1.1), and denoting by $\omega = \text{curl}u$ the vorticity, we get

$$\begin{cases} \omega_t - \triangle\omega + u\cdot\nabla\omega = \text{curl}(\theta f), & (x,t)\in R^2\times[0,T), \\ \theta_t - \Delta\theta + u\cdot\nabla\theta = 0, & (x,t)\in R^2\times[0,T), \\ \text{div}u = 0, & (x,t)\in R^2\times[0,T), \end{cases} \tag{1.2}$$

with the initial data

$$\omega(x,0) = \omega_0, \quad \theta(x,0) = \theta_0, \qquad x\in R^2. \tag{1.3}$$

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When $\theta = 0$ in (1.1), it is clear that (1.1) becomes the incompressible Navier-Stokes equation. Since then, in the case that $u_0 \in L^2(\Omega)$, the uniqueness and the regularity of the weak solutions and the global (in time) existence of strong solutions have been extensively investigated (see [1, 7–9, 11–13] and references therein). The strong well-posedness is only local in time if n=3. F.J. McGrath[10] proved the existence and uniqueness of classical solutions of the non-stationary Navier-Stokes and Euler equations in the entire plane R^2 when $\omega_0(x) \in L^1(R^2) \cap C^{2,\lambda}(R^2)$, $0 < \lambda < 1$. M. Ben-Artzi[3] constructed the unique smooth solutions to the Navier-Stokes equations of incompressible flow in the whole plane under the assumption that the initial vorticity belongs to $L^1(R^2)$ or the finite Radon measure space. Moreover, the large-time behavior was investigated in [3].

In the case that $u_0(x) \in L^2(R^2)$, $\theta_0(x) \in L^2(R^2)$, the well-posedness of the problem (1.1) was proved in [6]. Chae and Nam[4] proved the local existence of classical solutions in $H^m(R^2)$ with $\mu = \nu = 0$. The main purpose of this paper is to prove the well-posedness of (1.1) when the initial vorticity $\omega_0 \in L^1(R^2)$ (or the finite Radon measure space). Since the equations in (1.1) have more coupled nonlinear terms between the velocity and the temperature function, the problem becomes more difficult. In this paper, we use the stream function form of the equations and the Schauder fixed-point theorem to get the new proof of these results. The use of the stream function equations results in stronger differentiability requirements on the initial data.

The main result of this paper can be stated as

Theorem 1.1. *Assume that for some* $0 < \lambda < 1$, $\omega_0(x) \in L^1(R^2) \cap C^{2,\lambda}(R^2)$, $\theta_0(x) \in C^{2,\lambda}(R^2)$, $f \in W^{1,\infty}(\overline{Q}_T) \cap W^{1,1}(\overline{Q}_T)$. *Then there exists a solution* (ω, θ) *of (1.2)–(1.3) such that*

(a) The solution is classical: all derivatives appearing in (1.2) are continuous in $R^2 \times (0, \infty)$.

(b) $\omega(x,t), \theta(x,t), u(x,t)$ *are continuous and uniformly bounded in* $R^2 \times (0, \infty)$.

(c) $\omega(x,t), \theta(x,t) \in L^\infty(0,T; L^1(R^2))$.

(d) For any $T > 0$,

$$\sup_{0 \le t \le T, |x| > R} |u(x,t)| \to 0, \qquad \text{as} \quad R \to \infty.$$

$$\sup_{0 \le t \le T, |x| > R} |\theta(x,t)| \to 0, \qquad \text{as} \quad R \to \infty.$$

Moreover, under conditions (a)–(d) the solution is unique.

2 The Case of Smooth Initial Data

It is known that the velocity can be recovered by

$$u(x,t) = (K * \omega)(x,t) = \int_{R^2} K(x-y)\omega(y,t)dy, \tag{2.4}$$

where

$$K(x) = \frac{1}{2\pi}|x|^{-2}(-x_2, x_1). \tag{2.5}$$

The relation (2.5) is called Biot-Savart law. Note that $\nabla \cdot K = 0$, which implies the incompressibility condition $\nabla \cdot u = 0$.

Define

$$\begin{aligned} B = \big\{ &\omega : \omega \in C(\overline{Q}_T) \cap L^\infty(\overline{Q}_T) \cap L^\infty(0,T; L^1(R^2)), \|\omega\|_{L^\infty(\overline{Q}_T)} + \|\omega\|_{L^\infty(L^1(R^2))} \\ &\le \|\omega_0\|_{L^\infty(R^2)}| + \|\omega_0\|_{L^1(R^2)} + T\|\theta_0\|_{L^\infty(R^2)}\|\nabla f\|_{L^\infty(\overline{Q}_T)} + T^{\frac{1}{2}}\|f\|_{L^1(\overline{Q}_T)} \\ &+ T^{\frac{3}{2}}\|f\|_{L^\infty(\overline{Q}_T)} + \|\theta_0\|_{L^\infty(R^2)}\|\nabla f\|_{L^1(Q_T)} \big\} \end{aligned}$$

We should note that B is a closed convex subset of $C(\overline{Q}_T)$. We will construct a mapping A which maps B into itself in such a way that a fixed point of A is a solution of (1.2) .

To this end, for $\omega \in B$, we first define $A_1\omega = a$ in the following way

$$a(x,t) = \frac{-1}{2\pi}\int_{R^2} \omega(\xi,t)\frac{x_i - \xi_i}{|x-\xi|^2}d\xi$$

for all $(x,\ t) \in \overline{Q}_T$ and $i = 1,2$.

Then, for each $a \in A_1B$, let L_a be the linear parabolic operator

$$L_a = \partial_t - \Delta + a \cdot \nabla.$$

Define the operator N by $Na = \theta$ where $\theta \in C(\overline{Q}_T)$ is the solution of

$$\begin{cases} L_a\theta(x,t) = 0, \\ \theta(x,0) = \theta_0(x) \end{cases} \tag{2.6}$$

for all $x \in R^2,\ t > 0$.

Once θ is defined by (2.6), we define the operator A_2 by $A_2a = v$ where $v \in C(\overline{Q}_T)$ is the solution of

$$\begin{cases} L_a v(x,t) = \mathrm{curl}(\theta f), \\ v(x,0) = \omega_0(x), \end{cases} \tag{2.7}$$

for all $x \in R^2,\ t > 0$.

Finally, we define the operator A by $A = A_2A_1$. Then, we have

Theorem 1.2. *For each $a \in A_1B$, there exists a unique fundamental solution $\Gamma_a(x,t;\xi,s)$ corresponding to L_a which has the following properties:*

i) Γ_a is defined if (x,t), $(\xi,s) \in \overline{Q}_T$ and $t > s$.

ii) For any fixed $(\xi,s) \in \overline{Q}_T$, Γ_a satisfies $L_a\Gamma_a = 0$ as a function of (x,t) $(x \in R^2,\ s < t \leq T)$.

iii) If f is continuous on R^2, then

$$\lim_{t\to s}\int_{R^2}\Gamma_a(x,t;\xi,s)f(\xi)d\xi = f(x).$$

iv) $\Gamma_a(x,t;\xi,s) > 0$, for $t > s$.

v) $\displaystyle\int_{R^2}\Gamma_a(x,t;\xi,s)d\xi = 1$, for $t > s$.

vi) $v = A_2a$ is given by

$$v(x,t) = \int_{R^2}\Gamma_a(x,t;\xi,0)\omega_0(\xi)d\xi - \int_0^t\int_{R^2}\Gamma_a(x,t;\xi,s)\mathrm{curl}(\theta f)(\xi,s)d\xi ds.$$

And θ is given by

$$\theta(x,t) = \int_{R^2}\Gamma_a(x,t;\xi,0)\theta_0(\xi)d\xi.$$

vii) The second derivatives of v are bounded on Q_T.

Proof. The proof can be found in [10] and we omit it here. □

Now we state some properties of the fundamental solution.

Lemma 1.1. *Let $L_a^* = \partial_t + \Delta - a \cdot \nabla$ be the adjoint operators of L_a and $\Gamma_a^*(x,t;\xi,s)$ be the fundamental solution for L_a^*. Then we have*

$$\Gamma_a(x,t;\xi,s) = \Gamma_a^*(x,t;\xi,s)$$

for all $a \in A_1B; x, \xi \in R^2$ and $0 \le s < t \le T$.

Moreover,

$$|\Gamma_a(x,t;\xi,s)| < C(t-s)^{-1}\exp[-\overline{C}|x-\xi|^2/(t-s)], \tag{2.8}$$

$$\Big|\frac{\partial \Gamma_a}{\partial x_i}(x,t;\xi,s)\Big| < C(t-s)^{-\frac{3}{2}}\exp[-\overline{C}|x-\xi|^2/(t-s)], \tag{2.9}$$

$$|\Gamma_a^*(x,t;\xi,s)| < C(t-s)^{-1}\exp[-\overline{C}|x-\xi|^2/(t-s)], \tag{2.10}$$

$$\Big|\frac{\partial \Gamma_a^*}{\partial x_i}(x,t;\xi,s)\Big| < C(t-s)^{-\frac{3}{2}}\exp[-\overline{C}|x-\xi|^2/(t-s)]. \tag{2.11}$$

In these estimates the constants $C, \overline{C}$ can be chosen independently of $a \in A_1B$.

Proof. The proof can be seen in [10] and we omit it here. □

By the maximum principle for parabolic equations, it is easy to get

$$\|\theta\|_{L^\infty(Q_T)} \le \|\theta_0\|_{L^\infty(R^2)},$$

and

$$\begin{aligned}\|v\|_{L^\infty(Q_T)} \le & \|\omega_0\|_{L^\infty(R^2)} + T\|\nabla\theta f\|_{L^\infty(Q_T)} + T\|\theta\nabla f\|_{L^\infty(Q_T)}\\ \le & \|\omega_0\|_{L^\infty(R^2)} + T\|\nabla\theta\|_{L^\infty(Q_T)}\|f\|_{L^\infty(Q_T)} + T\|\theta\|_{L^\infty(Q_T)}\|\nabla f\|_{L^\infty(Q_T)}\\ \le & \|\omega_0\|_{L^\infty(R^2)} + T\|\nabla\theta\|_{L^\infty(Q_T)}\|f\|_{L^\infty(Q_T)} + T\|\theta_0\|_{L^\infty(R^2)}\|\nabla f\|_{L^\infty(Q_T)}.\end{aligned}$$

Lemma 1.2. *If $a \in A_1B, \theta_0 \in C^{2,\lambda}(R^2)$, then $\|\nabla\theta\|_{L^\infty(Q_T)} \le M$.*

Proof. For $a = A_1\omega$ with $\omega \in B$, we write

$$\theta(x,t) = \theta_1(x,t) + \theta_0(x,t)$$

where $\theta_1(x,t)$ satisfies

$$L_a\theta_1(x,t) = -L_a\theta_0(x), \qquad \theta_1(x,0) = 0.$$

Note that $-L_a\theta_0(x) \in C^{\lambda,0}(Q_T)$ and $\|-L_a\theta_0(x)\|_{L^\infty(Q_T)} < C$, where C is independent of $a \in A_1B$. Using Theorem 1.2, we obtain

$$\theta_1(x,t) = -\int_0^t\int_{R^2}\Gamma_a(x,t;\xi,s)[-L_a\theta_0](\xi,s)d\xi ds.$$

For $a \in A_1B$, it follows from (2.9) that

$$\begin{aligned}\Big|\frac{\partial\theta_1}{\partial x_i}(x,t)\Big| \le & C\int_0^T\int_{R^2}(t-s)^{-\frac{3}{2}}\exp\Big[-\frac{\overline{C}|x-\xi|^2}{t-s}\Big]d\xi ds\\ = & \frac{\pi C}{\overline{C}}\int_0^T(t-s)^{-\frac{1}{2}}ds\\ \le & \frac{2\pi C}{\overline{C}}T^{\frac{1}{2}},\end{aligned} \tag{2.12}$$

where C, $\overline{C}$ are constants and $i = 1, 2$. The proof of the lemma is finished.

By Lemma 1.2, we have

$$\|v\|_{L^\infty(Q_T)} \le \|\omega_0\|_{L^\infty(R^2)} + T^{\frac{3}{2}}\|f\|_{L^\infty(Q_T)} + T\|\theta_0\|_{L^\infty(R^2)}\|\nabla f\|_{L^\infty(Q_T)} < \infty. \tag{2.13}$$

□

This and the following Lemma imply that $AB \subset B$.

Lemma 1.3. *If $v \in AB$, then $v \in L^\infty([0,T];L^1(R^2))$ and*

$$\|v\|_{L^\infty([0,T];L^1(R^2))} \le \|\omega_0\|_{L^1(R^2)} + T^{\frac{1}{2}}\|f\|_{L^1(Q_T)} + \|\theta_0\|_{L^\infty(R^2)}\|\nabla f\|_{L^1(Q_T)}.$$

Proof. It follows from Theorem 1.2 that

$$\begin{aligned}
&\int_{R^2} |v(x,t)|dx \\
\le &\int_{R^2} |\omega_0(\xi)|d\xi \int_{R^2} \Gamma_a(x,t,\xi,0)dx \\
&+ \int_0^t \int_{R^2} |\mathrm{curl}(\theta f)(\xi,\tau)|d\xi d\tau \int_{R^2} \Gamma_a(x,t,\xi,\tau)dx \\
\le &\int_{R^2} |\omega_0(\xi)|d\xi \int_{R^2} \Gamma_a(x,t,\xi,0)dx + \int_0^t \int_{R^2} |\nabla\theta f|d\xi d\tau \\
&\times \int_{R^2} \Gamma_a(x,t,\xi,\tau)dx + \int_0^t \int_{R^2} |\theta\nabla f|d\xi d\tau \int_{R^2} \Gamma_a(x,t,\xi,\tau)dx \\
\le &\|\omega_0\|_{L^1(R^2)} + \|\nabla\theta\|_{L^\infty(Q_T)}\|f\|_{L^1(Q_T)} + \|\theta\|_{L^\infty(Q_T)}\|\nabla f\|_{L^1(Q_T)} \\
\le &\|\omega_0\|_{L^1(R^2)} + T^{\frac{1}{2}}\|f\|_{L^1(Q_T)} + \|\theta_0\|_{L^\infty(R^2)}\|\nabla f\|_{L^1(Q_T)} < \infty.
\end{aligned} \tag{2.14}$$

□

By (2.13), Lemma 1.3, the fact that $AB \subset B$ has been proved.

Lemma 1.4. *There exists a constant $\overline{M}$ such that for all $v \in AB$ and*

$$\|\nabla v\|_{L^\infty(Q_T)} < \overline{M}.$$

Proof. The proof is similar to that of [10], and we give the sketch here.

For $v = A_2 a \in AB$, we write

$$v(x,t) = v_1(x,t) + \omega_0(x),$$

where $v_1(x,t)$ satisfies

$$L_a v_1(x,t) = -L_a\omega_0(x), \qquad v_1(x,0) = 0.$$

Note that $-L_a\omega_0(x) \in C^{\lambda,0}(Q_T)$ and $\|-L_a\omega_0(x)\|_{L^\infty(Q_T)} < C$, where C is independent of $a \in A_1 B$. Using Theorem 1.2, we have

$$v_1(x,t) = -\int_0^t \int_{R^2} \Gamma_a(x,t,\xi,s)[-L_a\omega_0](\xi,s)d\xi ds.$$

Then

$$\begin{aligned}
\Big|\frac{\partial v_1}{\partial x_i}(x,t)\Big| \le &C\int_0^T \int_{R^2} (t-s)^{-\frac{3}{2}} \exp\Big[-\frac{\overline{C}|x-\xi|^2}{t-s}\Big]d\xi ds \\
= &\frac{\pi C}{\overline{C}}\int_0^T (t-s)^{-\frac{1}{2}}ds \\
\le &\frac{2\pi C}{\overline{C}}T^{\frac{1}{2}},
\end{aligned} \tag{2.15}$$

which implies

$$\|\nabla v\|_{L^\infty(Q_T)} \leq M = CT^{\frac{1}{2}}.$$

□

In the following, we will prove that AB is a relatively compact subset of $C(\overline{Q}_T)$, which is needed to apply the Schauder fixed point theorem. We first give the following lemma which extends the usual version of Ascoli's theorem to a class of continuous functions defined on an unbounded set (see [10]).

Lemma 1.5. *If $\{u_n\}_{n=1}^\infty$ is equicontinuous and uniformly bounded on $\overline{Q}_T$, and if given $\varepsilon > 0$ there exists $P > 0$ such that $(x,t) \in \overline{Q}_T$ and $|x| \geq P$ imply $|u_n(x,t)| \leq \varepsilon$ for $n = 1, 2, 3\cdots$, then there exists a subsequence of $\{u_n\}$ that convergence uniformly on $\overline{Q}_T$.*

Next we prove the equicontinuous.

Lemma 1.6. *For $a \in B$, let $v = A_2 a \in AB$. Then v is equicontinuous on $\overline{Q}_T$.*

Proof. Due to Lemma 1.4, we only need to prove the continuity of $v(x,t)$ with respect to t. Let $0 \leq t_2 \leq t_1 \leq T$. Using Theorem 1.2 v) yields

$$v(x,t_2) = v(x,t_2)\int_{R^2} \Gamma_a(x,t_1;\xi,t_2)d\xi.$$

Using Theorem 1.2 v) yields

$$v(x,t_1) = \int_{R^2} \Gamma_a(x,t_1;\xi,t_2)v(\xi,t_2)d\xi - \int_{t_2}^{t_1}\int_{R^2} \Gamma_a(x,t_1;\xi,s)\mathrm{curl}(\theta f)(\xi,s)d\xi ds.$$

Then for any $x \in R^2$, we have

$$\begin{aligned}|v(x,t_1) - v(x,t_2)| \leq &\int_{R^2} |\Gamma_a(x,t_1;\xi,t_2)||v(\xi,t_2) - v(x,t_2)|d\xi \\ &+ \int_{t_2}^{t_1}\int_{R^2} |\Gamma_a(x,t_1;\xi,s)||\mathrm{curl}(\theta f)(\xi,s)|d\xi ds \\ =&I_1 + I_2. \end{aligned} \tag{2.16}$$

Now we estimate I_1 and I_2. Using Theorem 1.2 iv) and v), we have

$$I_2 \leq \|\mathrm{curl}(\theta f)\|_{L^\infty(Q_T)}(t_1 - t_2).$$

By Lemma 1.4 and (2.8), we get

$$I_1 < C\overline{M}\int_{R^2}(t_1 - t_2)^{-1}|x-\xi|\exp[-\overline{C}|x-\xi|^2/(t_1-t_2)]d\xi,$$

where $C, \overline{C}$ and $\overline{M}$ are constants independent of $v \in AB$. Noting that

$$[|x-\xi|^2/(t_1-t_2)]^{\frac{1}{2}}\exp\Big[-\frac{1}{2}\overline{C}|x-\xi|^2/(t_1-t_2)\Big]d\xi$$

uniformly with respect to $x, \xi \in R^2$ and $0 \leq t_2 < t_1 \leq T$, we obtain

$$I_1 < C_1\int_{R^2}(t_1-t_2)^{-\frac{1}{2}}\exp\Big[-\frac{1}{2}\overline{C}|x-\xi|^2/(t_1-t_2)\Big]d\xi = C_1(t_1-t_2)^{\frac{1}{2}}.$$

The proof of the lemma is finished. □

Concerning the uniform behavior at infinity, we have

Lemma 1.7. *For any $\varepsilon > 0$, there exists a $n > 0$ such that*

$$\int_{R^2 - B(0,n(\varepsilon))} |v(x,t)| dx < \varepsilon$$

for all $v \in AB,\ 0 < t \leq T$.

Proof. Thanks to (2.8) and Theorem 1.2, there exists constants C and $\overline{C}$ such that for any all $v \in AB$ and all $(x,t) \in Q_T$, we have

$$|v(x,t)| \leq C[h_1(x,t) + h_2(x,t)],$$

where

$$h_1(x,t) = \int_{R^2} |\omega_0(\xi)| t^{-1} \exp[-\overline{C}|x-\xi|^2/t] d\xi$$

and

$$h_2(x,t) = \int_0^t \int_{R^2} (t-s)^{-1} \exp[-\overline{C}|x-\xi|^2/(t-s)]\, |\mathrm{curl}(\theta f)(\xi,s)| d\xi ds.$$

For $n > 0$ and $0 < t \leq T$, one has

$$\begin{aligned}
&\int_{R^2 - B(0,n)} h_1(x,t) dx \\
=&\int_{R^2} |\omega_0(\xi)| d\xi \int_{R^2 - B(0,n)} t^{-1} \exp[-\overline{C}|x-\xi|^2/t] dx \\
=&\int_{R^2 - B(0,n/2)} d\xi \int_{R^2 - B(0,n)} dx + \int_{B(0,n/2)} d\xi \int_{R^2 - B(0,n)} dx \\
=&I_1 + I_2.
\end{aligned} \tag{2.17}$$

Direct estimates yield

$$\begin{aligned}
I_1 &\leq \int_{R^2 - B(0,n/2)} d\xi \int_{R^2} dx \leq \frac{\pi}{C} \int_{R^2 - B(0,n/2)} |\omega_0(\xi)| d\xi, \\
I_2 &\leq t^{-1} \exp\Big[-\frac{1}{2}\overline{C}\Big(\frac{n}{2}\Big)^2 t^{-1}\Big] \int_{B(0,n/2)} |\omega_0(\xi)| d\xi \int_{R^2 - B(0,n)} t^{-1} \exp[-\overline{C}|x-\xi|^2/2t] dx \\
&\leq 8[e\overline{C}n^2]^{-1} \int_{R^2} |\omega_0(\xi)| d\xi \int_{R^2} t^{-1} \exp[-\overline{C}|x-\xi|^2/2t] dx \\
&\leq Cn^{-2} \|\omega_0\|_1.
\end{aligned} \tag{2.18}$$

Moreover, one has

$$\begin{aligned}
&\int_{R^2 - B(0,n(\varepsilon))} h_2(x,t) dx \\
=&\int_0^t \int_{R^2 - B(0,n)} dx \times \int_{R^2} (t-s)^{-1} \exp[-\overline{C}|x-\xi|^2/(t-s)]\, |\mathrm{curl}(\theta f)(\xi,s)| d\xi ds \\
\leq&\frac{\pi}{C} \int_0^t \Big[\int_{R^2 - B(0,n(\varepsilon))} |\mathrm{curl}(\theta f)(\xi,s)| d\xi + Cn^{-2} \Big] ds.
\end{aligned} \tag{2.19}$$

Using the estimates (2.17) and (2.19), we finish the proof of the lemma. □

Lemma 1.8. *The operator $A : B \to B$ is continuous.*

Proof. Let $\{\omega_n\}_{n=0}^{\infty} \subset B$ and $\|\omega_n - \omega_0\|_{L^\infty(\overline{Q}_T)} \to 0$ as $n \to \infty$. Let $a^n = A_1\omega_n$ and $v_n = A\omega_n$. Then

$$\|a^n - a^0\|_{L^\infty(\overline{Q}_T)} = \|A_1\omega_n - A_1\omega_0\|_{L^\infty(\overline{Q}_T)} \to 0$$

as $n \to \infty$.

Let θ_n, θ_0 satisfy

$$\begin{aligned} L_{a^n}\theta_n &= \partial_t\theta_n - \triangle\theta_n + a^n \cdot \nabla\theta_n = 0, \\ L_{a^0}\theta_0 &= \partial_t\theta_0 - \triangle\theta_0 + a^0 \cdot \nabla\theta_0 = 0, \end{aligned}$$

respectively.

Then

$$\begin{aligned} L_{a^n}(\theta_n - \theta_0) =& L_{a^n}\theta_n - L_{a^0}\theta_0 + L_{a^0}\theta_0 - L_{a^n}\theta_0 \\ =& L_{a^n}\theta_n - (L_{a^n} - L_{a^0})\theta_0 - L_{a^0}\theta_0 \\ =& (L_{a^0} - L_{a^n})\theta_0 = (a^0 - a^n) \cdot \nabla\theta_0 \end{aligned} \tag{2.20}$$

with initial data

$$(\theta_0 - \theta_n)|_{t=0} = 0.$$

By the maximum principle for parabolic equations and Lemma 1.2, one has

$$\|\theta_n - \theta_0\|_{L^\infty(\overline{Q}_T)} \le T\|a^0 - a^n\|_{L^\infty(\overline{Q}_T)}\|\nabla\theta_0\|_{L^\infty(\overline{Q}_T)} \to 0, \tag{2.21}$$

as $n \to \infty$.

By Theorem 1.2 vi), integrating by parts over R^2, we get

$$\begin{aligned} \nabla\theta_n(x) =& \int_{R^2} \nabla_x\Gamma_{a^n}(x,t;\xi,0)\widetilde{\theta}_0(\xi)d\xi = \int_{R^2} \nabla_\xi\Gamma_{a^n}(x,t;\xi,0)\widetilde{\theta}_0(\xi)d\xi \\ =& -\int_{R^2} \Gamma_{a^n}(x,t;\xi,0)\nabla_\xi\widetilde{\theta}_0(\xi)d\xi, \end{aligned} \tag{2.22}$$

where $\widetilde{\theta_0}$ is the initial data.

By (2.8) and the assumption that $\theta_0(x) \in C^{2,\lambda}(R^2)$, we get

$$\|\nabla\theta_n\|_{L^\infty(\overline{Q}_T)} \le C\int_{R^2}(t-s)^{-1}\exp[-\overline{C}|x-\xi|^2/(t-s)]d\xi \le C. \tag{2.23}$$

Similar arguments yield

$$\|\nabla\theta_0\|_{L^\infty(\overline{Q}_T)} \le C.$$

Let v_n, v_0 satisfy

$$\begin{aligned} \partial_t v_n - \triangle v_n + a^n \cdot \nabla v_n &= \mathrm{curl}(\theta_n f), \\ \partial_t v_0 - \triangle v_0 + a^0 \cdot \nabla v_0 &= \mathrm{curl}(\theta_0 f), \end{aligned}$$

respectively. Then we have

$$\begin{aligned} &\partial_t(v_n - v_0) - \triangle(v_n - v_0) + a^n \cdot \nabla(v_n - v_0) + (a^n - a^0) \cdot \nabla v_0 \\ =& \mathrm{curl}((\theta_n - \theta_0)f). \end{aligned} \tag{2.24}$$

Multiply (2.24) by $(v_n - v_0)$ and integrate over R^2 to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_n - v_0\|_2^2 + \mu\|\nabla(v_n - v_0)\|_2^2 \\
\leq&\|v_n - v_0\|_2\|\nabla v_0\|_2\|a^n - a^0\|_\infty + \|\theta_n - \theta_0\|_\infty\|f\|_2\|\nabla(v_n - v_0)\|_2 \\
\leq&C\|v_n - v_0\|_2^2\|a^n - a^0\|_\infty + C\|\nabla v_0\|_2^2\|a^n - a^0\|_\infty \\
&+ C\|\theta_n - \theta_0\|_\infty^2\|f\|_2^2 + \frac{\mu}{2}\|\nabla(v_n - v_0)\|_2^2.
\end{aligned} \tag{2.25}$$

Thus

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v_n - v_0\|_2^2 + \frac{\mu}{2}\|\nabla(v_n - v_0)\|_2^2 \\
\leq&C\|v_n - v_0\|_2^2\|a^n - a^0\|_\infty + C\|\nabla v_0\|_2^2\|a^n - a^0\|_\infty + C\|\theta_n - \theta_0\|_\infty^2\|f\|_2^2.
\end{aligned} \tag{2.26}$$

By Gronwall's inequality, we have

$$\|v_n - v_0\|_{L^\infty(L^2)} \to 0, \tag{2.27}$$
$$\|\nabla v_n - \nabla v_0\|_{L^2(\overline{Q}_T)} \to 0 \tag{2.28}$$

as $n \to \infty$.

Similar to (2.22), by Theorem 1.2 vi), we get

$$\begin{aligned}
\nabla_x v_n(x,t) =& \int_{R^2} \nabla_x \Gamma_{a^n}(x-y,t)\widetilde{v}_0(y)dy \\
&+ \int_0^t \int_{R^2} \nabla_x \Gamma_{a^n}(x-y,t-s)\mathrm{curl}(\theta_n f)(y,s)dyds \\
=& -\int_{R^2} \Gamma_{a^n}(x-y,t)\nabla_y \widetilde{v}_0(y)dy \\
&+ \int_0^t \int_{R^2} \nabla_x \Gamma_{a^n}(x-y,t-s)\mathrm{curl}(\theta_n f)(y,s)dyds,
\end{aligned} \tag{2.29}$$

where $\widetilde{v}_0$ is the initial data. Then we have

$$\begin{aligned}
&\|\nabla_x v_n\|_{L^\infty(\overline{Q}_T)} \\
\leq&C\|\nabla\widetilde{v}_0\|_\infty + \|\mathrm{curl}(\theta_n f)\|_{L^\infty(\overline{Q}_T)} \int_0^t \int_{R^2} \nabla_x \Gamma_{a^n}(x-y,t-s)dyds \\
\leq&C.
\end{aligned} \tag{2.30}$$

Similar estimates give

$$\|\nabla v_0\|_{L^\infty(Q_T)} \leq C.$$

By the Gagliardo-Nirenberg inequality, we get that

$$\begin{aligned}
\|v_n - v_0\|_{L^\infty(\overline{Q}_T)} \leq&\|\nabla v_n - \nabla v_0\|_{L^\infty(\overline{Q}_T)}^{\frac{1}{2}}\|v_n - v_0\|_{L^\infty(L^2)}^{\frac{1}{2}} \\
\leq&C(\|\nabla v_n\|_{L^\infty(\overline{Q}_T)} + \|\nabla v_0\|_{L^\infty(Q_T)})\|v_n - v_0\|_{L^\infty(L^2)}^{\frac{1}{2}} \\
\leq&C\|v_n - v_0\|_{L^\infty(L^2)}^{\frac{1}{2}}.
\end{aligned} \tag{2.31}$$

Hence, using (2.27), we get

$$\|v_n - v_0\|_{L^\infty(\overline{Q}_T)} \to 0$$

as $n \to \infty$.

The proof of the lemma is finished. □

Proof of Theorem 1.1. Since A is continuous, the Schauder fixed point theorem yields $\omega \in B$ such that $\omega = A\omega$. The uniqueness is shown by a similar argument to [10]. □

Remark 1.1. The maximum principle can be applied to (2.6) and its dual since $\nabla \cdot u = 0$. We can therefore conclude (for the solution of (1.2)) that

$$\|\omega(\cdot,t)\|_1 \le \|\omega_0\|_1, \qquad \|\theta(\cdot,t)\|_1 \le \|\theta_0\|_1,$$

and

$$\|\omega(\cdot,t)\|_\infty \le \|\omega_0\|_\infty + C(T)\|\theta_0\|_{W^{1,\infty}}, \qquad \|\theta(\cdot,t)\|_\infty \le \|\theta_0\|_\infty$$

for $t > 0$ and by interpolation,

$$\begin{aligned} &\|\omega(\cdot,t)\|_p \le \|\omega_0\|_p + C(T)\|\theta_0\|_{W^{1,\infty}}, \qquad 1 \le p \le \infty, \\ &\|\theta(\cdot,t)\|_p \le \|\theta_0\|_p, \qquad 1 \le p \le \infty. \end{aligned} \tag{2.32}$$

Remark 1.2. When the initial vorticity belongs to $L^1(R^2)$ (or the finite Radon measure space), whether (1.2) has global solution is still unknown. We will study it in future works.

References

[1] Beirão da Veiga, H. A new regularity class for the Navier-Stokes equations in R^n. *Chin. Ann. of Math.*, 16B(4): 407–412 (1995)

[2] Ben-Artzi, M. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rat. Mech. Anal.*, 128: 329–358 (1994)

[3] Ben-Artzi, M. Planar Navier-Stokes equations vorticity approach. Handbook of mathematical fluid dynamics, North-Holland, Amsterdam

[4] Chae, D., Nam, H.-S. Local existence and blow-up criterion for the Boussinesq equations. *Nagoya Math. J.*, 155: 55–80 (1999)

[5] Chae, D., Kim, S., Nam, H.-S. Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. *Nogaga Math. J.*, 155: 55–80 (1999)

[6] Kagei, Y. On weak solutions of nonstationary Boussinesq equations. *Diff and Integral Equ*, 9: 587–611 (1993)

[7] Kato, T. Strong L^p solutions of the Navier-Stokes equations in R^n, with application to weak solutions. *Math. Z.*, 187: 471–480 (1984)

[8] Leray, J. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63: 193–248 (1934)

[9] Lions, P.L. Mathematical topics in fluid mechanics, Vol.1: incompressible models. In: Oxford Lecture Series in Mathematics and Its Applications, Vol.10, 1996

[10] McGrath, F.J. Nonstationary plane flow of viscous and ideal fluids. *Arch. Rat. Mech. Anal.*, 27: 328–348 (1968)

[11] Masuda, K. Weak solutions of the Navier-Stokes equations. *Tohoku Math. J.*, 36: 623–646 (1984)

[12] Sohr, H. The Navier-Stokes Equations:an elementary functional analytic approach. Birkhäuser Verlag, 2001

[13] Temam, R. Navier-Stokes equations theory and numerical Analysis. North-Holland-Amsterdam, New York, 1984