



Effective elastic moduli of triangular lattice material with defects

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ABSTRACT

This paper presents an attempt to extend homogenization analysis for the effective elastic moduli of triangular lattice materials with microstructural defects. The proposed homogenization method adopts a process based on homogeneous strain boundary conditions, the micro-scale constitutive law and the micro-to-macro static operator to establish the relationship between the macroscopic properties of a given lattice material to its micro-discrete behaviors and structures. Further, the idea behind Eshelby's equivalent eigenstrain principle is introduced to replace a defect distribution by an imagining displacement field (eigendisplacement) with the equivalent mechanical effect, and the triangular lattice Green's function technique is developed to solve the eigendisplacement field. The proposed method therefore allows handling of different types of microstructural defects as well as its arbitrary spatial distribution within a general and compact framework. Analytical closed-form estimations are derived, in the case of the dilute limit, for all the effective elastic moduli of stretch-dominated triangular lattices containing fractured cell walls and missing cells, respectively. Comparison with numerical results, the Hashin–Shtrikman upper bounds and uniform strain upper bounds are also presented to illustrate the predictive capability of the proposed method for lattice materials. Based on this work, we propose that not only the effective Young's and shear moduli but also the effective Poisson's ratio of triangular lattice materials depend on the number density of fractured cell walls and their spatial arrangements.

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1. Introduction

Materials with a lattice-like structure have been extensively utilized in a variety of engineering applications of lightweight construction, thermal insulation and energy absorption due to their excellent properties including high stiffness-to-weight ratios and multi-functional adaptabilities (Gibson and Ashby, 1997). Some lattice materials including triangular honeycomb and triangulated truss structure possess repeating unit cells with fully triangulated elements, and can be effectively modeled as a triangular lattice with a periodic unit cell comprising a few beams or trusses. For an intact periodic triangular lattice, the effective elastic properties have been described through the application of simple beam theory (for example, Gibson and Ashby, 1997; Wang and McDowell, 2003), or have been deduced by using homogenization techniques (for example, Hohe and Becker, 1999; Kumar and McDowell, 2004; Gonella and Ruzzene, 2008).

In practice, many realistic lattice materials have non-periodic microstructures due to fractured/missing cell walls or missing cells during the manufacturing process. Considerable effort has been devoted for the analysis of properties of

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lattice materials containing microstructural defects. Silva and Gibson (1997) performed a finite element study to find the effect of randomly removing individual cell edges on the Young's modulus and the elastic and plastic collapse stress in regular hexagonal honeycombs as well as in Voronoi honeycombs. Albuquerque et al. (1999) proposed an experimentally based equation for the dependence of Young's modulus on defect concentration in the form of fractured cell walls in hexagonal honeycombs. Guo and Gibson (1999) investigate the effects of missing cells on the Young's moduli, the elastic buckling strength and the plastic collapse strength of regular honeycombs using the finite element method (FEM). Using a combination of analytical methods and FEM, Chen et al. (1999) investigated the influence of different types of morphological imperfection on the yielding of 2D cellular solids. Chen et al. (2001) also numerically studied the effects of holes and rigid inclusions on the elastic modulus and yield strength of regular honeycombs under bi-axial loading. Wallach and Gibson (2001) calculated numerically the effect of randomly removing members of a three-dimensional truss material on the Young's modulus and compressive strength. Wang and McDowell (2003) used finite element simulations to investigate the effects of fractured/missing cell walls on in-plane effective elastic stiffness and initial yield strength of square and triangular cell metal honeycombs.

An alternative to those numerical and experimental procedures is theoretical investigation of the macroscopic description of lattice materials with microstructural defects. Over the last several years many theoretical micromechanical models based on Eshelby's equivalent inclusion theory (Eshelby, 1957), which establishes the equivalency between an eigenstrain field and an inhomogeneity distribution, have been developed and applied to predict the effective elastic properties of heterogeneous materials with complex microstructures (see the reviews by Mura, 1987; Nemat-Nasser and Hori, 1993). In these classic homogenization theories the microscopic level is described in terms of continuum mechanics, therefore introducing a local or microscopic stress and strain distribution. Comparatively, the microscopic structure of lattice materials is more convenient to be treated as a lattice of beams or trusses connected in vertices, and nodal force and nodal displacement are considered as the local variables. Using the micro-continuous homogenization theories, it is often difficult to accurately estimate the effects of micro-discrete behaviors and structures on the macroscopic properties of lattice materials.

The goal of this paper is to present a micro-discrete homogenization method for derivation of the overall macroscopic response of triangular lattice materials with fractured cell walls and missing cells. To achieve our goal, we first will propose a general homogenization process for the modeling of lattice materials based on homogeneous strain boundary conditions, a micro-scale constitutive law and a micro-to-macro static operator. One can find similar homogenization process for discrete media (for example, Ostoja-Starzewski, 2002; Florence and Sab, 2006) and for granular material (for example, Cambou et al., 1995; Nicot and Darve, 2005). Second, we will generalize Eshelby's equivalent inclusion theory to establish the equivalency between an imaging displacement (eigendisplacement) field and a defect distribution in the triangular lattice, such that distribution of defects can be replaced by the eigendisplacement field with the equivalent mechanical effect. Third, we will develop the lattice Green's function technique, which has been studied in many problems of condensed matter physics (for example, Horiguchi, 1972; Economou, 1983; Cserti, 2000), for calculation of the eigendisplacement field. Using the homogenization process, the eigendeformation method and the triangular lattice Green's function, the dilute solutions of effective moduli of stretch-dominated triangular lattices containing fractured cell walls and missing cells will finally be derived and compared with some numerical results available in the literature and the results from micro-continuous homogenized models.

Both symbolic and index notations are used in the present paper; for instance, the strain tensor is denoted by either \mathbf{E} or E_{ij} . In the symbolic notation, '.' and ':' stand for the first- and second-order contractions, ' \otimes ' denotes the tensor product.

2. Micromechanics of a two-dimensional triangular truss framework

2.1. Two-scale description

Consider the homogenized macro-continuum under conditions of quasi-static equilibrium; see Fig. 1 for a schematic diagram. Associated with a material point is a microstructure which characterizes a representative volume element (RVE) of a two-dimensional triangular truss framework containing many truss defects (hereafter representing fractured cell walls) or cell defects (hereafter representing missing cells). The RVE contains a sufficient number of truss elements such that the material is statistically homogeneous. For the sake of simplicity, body forces will not be considered in this paper.

In the RVE, the important static quantity for a truss connecting two nearest-neighboring joints (nodes) p and q is the elastic force $\mathbf{f}^{(pq)}$ in the truss. The associated kinematic variable between nodes p and q is the relative displacement

$$\mathbf{u}^{(pq)} = \mathbf{u}^{(q)} - \mathbf{u}^{(p)} \quad (2.1)$$

where $\mathbf{u}^{(p)}$ and $\mathbf{u}^{(q)}$ are the displacement of nodes p and q , respectively. A linear elastic constitutive relation is considered, which relates the normal and tangential components $f_n^{(pq)}$ and $f_t^{(pq)}$ of the force $\mathbf{f}^{(pq)}$ to the normal and tangential components $u_n^{(pq)}$ and $u_t^{(pq)}$ of the relative displacement $\mathbf{u}^{(pq)}$ and to the axial stiffness $k_n^{(pq)}$ and the bending stiffness $k_t^{(pq)}$ of trusses in normal and tangential direction

$$f_n^{(pq)} = k_n^{(pq)} u_n^{(pq)}, \quad f_t^{(pq)} = k_t^{(pq)} u_t^{(pq)} \quad (2.2)$$

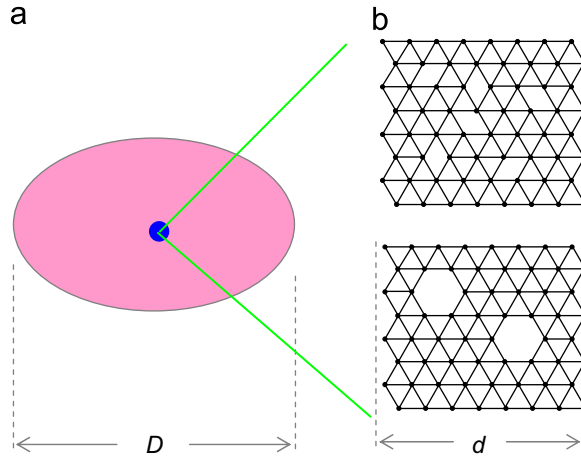


Fig. 1. An homogenized macro-continuum consisting of a two-dimensional triangular truss framework with microstructural defects (RVEs): (a) macrostructure, (b) microstructure (truss defects and cell defects).

The linear elastic constitutive relation can also be expressed by stiffness matrix $\mathbf{K}^{(pq)}$

$$\mathbf{f}^{(pq)} = \mathbf{K}^{(pq)} \times \mathbf{u}^{(pq)}, \quad \mathbf{K}^{(pq)} = k_n^{(pq)} \mathbf{n}^{(pq)} \otimes \mathbf{n}^{(pq)} + k_t^{(pq)} (\mathbf{I} - \mathbf{n}^{(pq)} \otimes \mathbf{n}^{(pq)}) \quad (2.3)$$

where $\mathbf{n}^{(pq)}$ are the unit vectors of the truss connecting nodes p and q , \mathbf{I} is the second order identity tensor (Kronecker delta symbol).

At the macroscopic level, the important kinematic and static quantities are the macro stress tensor $\boldsymbol{\Sigma}$ and the macro strain tensor \mathbf{E} . With this linear elastic constitutive relation in the RVE, the macroscopic constitutive relationship is also elastic. One then defines classically the effective modulus tensor $\bar{\mathbf{C}}$ at material point through the relationship between $\boldsymbol{\Sigma}$ and \mathbf{E}

$$\boldsymbol{\Sigma} = \bar{\mathbf{C}} : \mathbf{E} \quad (2.4)$$

2.2. Homogenization process

Within the framework of the micromechanics, the macroscopic stress–strain relationship for the RVE needs to be derived from the local constitutive relation for a truss. For this purpose it is necessary to link the global representative variables (i.e. macro stress $\boldsymbol{\Sigma}$ and macro strain \mathbf{E}) to the local variables (i.e., elastic force $\mathbf{f}^{(pq)}$ and relative displacement $\mathbf{u}^{(pq)}$). The micromechanical expression for the macro stress tensor in terms involving internal forces of the RVE, which has been documented by many authors (for example, Christoffersen et al., 1981; Donev and Torquato, 2003; Nicot and Darve, 2005), is given by

$$\boldsymbol{\Sigma} = \frac{1}{2S} \sum_c (\mathbf{f}^{(c)} \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{f}^{(c)}) \quad (2.5)$$

where $\mathbf{f}^{(c)}$ is the elastic force in the c th truss, $\mathbf{l}^{(c)}$ is the branch vector connecting the two nodes of the c th truss. The sum is extended to all the trusses over the area S of the RVE.

Dissimilar to the micromechanical interpretation of macroscopic stress tensor, an accepted general way of averaging the nodal displacement in order to get the macroscopic strain tensor has been lacking. The underlying reason is that the concept of kinematic compatibility does not exist naturally inside discrete structures. In order to circumvent this difficulty homogeneous strain boundary conditions are assumed to be associated with prescribed displacements $\mathbf{u}^{(\beta)}$ of boundary node β , i.e.

$$\mathbf{u}^{(\beta)} = \boldsymbol{\varepsilon}^0 \times \mathbf{x}^{(\beta)} \quad (2.6)$$

where $\boldsymbol{\varepsilon}^0$ is a constant strain tensor, $\mathbf{x}^{(\beta)}$ is the position of node β in the original configuration. Based on the equivalency between the lattice and the homogenized continuum, the displacement at the boundary of the homogenization continuum must be prescribed by

$$\mathbf{u}^0 = \boldsymbol{\varepsilon}^0 \times \mathbf{x} \quad (2.7)$$

Note that applying Gauss's theorem, \mathbf{E} is associated to prescribed displacement \mathbf{u}^0 at the boundary given by

$$\mathbf{E} = \frac{1}{2S} \int_{\partial S} (\mathbf{u}^0 \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}^0) d\Gamma \quad (2.8)$$

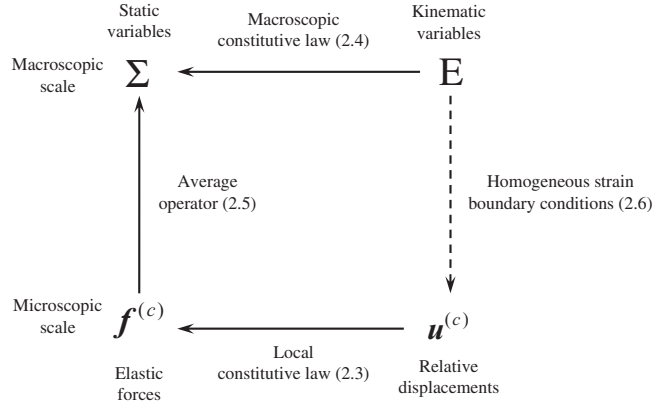


Fig. 2. General homogenization process for lattice materials.

This leads, according to the boundary conditions (2.7), to

$$\mathbf{E} = \frac{1}{2S} \int_{\partial S} (\mathbf{u}^0 \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{u}^0) d\Gamma = \frac{1}{2S} \int_{\partial S} ((\boldsymbol{\varepsilon}^0 \times \mathbf{x}) \otimes \mathbf{n} + \mathbf{n} \otimes (\boldsymbol{\varepsilon}^0 \times \mathbf{x})) d\Gamma = \boldsymbol{\varepsilon}^0 \quad (2.9)$$

Now that the macro strain tensor of the RVE \mathbf{E} equals $\boldsymbol{\varepsilon}^0$ for any compatible displacement field obeying (2.6), one can define an homogenization process combining homogeneous strain boundary conditions (2.6), the local constitutive equations (2.3) and the static average operator (2.5) to allow the macro stress tensor and the macro strain tensor to be related (Fig. 2).

2.3. Effective moduli of a perfect periodic triangular lattice of trusses

For a two-dimensional perfect periodic triangular lattice of trusses subjected to homogeneous strain \mathbf{E} , it can be easily proved that the discrete displacement field given by

$$\mathbf{u}_0^{(p)} = \mathbf{E} \times \mathbf{x}^{(p)} \quad (2.10)$$

satisfies the quasi-static force equilibrium equations when node p is inside the lattice, or obeys boundary conditions (2.6) if node p is at the boundary of the lattice. Here $\mathbf{x}^{(p)}$ is the position vector of node p in the original microstructural configuration. Therefore, (2.10) is the discrete solution of the perfect triangular lattice. Using the homogenization process shown in Fig. 2, closed-form solutions for the elastic tensor of the lattice are written as

$$\begin{aligned} \bar{C}_{ijst}^0 &= \frac{l_0^2(k_n - k_t)}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} \\ &+ \frac{l_0^2 k_t}{4S} \sum_c (\delta_{is} n_j^{(c)} n_t^{(c)} + \delta_{it} n_j^{(c)} n_s^{(c)} + \delta_{js} n_t^{(c)} n_i^{(c)} + \delta_{jt} n_i^{(c)} n_s^{(c)}) \quad i, j, s, t = 1, 2 \end{aligned} \quad (2.11)$$

where l_0 is the distance between two nodes. When the size of the lattice is far greater than l_0 we see that

$$\bar{C}_{1111}^0 = \bar{C}_{2222}^0 = \frac{\sqrt{3}(3k_n + k_t)}{4}, \quad \bar{C}_{1122}^0 = \bar{C}_{2211}^0 = \frac{\sqrt{3}(k_n - k_t)}{4}, \quad \bar{C}_{1212}^0 = \frac{\sqrt{3}(k_n + k_t)}{4} \quad (2.12)$$

(2.12) shows that the equivalent continuum is isotropic and has two independent effective elastic constants. Especially for a stretch-dominated triangular structure that only deform along their longitudinal direction, its stiffness matrix takes the form

$$k_n = \frac{E_s t}{l}, \quad k_t = 0 \quad (2.13)$$

where E_s is the Young's modulus of the truss, t the thickness and l the length. Under the plane stress deformations, the effective Young's modulus E_0 , the shear modulus μ_0 and the effective Poisson's ratio ν_0 of the stretch-dominated triangular lattice is of form

$$\begin{aligned} \bar{C}_{1111}^0 = \bar{C}_{2222}^0 &= \frac{E_0}{1 - (\nu_0)^2} = \frac{3\sqrt{3}k_n}{4} \\ \bar{C}_{1122}^0 = \bar{C}_{2211}^0 &= \frac{E_0 \nu_0}{1 - (\nu_0)^2} = \frac{\sqrt{3}k_n}{4} \end{aligned}$$

$$\bar{C}_{1212}^0 = \mu_0 = \frac{E_0}{2(1+\nu_0)} = \frac{\sqrt{3}k_n}{4} \quad (2.14)$$

From (2.13) and (2.14), the elastic moduli E_0 , μ_0 and ν_0 are given by

$$\frac{E_0}{E_s} = \frac{2\sqrt{3}t}{3l} \quad (2.15)$$

$$\frac{\mu_0}{E_s} = \frac{\sqrt{3}t}{4l} \quad (2.16)$$

$$\nu_0 = \frac{1}{3} \quad (2.17)$$

It is important to point out that (2.15)–(2.17) have been obtained by applying the simple beam theory (for example, Gibson and Ashby, 1997; Wang and McDowell, 2003; Fleck and Qiu, 2007).

3. Eigendeformation method for triangular lattice with defects

For a triangular lattice with microstructural defects, it may not be feasible to directly derive the discrete displacement field under homogeneous strain boundary conditions (2.6). The difficulty arises because of non-periodic defects, which produces nonuniform distribution of internal forces and deformation. To this end, we shall introduce the concept of eigenstrains in continuum mechanics into micromechanical modeling of a triangular lattice with defects. The heterogeneous problem of triangular lattice structure can be reduced to another homogeneous medium problem which could be analytically solved in principle.

3.1. Eigendeformation method

Consider a perfect triangular truss structure subjected to homogeneous strain boundary conditions (2.6), and let B denote the set of trusses in the lattice. According to (2.10), the force $\mathbf{f}_0^{(c)}$ and the deformation $\mathbf{u}_0^{(c)}$ of the c th truss with length vector $\mathbf{l}^{(c)}$ are given by

$$\mathbf{f}_0^{(c)} = \mathbf{K}^{(c)} \times \mathbf{u}_0^{(c)}, \quad \mathbf{u}_0^{(c)} = \mathbf{E} \times \mathbf{l}^{(c)} \quad (3.1)$$

When some trusses are removed from the lattice at random, the total deformation within the lattice is addition of the original deformation $\mathbf{u}_0^{(c)}$ and a disturbance deformation field $\mathbf{u}^{(c)}$, which is unknown at present and to be determined later. Let B' denote the set of all removed trusses. The force field within the lattice with defects can be written as

$$\bar{\mathbf{f}}^{(c)} = \begin{cases} \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) & c \in B - B' \\ \mathbf{0} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) & c \in B' \end{cases} \quad (3.2)$$

In analogy with the Eshelby's equivalent eigenstrain method, a suitable transformation deformation (eigendeformation) field

$$\bar{\mathbf{u}}^{(c)} = \begin{cases} \mathbf{0} & c \in B - B' \\ \tilde{\mathbf{u}}^{(c)} & c \in B' \end{cases} \quad (3.3)$$

is chosen to superpose with the deformation field of the perfect triangular lattice, such that the resulting force field within the lattice

$$\bar{\mathbf{f}}^{(c)} = \begin{cases} \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) & c \in B - B' \\ \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)} - \tilde{\mathbf{u}}^{(c)}) & c \in B' \end{cases} \quad (3.4)$$

is the same as that of the lattice with defects under the same boundary conditions (2.6), i.e.,

$$\bar{\mathbf{f}}^{(c)} = \begin{cases} \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) \\ \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)} - \tilde{\mathbf{u}}^{(c)}) \end{cases} = \begin{cases} \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) & c \in B - B' \\ \mathbf{0} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) & c \in B' \end{cases} \quad (3.5)$$

From (3.5) the eigendeformation needs to satisfy the consistency condition

$$\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)} - \tilde{\mathbf{u}}^{(c)} = \mathbf{0} \quad \forall c \in B' \quad (3.6)$$

Based on the mechanical equivalence between the triangular lattice with defects and the perfect triangular lattice with the eigendeformation, the macro stress tensor of the lattice with defects has the form

$$\Sigma = \frac{1}{2S} \sum_{c \in B} [\mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)})] - \frac{1}{2S} \sum_{c \in B'} (\mathbf{K}^{(c)} \times \tilde{\mathbf{u}}^{(c)} \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{K}^{(c)} \times \tilde{\mathbf{u}}^{(c)}) \quad (3.7)$$

Divide the set of trusses in the perfect triangular lattice into a number of subsets, $B=B_1\cup B_2\cdots\cup B_n$. Each subset composes of the trusses with the same stiffness tensor and length vector, and these trusses can be linked into an one-dimensional chain whose two endpoints lies on the boundary of the lattice. As a result, the summation expression of the first item on the right-hand side of (3.7) has the form

$$\begin{aligned} & \frac{1}{2S} \sum_{c \in B} \left[\mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) \right] \\ &= \frac{1}{2S} \sum_{i=1}^n \left[\mathbf{K}^{(i)} \times \left(\sum_{c \in B_i} (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) \right) \otimes \mathbf{l}^{(i)} + \mathbf{l}^{(i)} \otimes \mathbf{K}^{(i)} \times \left(\sum_{c \in B_i} (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) \right) \right] \end{aligned} \quad (3.8)$$

in which $\mathbf{K}^{(i)}$ and $\mathbf{l}^{(i)}$ are the stiffness tensor and the length vector of the trusses in the subset B_i , respectively. To satisfy displacement boundary conditions (2.6), one can derive that

$$\sum_{c \in B_i} (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)}) = \mathbf{E} \times (\mathbf{x}_{\text{end}}^{(i)} - \mathbf{x}_{\text{start}}^{(i)}), \quad i = 1, \dots, n \quad (3.9)$$

where $\mathbf{x}_{\text{start}}^{(i)}$ and $\mathbf{x}_{\text{end}}^{(i)}$ are the two endpoints of the one-dimensional chain constructed by the trusses of subset B_i , whereas according to (2.10) we have

$$\sum_{c \in B_i} \mathbf{u}^{(c)} = \mathbf{E} \times (\mathbf{x}_{\text{end}}^{(i)} - \mathbf{x}_{\text{start}}^{(i)}) \quad i = 1, \dots, n \quad (3.10)$$

Comparison of (3.9) and (3.10) leads to

$$\sum_{c \in B_i} \mathbf{u}^{(c)} = 0 \quad i = 1, \dots, n \quad (3.11)$$

From (3.7), (3.8) and (3.11), the macro stress tensor in (3.7) is simplified as

$$\boldsymbol{\Sigma} = \frac{1}{2S} \sum_{c \in B} (\mathbf{K}^{(c)} \times \mathbf{u}_0^{(c)} \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{K}^{(c)} \times \mathbf{u}_0^{(c)}) - \frac{1}{2S} \sum_{c \in B} (\mathbf{K}^{(c)} \times \tilde{\mathbf{u}}^{(c)} \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{K}^{(c)} \times \tilde{\mathbf{u}}^{(c)}) \quad (3.12)$$

Consequently, the key problem of estimation of effective properties of a lattice with defects is transferred to solve the eigendeformation field satisfying the consistency condition (3.6).

To find the relationship between the eigendeformation and the corresponding perturbation deformation, we extend the definition of the eigendeformation prescribed on the trusses in the subset B' to the whole lattice, i.e.

$$\tilde{\mathbf{u}}^{(c)} = \begin{cases} 0 & c \in B - B' \\ \tilde{\mathbf{u}}^{(c)} & c \in B' \end{cases} \quad (3.13)$$

Then (3.4) is rewritten as a unified fashion over the set B

$$\tilde{\mathbf{f}}^{(c)} = \mathbf{K}^{(c)} \times (\mathbf{u}_0^{(c)} + \mathbf{u}^{(c)} - \tilde{\mathbf{u}}^{(c)}) \quad c \in B \quad (3.14)$$

and the force equilibrium equations for a generic node p take a form

$$\sum_{\alpha=1}^6 \tilde{\mathbf{f}}^{(pq_\alpha)} = \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \mathbf{u}_0^{(pq_\alpha)} + \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \mathbf{u}^{(pq_\alpha)} - \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \tilde{\mathbf{u}}^{(pq_\alpha)} = 0 \quad (3.15)$$

According to (2.7), the deformation field $\mathbf{u}_0^{(pq_\alpha)}$ in a perfect triangular lattice is given by $\mathbf{u}_0^{(pq_\alpha)} = \mathbf{E} \times \mathbf{l}^{(pq_\alpha)}$, which leads to $\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \mathbf{u}_0^{(pq_\alpha)} = 0$. The force equilibrium Equations (3.15) reduces to

$$\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \mathbf{u}^{(pq_\alpha)} - \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \tilde{\mathbf{u}}^{(pq_\alpha)} = 0 \quad (3.16)$$

which implies the disturbance deformation field can be related to the eigendeformation distribution via (3.16). Considering the contribution of eigendeformation distribution to the equations of equilibrium is similar to that of a type of body force, i.e.

$$\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \mathbf{u}^{(pq_\alpha)} + \mathbf{X}^{(p)} = 0, \quad \mathbf{X}^{(p)} = - \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \tilde{\mathbf{u}}^{(pq_\alpha)} \quad (3.17)$$

the triangular lattice Green's function can be used to derive the relationship between the eigendeformation and the corresponding perturbation deformation.

3.2. Lattice Green’s function for an infinite and perfect triangular lattice of trusses

Consider an infinite and perfect triangular lattice which consists of all nodes specified by position vector \mathbf{x} given in the form

$$\mathbf{x} = l_1 \mathbf{a}_1 + l_2 \mathbf{a}_2 \tag{3.18}$$

where \mathbf{a}_1 and \mathbf{a}_2 are independent primitive translation vectors, such that the position vectors of the nearest neighbors of a generic node p are rewritten as follows

$$\begin{aligned} \mathbf{x}^{(q_1)} &= \mathbf{x}^{(p)} + \mathbf{l}^{(1)} = \mathbf{x}^{(p)} + 2\mathbf{a}_1, & \mathbf{x}^{(q_4)} &= \mathbf{x}^{(p)} + \mathbf{l}^{(4)} = \mathbf{x}^{(p)} - 2\mathbf{a}_1, \\ \mathbf{x}^{(q_2)} &= \mathbf{x}^{(p)} + \mathbf{l}^{(2)} = \mathbf{x}^{(p)} + \mathbf{a}_1 + \mathbf{a}_2, & \mathbf{x}^{(q_5)} &= \mathbf{x}^{(p)} + \mathbf{l}^{(5)} = \mathbf{x}^{(p)} - \mathbf{a}_1 - \mathbf{a}_2, \\ \mathbf{x}^{(q_3)} &= \mathbf{x}^{(p)} + \mathbf{l}^{(3)} = \mathbf{x}^{(p)} - \mathbf{a}_1 + \mathbf{a}_2, & \mathbf{x}^{(q_6)} &= \mathbf{x}^{(p)} + \mathbf{l}^{(6)} = \mathbf{x}^{(p)} + \mathbf{a}_1 - \mathbf{a}_2 \end{aligned} \tag{3.19}$$

Without loss of generality, one may assume that the lattice is in static balance by applying the external nodal forces $\{\mathbf{f}^{\text{ext}}\}$ to all nodes of the lattice. Then, the elastic force on node p induced by the displacements of its neighbors (or itself) must be balanced by an external force as given in the equation

$$\sum_{\alpha=1}^6 \mathbf{f}^{(pq_\alpha)} = \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times (\mathbf{u}^{(q_\alpha)} - \mathbf{u}^{(p)}) = -\mathbf{f}^{\text{ext}(p)} \tag{3.20}$$

Introducing the lattice Green’s function to represent the displacement of the lattice at one node resulting from a unit external force acting on the lattice at another node, the total displacement on node p can be rewritten as

$$\mathbf{u}^{(p)} = \sum_{p'} \mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)}) \times \mathbf{f}^{\text{ext}(p')} \tag{3.21}$$

where the summation with respect to p' is referred to all nodes exerted by external forces. Substitution of (3.21) into (3.20) gives the equilibrium equation in terms of the lattice Green’s function

$$\begin{aligned} \mathbf{f}^{\text{ext}(p)} &= - \sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times \left[\left(\sum_{p'} \mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)} - \mathbf{l}^{(\alpha)}) - \sum_{p'} \mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)}) \right) \times \mathbf{f}^{\text{ext}(p')} \right] \\ &= - \sum_{p'} \left[\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times (\mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)} - \mathbf{l}^{(\alpha)}) - \mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)})) \right] \times \mathbf{f}^{\text{ext}(p')} \end{aligned} \tag{3.22}$$

We find from (3.22) the lattice Green’s function for an infinite and perfect triangular lattice of trusses is the solution of the following equation

$$\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times (\mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)} - \mathbf{l}^{(\alpha)}) - \mathbf{G}(\mathbf{x}^{(p')} - \mathbf{x}^{(p)})) = -\mathbf{I} \delta(\mathbf{x}^{(p)}, \mathbf{x}^{(p')}) \tag{3.23}$$

or the compact form

$$\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} \times (\mathbf{G}(\mathbf{l} - \mathbf{l}^{(\alpha)}) - \mathbf{G}(\mathbf{l})) = -\mathbf{I} \delta(\mathbf{l}) \tag{3.24}$$

where δ is the Dirac delta function.

To derive the lattice Green’s function, we take periodic boundary conditions along the directions parallel to the two primitive translation vectors. Consider a perfect triangular lattice with L_1 and L_2 lattice nodes in the directions \mathbf{a}_1 and \mathbf{a}_2 , respectively. L_1 and L_2 are an even integer so as to be irrelevant in the limit $L_1 \rightarrow \infty$ and $L_2 \rightarrow \infty$, respectively. The total number of nodes in the lattice is $L_1 L_2$. It is convenient, therefore, introduce the Fourier transform

$$\mathbf{G}(\mathbf{l}) = \frac{1}{L_1 L_2} \sum_{\lambda} \hat{\mathbf{G}}(\lambda) e^{i\lambda \cdot \mathbf{l}} \tag{3.25}$$

$$\delta(\mathbf{l}) = \frac{1}{L_1 L_2} \sum_{\lambda} e^{i\lambda \cdot \mathbf{l}} \tag{3.26}$$

of the lattice Green’s function and the delta function into (3.24), and we can find

$$\hat{\mathbf{G}}(\lambda) = \left[\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} (1 - e^{-i\lambda \cdot \mathbf{l}^{(\alpha)}}) \right]^{-1} \tag{3.27}$$

Substituting (3.27) into (3.25) yields the solution of equation (3.24)

$$\mathbf{G}(\mathbf{l}) = \frac{1}{L_1 L_2} \sum_{\lambda} \left[\sum_{\alpha=1}^6 \mathbf{K}^{(\alpha)} (1 - e^{-i\lambda \cdot \mathbf{l}^{(\alpha)}}) \right]^{-1} e^{i\lambda \cdot \mathbf{l}} \tag{3.28}$$

Owing to the periodic boundary conditions, the wave vector λ in (3.28) is limited to the first Brillouin zone and is given by

$$\lambda = \frac{m_1}{L_1} \mathbf{b}_1 + \frac{m_2}{L_2} \mathbf{b}_2 \tag{3.29}$$

where m_1 and m_2 are integers such that $-L_i/2 \leq m_i \leq L_i/2$ for $i=1,2$, and \mathbf{b}_j are the reciprocal lattice vectors defined by

$$\mathbf{a}_i \times \mathbf{b}_j = 2\pi\delta_{ij}, \quad i, j = 1, 2 \tag{3.30}$$

When combining (3.28), (3.29) and (3.30) and setting

$$\xi_1 = \frac{2\pi m_1}{L_1}, \quad \xi_2 = \frac{2\pi m_2}{L_2}, \tag{3.31}$$

the lattice Green's function for an infinite and perfect triangular lattice takes the form

$$\mathbf{G}(\mathbf{l}) = \frac{1}{L_1 L_2} \sum_{(\xi_1, \xi_2)} \omega(\xi_1, \xi_2)^{-1} e^{i(l_1 \xi_1 + l_2 \xi_2)} \tag{3.32}$$

with

$$\omega(\xi_1, \xi_2) = \mathbf{K}^{(1)}(2 - e^{i2\xi_1} - e^{-i2\xi_1}) + \mathbf{K}^{(2)}(2 - e^{i(\xi_1 + \xi_2)} - e^{-i(\xi_1 + \xi_2)}) + \mathbf{K}^{(3)}(2 - e^{i(\xi_1 - \xi_2)} - e^{-i(\xi_1 - \xi_2)}) \tag{3.33}$$

Furthermore, we set $2\pi/L_1 = d\xi_1$, $2\pi/L_2 = d\xi_2$ and let $L_1 \rightarrow \infty$ and $L_2 \rightarrow \infty$. The discrete summation over (m_1, m_2) in (3.32) may be recast into an integral form as follows

$$\mathbf{G}(\mathbf{l}) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [\mathbf{K}^{(1)}(2 - e^{i2\xi_1} - e^{-i2\xi_1}) + \mathbf{K}^{(2)}(2 - e^{i(\xi_1 + \xi_2)} - e^{-i(\xi_1 + \xi_2)}) + \mathbf{K}^{(3)}(2 - e^{i(\xi_1 - \xi_2)} - e^{-i(\xi_1 - \xi_2)})]^{-1} e^{i(l_1 \xi_1 + l_2 \xi_2)} d\xi_1 d\xi_2 \tag{3.34}$$

The components of the integral expression of the lattice Green's function are, respectively,

$$\begin{aligned} G_{11}(l_1, l_2) &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{[2k_t(1 - \cos 2\xi_1) + (3k_n + k_t)(1 - \cos \xi_1 \cos \xi_2)] \cos l_1 \xi_1 \cos l_2 \xi_2 d\xi_1 d\xi_2}{|\omega(\xi_1, \xi_2)|} \\ G_{12}(l_1, l_2) = G_{21}(l_1, l_2) &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{\sqrt{3}(k_n - k_t) \sin \xi_1 \sin \xi_2 \sin l_1 \xi_1 \sin l_2 \xi_2 d\xi_1 d\xi_2}{|\omega(\xi_1, \xi_2)|} \\ G_{22}(l_1, l_2) &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{[2k_n(1 - \cos 2\xi_1) + (k_n + 3k_t)(1 - \cos \xi_1 \cos \xi_2)] \cos l_1 \xi_1 \cos l_2 \xi_2 d\xi_1 d\xi_2}{|\omega(\xi_1, \xi_2)|} \\ |\omega(\xi_1, \xi_2)| &= [2k_n(1 - \cos 2\xi_1) + (k_n + 3k_t)(1 - \cos \xi_1 \cos \xi_2)] \\ &\quad \times [2k_t(1 - \cos 2\xi_1) + (3k_n + k_t)(1 - \cos \xi_1 \cos \xi_2)] - 3(k_n - k_t)^2 \sin^2 \xi_1 \sin^2 \xi_2 \end{aligned} \tag{3.35}$$

3.3. General expressions of effective elastic moduli of triangular lattice with defects

With the help of the lattice Green's function, the contribution of the eigendeformation field to the disturbance deformation of the c th truss is

$$\begin{aligned} \mathbf{u}^{(c)} &= \sum_{c' \in B'} (\mathbf{G}(\mathbf{x}^{(q)} - \mathbf{x}^{(q)}) - \mathbf{G}(\mathbf{x}^{(p)} - \mathbf{x}^{(q)}) + \mathbf{G}(\mathbf{x}^{(p)} - \mathbf{x}^{(p)}) - \mathbf{G}(\mathbf{x}^{(q)} - \mathbf{x}^{(p)})) \times \mathbf{K}^{(c)} \cdot \tilde{\mathbf{u}}^{(c)} \\ &= \sum_{c' \in B'} (\mathbf{G}(\mathbf{l}^{(qq)}) - \mathbf{G}(\mathbf{l}^{(qp)}) + \mathbf{G}(\mathbf{l}^{(pp)}) - \mathbf{G}(\mathbf{l}^{(pq)})) \times \mathbf{K}^{(c)} \times \tilde{\mathbf{u}}^{(c)} \end{aligned} \tag{3.36}$$

where $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(q)}$ are the position vectors of the two ends of the c' th truss, $\mathbf{x}^{(p)}$ and $\mathbf{x}^{(q)}$ the position vectors of the c th truss. Substituting (3.36) into (3.6) yields

$$\tilde{\mathbf{u}}^{(c)} - \sum_{c' \in B'} (\mathbf{G}(\mathbf{l}^{(qq)}) - \mathbf{G}(\mathbf{l}^{(qp)}) + \mathbf{G}(\mathbf{l}^{(pp)}) - \mathbf{G}(\mathbf{l}^{(pq)})) \times \mathbf{K}^{(c)} \times \tilde{\mathbf{u}}^{(c)} = \mathbf{u}_0^{(c)} \quad \forall c \in B' \tag{3.37}$$

Combining equation (3.37) for each truss in the subset B' and solving the system of linear equations, the eigendeformation field will have the following form

$$\tilde{\mathbf{u}}^{(c)} = \sum_{c' \in B'} \mathbf{T}^{(c,c')} \times \mathbf{u}_0^{(c')} \quad \forall c \in B' \tag{3.38}$$

From (3.38), the macro stress tensor of the triangular lattice with defects (3.12) can be rewritten as

$$\Sigma = \frac{1}{2S} \sum_{c \in B} (\mathbf{K}^{(c)} \times \mathbf{u}_0^{(c)} \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \mathbf{K}^{(c)} \times \mathbf{u}_0^{(c)}) - \frac{1}{2S} \sum_{c \in B} \sum_{c' \in B'} (\bar{\mathbf{K}}^{(c,c')} \times \mathbf{u}_0^{(c')} \otimes \mathbf{l}^{(c)} + \mathbf{l}^{(c)} \otimes \bar{\mathbf{K}}^{(c,c')} \times \mathbf{u}_0^{(c)}) \tag{3.39}$$

with

$$\bar{\mathbf{K}}^{(c,c')} = \mathbf{K}^{(c)} \times \mathbf{T}^{(c,c')} \tag{3.40}$$

Consequently, the effective elastic stiffness tensor of a triangular lattice with arbitrarily shaped defects is obtained in a unified way as

$$\bar{\mathbf{C}}_{ijst} = \bar{\mathbf{C}}_{ijst}^0 - \frac{l_0^2}{4S} \sum_{c \in B} \sum_{c' \in B'} (\bar{\mathbf{K}}_{is}^{(c,c')} n_j^{(c)} n_t^{(c')} + \bar{\mathbf{K}}_{it}^{(c,c')} n_j^{(c)} n_s^{(c')} + \bar{\mathbf{K}}_{js}^{(c,c')} n_i^{(c)} n_t^{(c')} + \bar{\mathbf{K}}_{jt}^{(c,c')} n_i^{(c)} n_s^{(c')}) \tag{3.41}$$

where $\bar{\mathbf{C}}_{ijst}^0$ is given by (2.11).

4. Elastic moduli of stretch-dominated triangular lattice with defects

In this section, we apply the general procedure of Section 3 to estimate the effective elastic properties of an infinite triangular lattice containing a dilute concentration of defects. Let us consider an infinite perfect stretch-dominated triangular structure with cell size l_0 and axial stiffness k_n , and prescribe the homogeneous strain \mathbf{E} at infinity. Some trusses with the number density ϕ are fractured and removed from the lattice. In order to investigate the contribution of the interaction between adjacent fractured trusses around each lattice node into the effective moduli of lattice structure, the following four types of defect modes will be discussed: (i) single-truss defect, i.e., only one truss is allowed to be fractured at a node; (ii) double-truss defect, i.e., a couple of trusses with 180° angles are allowed to be fractured at a node; (iii) triple-truss defect, i.e., three trusses which form 120° angles with each other are allowed to be fractured at a node; (iv) cell defect, i.e., all six trusses connecting a node and its nearest neighbors are fractured; see Fig. 3 for a schematic diagram. For each defect mode, the predictions are obtained under the condition of dilute distribution. The dilute solutions are rigorous at small defect densities.

4.1. Case 1: Single-truss defects

Consider a truss connecting two nearest-neighboring nodes p and q and remove it from the infinite perfect stretch-dominated triangular lattice, as shown in Fig. 3(a), the eigendeformation prescribed on the truss can be given from (3.37) by

$$\tilde{\mathbf{u}}^{(pq)} = [\mathbf{I} - 2(\mathbf{G}(0) - \mathbf{G}(l_0 \mathbf{n}^{(pq)}))] \times k_n \mathbf{n}^{(pq)} \otimes \mathbf{n}^{(pq)}]^{-1} \times \mathbf{E} \times l_0 \mathbf{n}^{(pq)} \tag{4.1}$$

where $\mathbf{n}^{(pq)}$ is the unit vector of the truss defect. Without loss of generality, we assume $\mathbf{n}^{(pq)}$ have the following three directions

$$\begin{aligned} \theta^{(\alpha=1)} = 0^\circ : \quad \mathbf{n}^{(\alpha=1)} &= (1, 0); \\ \theta^{(\alpha=2)} = 60^\circ : \quad \mathbf{n}^{(\alpha=2)} &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right); \\ \theta^{(\alpha=3)} = 120^\circ : \quad \mathbf{n}^{(\alpha=3)} &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right); \end{aligned} \tag{4.2}$$

Combining (4.2), (4.1), (3.35), (3.40) and (3.41) and omitting the details of derivation for brevity, the overall elastic stiffness tensor of the lattice containing a single-truss defect is derived as follows:

$$\bar{\mathbf{C}}_{ijst}^{(b)} = \frac{k_n l_0^2}{S} \left(\sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - \frac{1}{A^{(\alpha)}} n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} \right) \tag{4.3}$$

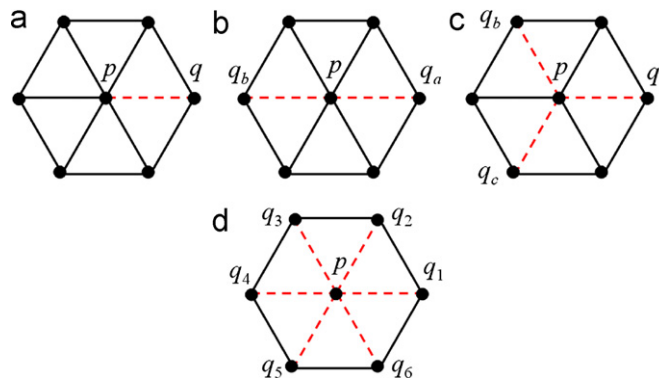


Fig. 3. The four types of defect modes in a triangular cell: (a) single-truss defect, (b) double-truss defect, (c) triple-truss defect, (d) cell defect.

where

$$\Delta^{(\alpha)} = \begin{cases} 1 - \frac{2}{\pi^2} \int_0^\pi \int_0^\pi \frac{(1 - \cos 2\xi_1)(1 - \cos \xi_1 \cos \xi_2) d\xi_1 d\xi_2}{(3 - 2\cos 2\xi_1 - \cos \xi_1 \cos \xi_2)(1 - \cos \xi_1 \cos \xi_2) - \sin^2 \xi_1 \sin^2 \xi_2} & \text{if } \alpha = 1 \\ \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{(1 - \cos 2\xi_1)(1 - \cos \xi_1 \cos \xi_2) d\xi_1 d\xi_2}{(3 - 2\cos 2\xi_1 - \cos \xi_1 \cos \xi_2)(1 - \cos \xi_1 \cos \xi_2) - \sin^2 \xi_1 \sin^2 \xi_2} & \text{if } \alpha = 2, 3 \end{cases} \quad (4.4)$$

Taking into account the symmetry of the lattice, $\Delta^{(\alpha)}$ is independent of the orientation, that is

$$\frac{1}{\pi^2} \int_0^\pi \int_0^\pi \frac{(1 - \cos 2\xi_1)(1 - \cos \xi_1 \cos \xi_2) d\xi_1 d\xi_2}{(3 - 2\cos 2\xi_1 - \cos \xi_1 \cos \xi_2)(1 - \cos \xi_1 \cos \xi_2) - \sin^2 \xi_1 \sin^2 \xi_2} = \frac{1}{3} \quad (4.5)$$

The numerical calculation using package DECUHR for automatic integration of singular functions over a hyper-rectangular region (Espelid and Genz, 1994) also gives the same result as (4.5).

When some single-truss defects are evenly distributed and randomly oriented throughout an infinite triangular lattice and the system is “dilute”, i.e. each of the single-truss defects can be treated independently, without any contribution from each other. In this case, averaging over the single-truss defect array is thus reduced simply to summation over the three directions ($\alpha=1,2,3$). From (4.3), analytical solutions of the effective moduli of triangular lattice with single-truss defects have the form

$$\bar{c}_{ijst} = \frac{k_n l_0^2}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - \frac{3k_n l_0^2 N}{S} \sum_{\alpha=1}^3 \phi^{(\alpha)} n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} \quad (4.6)$$

where N is the total number of trusses of the perfect lattice, $\phi^{(\alpha)}$ is the number density of the single-truss defects oriented in $\mathbf{n}^{(\alpha)}$ ($\alpha=1,2,3$). Considering the number densities of single-truss defects are the same in the three directions (i.e., $\phi^{(\alpha)} = \phi/3$) and the lattice structure is infinite (i.e., $N \rightarrow \infty$), one has the following set of equations

$$\bar{c}_{1111} = \bar{c}_{2222} = \frac{3\sqrt{3}k_n(1-3\phi)}{4}, \quad \bar{c}_{1122} = \bar{c}_{2211} = \bar{c}_{1212} = \frac{\sqrt{3}k_n(1-3\phi)}{4} \quad (4.7)$$

For plane stress, the effective Young’s modulus E^* , the shear modulus μ^* and the effective Poisson’s ratio ν^* are

$$\frac{E^*}{E_0} = 1 - 3\phi \quad \text{or} \quad \frac{E^*}{E_s} = (1 - 3\phi) \frac{2\sqrt{3}t}{3l} \quad (4.8)$$

$$\frac{\mu^*}{\mu_0} = 1 - 3\phi \quad \text{or} \quad \frac{\mu^*}{E_s} = (1 - 3\phi) \frac{\sqrt{3}t}{4l} \quad (4.9)$$

$$\frac{\nu^*}{\nu_0} = 1 \quad \text{or} \quad \nu^* = \frac{1}{3} \quad (4.10)$$

To verify the formulation presented above and evaluate the suitability of the dilute model, an intact periodic triangular lattice with the relative density

$$\bar{\rho} = \frac{2\sqrt{3}t}{l} = 0.15 \quad (4.11)$$

is weakened by the truss defects whose number density ϕ is varied in the range of 0–0.32. Substituting (4.11) into (4.8) and (4.9), the normalized Young’s modulus and shear modulus are then plotted in Fig. 4 against the number density of the truss defects. For comparison, the reference results which have been taken from Wang and McDowell (2003) using the FEM are included in Fig. 4. One can show that the dilute solutions for the Young’s modulus and the shear modulus are in excellent agreement with the numerical results up to the number density $\phi=0.16$. Additionally, the successfulness of the comparison implies that the effective Poisson’s ratio of a stretch-dominated triangular lattice containing randomly dispersed truss defects keeps 1/3 for the number density $0 < \phi < 0.16$. From Fig. 4 it is also seen that the dilute results are lower than the FEM data at a number density of about 0.32. It is due to the reason that the interactions between truss defects are ignored. Though this is an open problem of the dilute approximation, it may fall outside practical application limits considering the approximation of non-interacting truss defects remains accurate at relatively high number density of the truss defects.

4.2. Case 2: Double-truss defects

If a couple of trusses in the infinite perfect stretch-dominated triangular lattice are fractured, which connect node p and its nearest-neighboring nodes q_a and q_b and form 180° angles with each other, as shown in Fig. 3(b), (3.37) can be rewritten as

$$\begin{aligned} \hat{\mathbf{u}}^{(pq_a)} - 2(\mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_a)})) \times \mathbf{K}^{(pq_a)} \times \hat{\mathbf{u}}^{(pq_a)} - (\mathbf{G}(\mathbf{l}^{(q_a q_b)}) - \mathbf{G}(\mathbf{l}^{(q_a p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_b)})) \times \mathbf{K}^{(pq_b)} \times \hat{\mathbf{u}}^{(pq_b)} &= \mathbf{E} \times \mathbf{l}^{(pq_a)} \\ \hat{\mathbf{u}}^{(pq_b)} - 2(\mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_b)})) \times \mathbf{K}^{(pq_b)} \times \hat{\mathbf{u}}^{(pq_b)} - (\mathbf{G}(\mathbf{l}^{(q_b q_a)}) - \mathbf{G}(\mathbf{l}^{(q_b p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_a)})) \times \mathbf{K}^{(pq_a)} \times \hat{\mathbf{u}}^{(pq_a)} &= \mathbf{E} \times \mathbf{l}^{(pq_b)} \end{aligned} \quad (4.12)$$

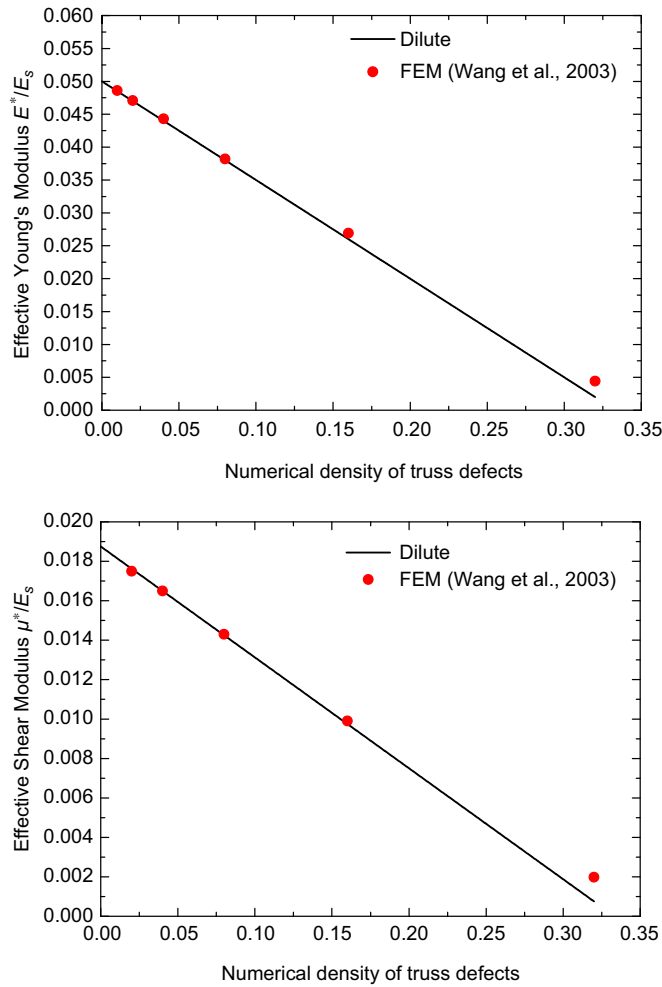


Fig. 4. Effects of number density of truss defects on the effective Young's modulus and shear modulus of triangular lattice.

Here where $\mathbf{l}^{(pq_i)}$, $\mathbf{K}^{(pq_i)}$ and $\tilde{\mathbf{u}}^{(pq_i)}(i=a,b)$ are the length vector, the stiffness matrix and the eigendeformation of truss (pq_i) , respectively. By the requirement of symmetry of nodes q_a and q_b with respect to node p , we find easily that

$$\mathbf{l}^{(pq_a)} = -\mathbf{l}^{(pq_b)} = l_0 \mathbf{n}^{(pq_a)}, \mathbf{K}^{(pq_a)} = \mathbf{K}^{(pq_b)} = k_n \mathbf{n}^{(pq_a)} \otimes \mathbf{n}^{(pq_a)}, \mathbf{K}^{(pq_a)} = \mathbf{K}^{(pq_b)} = k_n \mathbf{n}^{(pq_a)} \otimes \mathbf{n}^{(pq_a)} \tag{4.13}$$

where $\mathbf{n}^{(pq_a)}$ is the unit vector of truss (pq_a) , which is without loss of generality assumed to be one of the three directions in (4.2). Substituting (4.13) into (4.12), the eigendeformations prescribed on the trusses are given by

$$\tilde{\mathbf{u}}^{(pq_a)} = -\tilde{\mathbf{u}}^{(pq_b)} = [\mathbf{I} - (\mathbf{G}(0) - \mathbf{G}(2l_0 \mathbf{n}^{(pq_a)})) \times k_n \mathbf{n}^{(pq_a)} \otimes \mathbf{n}^{(pq_a)}]^{-1} \times \mathbf{E} \times l_0 \mathbf{n}^{(pq_a)} \tag{4.14}$$

Since it is not easy to analytically calculate the Green's functions in (4.14), a numerical scheme is then used. With the help of package DECUHR, we have

$$[\mathbf{I} - (\mathbf{G}(0) - \mathbf{G}(2l_0 \mathbf{n}^{(pq_a)})) \times k_n \mathbf{n}^{(pq_a)} \otimes \mathbf{n}^{(pq_a)}] = \begin{cases} \begin{bmatrix} 0.494 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} & \text{if } \mathbf{n}^{(pq_a)} = (1,0) \\ \begin{bmatrix} 0.873 & -0.219 \\ -0.219 & 0.62 \end{bmatrix} & \text{if } \mathbf{n}^{(pq_a)} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ \begin{bmatrix} 0.873 & 0.219 \\ 0.219 & 0.62 \end{bmatrix} & \text{if } \mathbf{n}^{(pq_a)} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{cases} \tag{4.15}$$

Combing (4.14), (4.15), (3.40) and (3.41), approximate analytical solutions of the effective moduli of the lattice containing a double-truss defect is derived as follows

$$C_{ijst}^{(b)} = \frac{k_n l_0^2}{S} \left(\sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - 4.05 n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} \right) \quad (\alpha = 1,2,3) \tag{4.16}$$

When an infinite triangular lattice contains a random distribution of randomly oriented double-truss defects and the interaction among the double-truss defects can be ignored, the overall moduli for a lattice with double-truss defects are given from (4.16) by

$$\bar{C}_{ijst} = \frac{k_n l_0^2}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - \frac{4.05 k_n l_0^2 N}{S} \sum_{\alpha=1}^3 \phi^{(\alpha)} n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} \quad (4.17)$$

where $\phi^{(\alpha)}$ is the number density of the double-truss defects oriented in $\mathbf{n}^{(\alpha)}$ ($\alpha=1,2,3$). As $\phi^{(\alpha)}$ are the same in the three directions (i.e., $\phi^{(\alpha)} = \phi/6$) and $N \rightarrow \infty$, we obtain

$$\bar{C}_{1111} = \bar{C}_{2222} = \frac{3\sqrt{3}k_n(1-2.025\phi)}{4}, \quad \bar{C}_{1122} = \bar{C}_{2211} = \bar{C}_{1212} = \frac{\sqrt{3}k_n(1-2.025\phi)}{4} \quad (4.18)$$

For plane stress, the normalized Young’s modulus, the shear modulus and the effective Poisson’s ratio are, respectively,

$$\frac{E^*}{E_0} = 1 - 2.025\phi \quad (4.19)$$

$$\frac{\mu^*}{\mu_0} = 1 - 2.025\phi \quad (4.20)$$

$$\frac{\nu^*}{\nu_0} = 1 \quad (4.21)$$

4.3. Case 3: Triple-truss defects

If three trusses in the infinite perfect stretch-dominated triangular lattice are fractured, which connect node p and its nearest-neighboring nodes q_a , q_b and q_c and form 120° angles with each other, as shown in Fig. 3(c), (3.37) can be rewritten as

$$\begin{aligned} &\tilde{\mathbf{u}}^{(pq_a)} - 2(\mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_a)})) \times \mathbf{K}^{(pq_a)} \times \tilde{\mathbf{u}}^{(pq_a)} - (\mathbf{G}(\mathbf{l}^{(q_a q_b)}) - \mathbf{G}(\mathbf{l}^{(q_a p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_b)})) \times \mathbf{K}^{(pq_b)} \times \tilde{\mathbf{u}}^{(pq_b)} \\ &\quad - (\mathbf{G}(\mathbf{l}^{(q_a q_c)}) - \mathbf{G}(\mathbf{l}^{(q_a p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_c)})) \times \mathbf{K}^{(pq_c)} \times \tilde{\mathbf{u}}^{(pq_c)} = \mathbf{E} \times \mathbf{l}^{(pq_a)} \\ &\tilde{\mathbf{u}}^{(pq_b)} - 2(\mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_b)})) \times \mathbf{K}^{(pq_b)} \times \tilde{\mathbf{u}}^{(pq_b)} - (\mathbf{G}(\mathbf{l}^{(q_b q_a)}) - \mathbf{G}(\mathbf{l}^{(q_b p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_a)})) \times \mathbf{K}^{(pq_a)} \times \tilde{\mathbf{u}}^{(pq_a)} \\ &\quad - (\mathbf{G}(\mathbf{l}^{(q_b q_c)}) - \mathbf{G}(\mathbf{l}^{(q_b p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_c)})) \times \mathbf{K}^{(pq_c)} \times \tilde{\mathbf{u}}^{(pq_c)} = \mathbf{E} \times \mathbf{l}^{(pq_b)} \\ &\tilde{\mathbf{u}}^{(pq_c)} - 2(\mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_c)})) \times \mathbf{K}^{(pq_c)} \times \tilde{\mathbf{u}}^{(pq_c)} - (\mathbf{G}(\mathbf{l}^{(q_c q_a)}) - \mathbf{G}(\mathbf{l}^{(q_c p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_a)})) \times \mathbf{K}^{(pq_a)} \times \tilde{\mathbf{u}}^{(pq_a)} \\ &\quad - (\mathbf{G}(\mathbf{l}^{(q_c q_b)}) - \mathbf{G}(\mathbf{l}^{(q_c p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_b)})) \times \mathbf{K}^{(pq_b)} \times \tilde{\mathbf{u}}^{(pq_b)} = \mathbf{E} \times \mathbf{l}^{(pq_c)} \end{aligned} \quad (4.22)$$

where $\mathbf{l}^{(pq_i)}$, $\mathbf{K}^{(pq_i)}$ and $\tilde{\mathbf{u}}^{(pq_i)}$ ($i=a,b,c$) are the length vector, the stiffness matrix and the eigendeformation of truss (pq_i) respectively. Without loss of generality, we assume the unit vector of truss (pq_a), $\mathbf{n}^{(pq_a)}$, can have the following two directions: (1,0) or $(1/2, \sqrt{3}/2)$. By using package DECUHR, if $\mathbf{n}^{(pq_a)} = (1,0)$, the numerical solution of (4.22) is given by

$$\begin{Bmatrix} \tilde{\mathbf{u}}^{(pq_a)} \\ \tilde{\mathbf{u}}^{(pq_b)} \\ \tilde{\mathbf{u}}^{(pq_c)} \end{Bmatrix} = \begin{bmatrix} 0.333 & 0.0 & -0.057 & 0.1 & -0.057 & -0.1 \\ 0.0 & 1.0 & 0.113 & -0.195 & -0.113 & -0.195 \\ -0.253 & 0.0 & 0.833 & 0.288 & -0.069 & -0.119 \\ -0.013 & 0.0 & 0.288 & 0.5 & -0.106 & -0.184 \\ -0.253 & 0.0 & -0.069 & 0.119 & 0.833 & -0.288 \\ 0.013 & 0.0 & 0.106 & -0.184 & -0.288 & 0.5 \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{E} \times l_0 \mathbf{n}^{(1)} \\ \mathbf{E} \times l_0 \mathbf{n}^{(3)} \\ -\mathbf{E} \times l_0 \mathbf{n}^{(2)} \end{Bmatrix} \quad (4.23)$$

and if $\mathbf{n}^{(pq_a)} = (1/2, \sqrt{3}/2)$, the numerical solution of (4.22) is given by

$$\begin{Bmatrix} \tilde{\mathbf{u}}^{(pq_a)} \\ \tilde{\mathbf{u}}^{(pq_b)} \\ \tilde{\mathbf{u}}^{(pq_c)} \end{Bmatrix} = \begin{bmatrix} 0.833 & -0.288 & -0.253 & 0.0 & -0.069 & 0.119 \\ -0.288 & 0.5 & 0.013 & 0.0 & 0.106 & -0.184 \\ -0.057 & -0.1 & 0.333 & 0.0 & -0.057 & 0.1 \\ -0.113 & -0.195 & 0.0 & 1.0 & 0.113 & -0.195 \\ -0.069 & -0.119 & -0.253 & 0.0 & 0.833 & 0.288 \\ -0.106 & -0.184 & -0.013 & 0.0 & 0.288 & 0.5 \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{E} \times l_0 \mathbf{n}^{(2)} \\ -\mathbf{E} \times l_0 \mathbf{n}^{(1)} \\ -\mathbf{E} \times l_0 \mathbf{n}^{(3)} \end{Bmatrix} \quad (4.24)$$

Substituting (4.23) or (4.24) into (3.40) and combining with (3.41), we found that whichever direction $\mathbf{n}^{(pq_a)}$ takes, approximate analytical solutions of the effective moduli of the lattice containing a triple-truss defect is of the form:

$$C_{ijst}^{(b)} = \frac{k_n l_0^2}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - 3.65 \frac{k_n l_0^2}{S} \sum_{\alpha=1}^3 n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)}$$

$$\begin{aligned}
 &+ 0.938 \frac{k_n l_0^2}{S} (n_i^{(1)} n_j^{(1)} n_s^{(2)} n_t^{(2)} + n_i^{(2)} n_j^{(2)} n_s^{(1)} n_t^{(1)} + n_i^{(1)} n_j^{(1)} n_s^{(3)} n_t^{(3)} \\
 &+ n_i^{(3)} n_j^{(3)} n_s^{(1)} n_t^{(1)} + n_i^{(2)} n_j^{(2)} n_s^{(3)} n_t^{(3)} + n_i^{(3)} n_j^{(3)} n_s^{(2)} n_t^{(2)})
 \end{aligned} \tag{4.25}$$

When the condition of dilute distribution is assumed and the interaction among the three-truss defects is neglected, the overall moduli of an infinite triangular lattice with triple-truss defects is approximated from (4.25) by

$$\begin{aligned}
 \bar{C}_{ijst} = & \frac{k_n l_0^2}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - 3.65 \frac{k_n l_0^2 N \phi}{3S} \sum_{\alpha=1}^3 n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} + 0.938 \frac{k_n l_0^2 N \phi}{3S} (n_i^{(1)} n_j^{(1)} n_s^{(2)} n_t^{(2)} + n_i^{(2)} n_j^{(2)} n_s^{(1)} n_t^{(1)} + n_i^{(1)} n_j^{(1)} n_s^{(3)} n_t^{(3)} \\
 & + n_i^{(3)} n_j^{(3)} n_s^{(1)} n_t^{(1)} + n_i^{(2)} n_j^{(2)} n_s^{(3)} n_t^{(3)} + n_i^{(3)} n_j^{(3)} n_s^{(2)} n_t^{(2)})
 \end{aligned} \tag{4.26}$$

Let $N \rightarrow \infty$, one has the following set of equations

$$\begin{aligned}
 \bar{C}_{1111} = \bar{C}_{2222} &= \frac{3\sqrt{3}k_n(1-2.712\phi)}{4}, \\
 \bar{C}_{1122} = \bar{C}_{2211} &= \frac{\sqrt{3}k_n(1+1.04\phi)}{4}, \quad \bar{C}_{1212} = \frac{\sqrt{3}k_n(1-4.588\phi)}{4}
 \end{aligned} \tag{4.27}$$

For plane stress, the normalized Young’s modulus, the shear modulus and the effective Poisson’s ratio are, respectively,

$$\frac{E^*}{E_0} = \frac{9(1-2.712\phi)(1-(v^*)^2)}{8} \approx 1-3.65\phi + O(\phi^2) \tag{4.28}$$

$$\frac{\mu^*}{\mu_0} = 1-4.588\phi \tag{4.29}$$

$$\frac{v^*}{v_0} = \frac{1+1.04\phi}{1-2.712\phi} \approx 1+3.752\phi + O(\phi^2) \tag{4.30}$$

4.4. Case 4: Cell defects

If all six trusses connecting a typical node p and its nearest neighbors q_α ($\alpha=1, \dots, 6$) are fractured and removed from the infinite perfect stretch-dominated triangular lattice, as shown in Fig. 3(d), (3.37) can be rewritten as

$$\tilde{\mathbf{u}}^{(pq_\alpha)} - \sum_{\alpha'=1}^6 (\mathbf{G}(\mathbf{l}^{(q_\alpha q_{\alpha'})}) - \mathbf{G}(\mathbf{l}^{(q_\alpha p)}) + \mathbf{G}(0) - \mathbf{G}(\mathbf{l}^{(pq_{\alpha'})})) \times \mathbf{K}^{(\alpha')} \times \tilde{\mathbf{u}}^{(pq_{\alpha'})} = \mathbf{E} \times \mathbf{l}^{(pq_\alpha)}, \quad (\alpha = 1, \dots, 6) \tag{4.31}$$

It is important to emphasize that the above resulting system of linear equations is ill-conditioned. The reason is that since node p is not connect by trusses with any of its nearest neighbors in the cell defect, the disturbance displacement $\mathbf{u}^{(p)}$ of node p can then be arbitrary value. In this case, we add an additional constraint conditions to the disturbance displacement field, i.e.,

$$\mathbf{u}^{(p)} = \sum_{\alpha'=1}^6 (\mathbf{G}(\mathbf{l}^{(pq_{\alpha'})}) - \mathbf{G}(0)) \times \mathbf{K}^{(\alpha')} \times \tilde{\mathbf{u}}^{(pq_{\alpha'})} = 0 \tag{4.32}$$

Substituting (4.32) into (4.31), the eigendeformation of the six truss removed can be rewritten as

$$\tilde{\mathbf{u}}^{(pq_\alpha)} - \sum_{\alpha'=1}^6 (\mathbf{G}(\mathbf{l}^{(q_\alpha q_{\alpha'})}) - \mathbf{G}(\mathbf{l}^{(q_\alpha p)})) \times \mathbf{K}^{(\alpha')} \times \tilde{\mathbf{u}}^{(pq_{\alpha'})} = \mathbf{E} \times \mathbf{l}^{(pq_\alpha)} \quad (\alpha = 1, \dots, 6) \tag{4.33}$$

Without loss of generality, we assume the unit vector of truss (pq_1) , $\mathbf{n}^{(pq_1)}$, to be $(1, 0)$. After using package DECUHR for numerical integral of Green’s functions in (4.33), the numerical solution of the eigendeformation field is derived as follows

$$\begin{pmatrix} \tilde{\mathbf{u}}^{(pq_1)} \\ \tilde{\mathbf{u}}^{(pq_2)} \\ \tilde{\mathbf{u}}^{(pq_3)} \\ \tilde{\mathbf{u}}^{(pq_4)} \\ \tilde{\mathbf{u}}^{(pq_5)} \\ \tilde{\mathbf{u}}^{(pq_6)} \end{pmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix}^{-1} \begin{pmatrix} \mathbf{E} \times l_0 \mathbf{n}^{(1)} \\ \mathbf{E} \times l_0 \mathbf{n}^{(2)} \\ \mathbf{E} \times l_0 \mathbf{n}^{(3)} \\ -\mathbf{E} \times l_0 \mathbf{n}^{(1)} \\ -\mathbf{E} \times l_0 \mathbf{n}^{(2)} \\ -\mathbf{E} \times l_0 \mathbf{n}^{(3)} \end{pmatrix} \tag{4.34}$$

where

$$\mathbf{A} = \begin{bmatrix} 0.667 & 0.0 & 0.055 & 0.094 & 0.026 & -0.045 \\ 0.0 & 1.0 & -0.032 & -0.055 & -0.032 & 0.055 \\ 0.0 & 0.0 & 0.917 & -0.144 & -0.055 & 0.094 \\ 0.126 & 0.0 & -0.144 & 0.75 & -0.032 & 0.055 \\ 0.08 & 0.0 & -0.055 & -0.094 & 0.917 & 0.144 \\ -0.013 & 0.0 & 0.032 & 0.055 & 0.144 & 0.75 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 0.173 & 0.0 & 0.026 & 0.045 & 0.055 & -0.094 \\ 0.0 & 0.0 & 0.032 & 0.055 & 0.032 & -0.055 \\ 0.08 & 0.0 & 0.043 & 0.075 & 0.014 & -0.025 \\ 0.013 & 0.0 & 0.075 & 0.13 & -0.038 & 0.066 \\ 0.0 & 0.0 & 0.014 & 0.025 & 0.043 & -0.075 \\ -0.126 & 0.0 & 0.038 & 0.066 & -0.075 & 0.13 \end{bmatrix}$$

Combining (4.34), (3.40) and (3.41), the overall elastic stiffness tensor of the lattice is given by

$$\begin{aligned} C_{ijst}^{(p)} &= \frac{k_n l_0^2}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - 4.1528 \frac{k_n l_0^2}{S} \sum_{\alpha=1}^3 n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} \\ &+ 0.4344 \frac{k_n l_0^2}{S} (n_i^{(1)} n_j^{(1)} n_s^{(2)} n_t^{(2)} + n_i^{(2)} n_j^{(2)} n_s^{(1)} n_t^{(1)} + n_i^{(1)} n_j^{(1)} n_s^{(3)} n_t^{(3)} \\ &+ n_i^{(3)} n_j^{(3)} n_s^{(1)} n_t^{(1)} + n_i^{(2)} n_j^{(2)} n_s^{(3)} n_t^{(3)} + n_i^{(3)} n_j^{(3)} n_s^{(2)} n_t^{(2)}) \end{aligned} \tag{4.35}$$

In the case of the dilute limit, the interactions between different cell defects are negligible. The effective elastic stiffness tensor of an infinite triangular lattice containing randomly dispersed cell defects is approximated from (4.35) by

$$\begin{aligned} \bar{C}_{ijst} &= \frac{k_n l_0^2}{S} \sum_c n_i^{(c)} n_j^{(c)} n_s^{(c)} n_t^{(c)} - 4.1528 \frac{k_n l_0^2 N \phi}{6S} \sum_{\alpha=1}^3 n_i^{(\alpha)} n_j^{(\alpha)} n_s^{(\alpha)} n_t^{(\alpha)} \\ &+ 0.4344 \frac{k_n l_0^2 N \phi}{6S} (n_i^{(1)} n_j^{(1)} n_s^{(2)} n_t^{(2)} + n_i^{(2)} n_j^{(2)} n_s^{(1)} n_t^{(1)} + n_i^{(1)} n_j^{(1)} n_s^{(3)} n_t^{(3)} \\ &+ n_i^{(3)} n_j^{(3)} n_s^{(1)} n_t^{(1)} + n_i^{(2)} n_j^{(2)} n_s^{(3)} n_t^{(3)} + n_i^{(3)} n_j^{(3)} n_s^{(2)} n_t^{(2)}) \end{aligned} \tag{4.36}$$

where ϕ is the volume fraction of all removed trusses. Let $N \rightarrow \infty$, one has the following set of equations

$$\begin{aligned} \bar{C}_{1111} = \bar{C}_{2222} &= \frac{3\sqrt{3}k_n(1-1.8592\phi)}{4}, \\ \bar{C}_{1122} = \bar{C}_{2211} &= \frac{\sqrt{3}k_n(1-0.9904\phi)}{4}, \quad \bar{C}_{1212} = \frac{\sqrt{3}k_n(1-2.2936\phi)}{4} \end{aligned} \tag{4.37}$$

For plane stress, one obtains the normalized Young’s modulus, the shear modulus and the effective Poisson’s ratio

$$\frac{E^*}{E_0} = \frac{9(1-1.859\phi)(1-(v^*)^2)}{8} \approx 1-2.076\phi + O(\phi^2) \tag{4.38}$$

$$\frac{\mu^*}{\mu_0} = 1-2.294\phi \tag{4.39}$$

$$\frac{v^*}{v_0} = \frac{1-0.99\phi}{1-1.859\phi} \approx 1+0.869\phi + O(\phi^2) \tag{4.40}$$

5. Results and discussions

The effective elastic moduli predicted in the previous section are compared with the Hashin-Shtrikman upper bounds and uniform strain upper bounds in this section. Furthermore, Section 4 together with this Section then reveals the effect of the interactions between adjacent fractured trusses on the properties of the lattice.

5.1. Hashin–Shtrikman (HS) upper bounds

Torquato et al. (1998) treated a perfect triangular lattice as a two-dimensional porous solid in the limit that the solid volume fraction is very small, and derived the Hashin–Shtrikman (HS) upper bounds given by

$$\frac{E_0}{E_s} \leq \frac{2\sqrt{3}t}{3l}, \quad \frac{\mu_0}{E_s} \leq \frac{\sqrt{3}t}{4l} \quad (5.1)$$

Following the same assumptions, the HS upper bounds can be deduced for a triangular lattice containing randomly dispersed truss defects. When trusses with the number density ϕ are fractured, the upper bounds for the Young's and shear moduli of the lattice are given by

$$\frac{E^*}{E_0} \leq 1-\phi \quad \text{or} \quad \frac{E^*}{E_s} \leq (1-\phi) \frac{2\sqrt{3}t}{3l} \quad (5.2)$$

$$\frac{\mu^*}{\mu_0} \leq 1-\phi \quad \text{or} \quad \frac{\mu^*}{E_s} \leq (1-\phi) \frac{\sqrt{3}t}{4l} \quad (5.3)$$

5.2. Uniform strain upper bounds

On the other hand, the upper bounds for the effective elastic moduli of granular materials are derived using the principle of minimum potential energy and the uniform strain assumption (for example, Henderson et al., 2001; Krut and Rothenburg, 2002). Analogous methods can be used to estimate the elastic moduli of triangular lattices containing randomly dispersed truss defects. Assuming every node within the triangular lattice displaces in accordance with the macro strain tensor \mathbf{E}

$$\mathbf{u}^{(p)} = \mathbf{E} \times \mathbf{x}^{(p)} \quad (5.4)$$

and using the homogenization process shown in Fig. 2, we can derive another set of solutions

$$\bar{C}_{1111} = \bar{C}_{2222} = \frac{3\sqrt{3}k_n(1-\phi)}{4}, \quad \bar{C}_{1122} = \bar{C}_{2211} = \bar{C}_{1212} = \frac{\sqrt{3}k_n(1-\phi)}{4} \quad (5.5)$$

For plane stress, the effective Young's modulus E_{as}^* and the shear modulus μ_{as}^* are

$$\frac{E_{as}^*}{E_0} = 1-\phi \quad \text{or} \quad \frac{E_{as}^*}{E_s} = (1-\phi) \frac{2\sqrt{3}t}{3l} \quad (5.6)$$

$$\frac{\mu_{as}^*}{\mu_0} = 1-\phi \quad \text{or} \quad \frac{\mu_{as}^*}{E_s} = (1-\phi) \frac{\sqrt{3}t}{4l} \quad (5.7)$$

Comparison of (5.2), (5.3) and (5.6), (5.7) shows that the HS upper bounds coincide with those predicted on the base of the uniform strain assumption. It should be pointed out that Durand (2005) derived the upper-bound for the bulk modulus of 2D cellular materials by imposing a uniform radially oriented displacement on the boundary of a circular portion of a cellular network and using the principle of minimum potential energy. He found the bulk modulus recovers exactly the expression of the HS upper bounds in the low-density limit. It can be easily seen that the boundary displacement defined by Durand is a special case of the uniform strain assumption.

The variation of the normalized Young's modulus E^*/E_0 and shear modulus G^*/G_0 in (5.6) and (5.7) with the number density of fractured trusses ϕ for plane-stress state of deformation are shown in Figs. 5 and 6 respectively. Also shown in these figures are the results of the present model for the four types of defect modes. It can be seen that the present results are much lower than the HS upper bound solutions. This observation implies that more rigorous bounds of the Hashin–Shtrikman type could be obtained by taking advantage of the solution to the auxiliary problem of an isolated defect in an infinite triangular lattice.

5.3. Interactions among adjacent fractured trusses

Figs. 5 and 6 also show that the effective elastic moduli of a triangular lattice with defects strongly depend on the interactions between adjacent fractured trusses. As the same number density of trusses is fractured, compared with the single-truss defect mode, the double-truss defect mode has much higher stiffness while the triple-truss defect mode has much lower stiffness. A reasonable explanation of the above results is that there exist amplification and shielding effects between adjacent fractured trusses sharing a common node. If adjacent fractured trusses are collinear, each fractured truss is subjected to the shielding effect from the remaining fractured truss, and the interactions will cause less reduction in the effective Young's and shear modulus. Conversely, if adjacent fractured trusses are non-collinear, each fractured truss is subjected to the amplification effect from the remaining fractured truss, and the interactions will cause more reduction in the effective Young's and shear modulus. For the cell defect mode, it seems that the stress amplifying effect plays a more

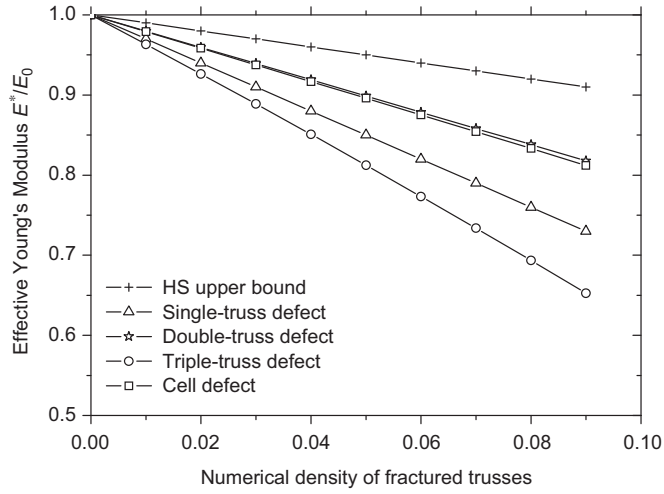


Fig. 5. Variation of normalized Young's modulus with numerical density of fractured trusses.

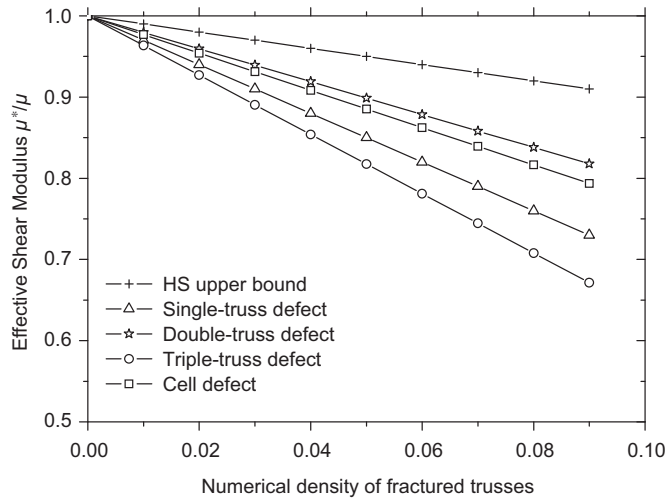


Fig. 6. Variation of normalized shear modulus with numerical density of fractured trusses.

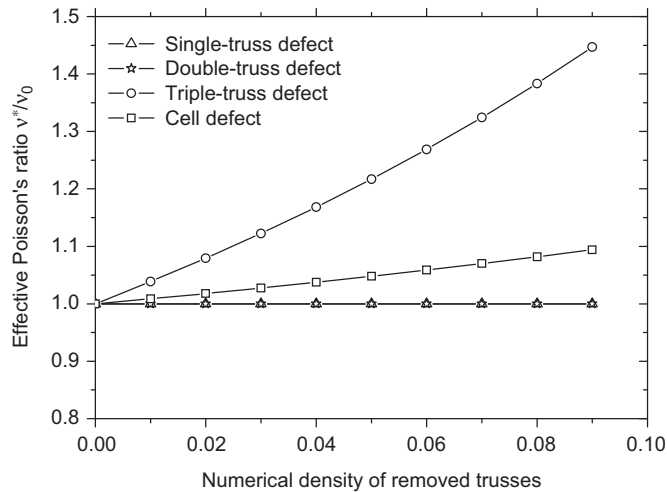


Fig. 7. Variation of normalized Poisson's ratio with numerical density of fractured trusses.

significant role than the stress shielding effect in the effective Young's and shear modulus. In this case, a triangular lattice with cell defects exhibits more slow reduction of the effective Young's and shear modulus than a triangular lattice with single-truss defects.

Another important aspect of our analytical results is the variation in the Poisson ratio for the four types of defect modes, as shown in Fig. 7. It is interesting to see that the effective Poisson's ratio for the lattice with double-truss defects remains unchanged with increasing defect density, which is the same as the case of the lattice with single-truss defects. On the other hand, the presence of triple-truss defects produces a significant increase in the effective Poisson ratio. Considering a hexagonal lattice can be constructed from a triangular lattice by removing three trusses at 120° to each other from a triangular cell, and the overall Poisson ratio of a hexagonal lattice is 1 (for example, Wang and McDowell, 2003; Fleck and Qiu, 2007), it seems reasonable that the overall Poisson ratio of a triangular lattice increases gradually with increasing the number of three-truss defects, since the appearance of three-truss defects causes that a triangular lattice gradually becomes a hexagonal lattice. For the cell defect mode, the competition of collinear and non-collinear effects of adjacent fractured trusses results in the slight increase in the effective Poisson's ratio with increasing the number density of the removed trusses.

6. Conclusions

This paper presents a micro-discrete homogenization method to predict the overall macroscopic response of triangular lattice materials with microstructural defects. In Section 2, a general homogenization process for the modeling of lattice materials is proposed based on homogeneous strain boundary conditions (2.6), the local constitutive law (2.3) and the static average operator (2.5). Using the proposed homogenization process, an explicit closed-form solution of the effective moduli (2.12) is directly derived for intact periodic triangular lattices. Accounting for the fact that non-periodic defects lead to the difficulties of deriving the elastic displacement fields, an eigendeformation method is developed to establish the equivalency between an imagining displacement field (eigendisplacement) and a defect distribution in Section 3. The mechanical response of a triangular lattice with defects, under uniform far-field loading, is proved to be equivalent to that of a perfect periodic lattice with an eigendisplacement field. In order to solve the equivalent eigendeformation field via Green's function technique, closed-form expression for the Green's function for triangular lattice is given by (3.35), which correlates the displacement at one lattice site to a unit force acting on another lattice site. With the help of a combination of the homogenization process, the eigendeformation method and the lattice Green's function, a compact analytical formula (3.41) is deduced to calculate the effective elastic moduli for triangular lattice containing different types of defects.

In Section 4, the present homogenization method is applied in particular to investigate the influence of fractured cell walls (truss defects) and missing cells (cell defects) on the overall elastic moduli of stretch-dominated triangular lattices in the case of the dilute limit. It is shown that the predicted effective Young's and shear moduli of triangular lattices with the single-truss defect mode are in good agreement with FE calculations reported by Wang and McDowell (2003) in the dilute limit, see Fig. 4. Comparisons with the Hashin–Shtrikman upper bounds are given in Section 5. It is proved that the Hashin–Shtrikman upper bounds are identical to those resulting from the assumption of uniform strain but much larger than the results of the present model for the four types of defect modes, see Figs. 5 and 6.

Section 4 together with Section 5 also reveals the interactions between adjacent fractured trusses have strongly effects on the effective elastic moduli of the lattice. On the one hand, as the same number density of trusses is fractured, the double-truss defect mode has the highest stiffness while the triple-truss defect mode has the lowest stiffness. The stiffness of the single-truss defect mode and the cell defect mode lie between the former two cases, see Figs. 5 and 6. On the other hand, with increasing the numerical density of fractured trusses, the effective Poisson's ratio remains unchanged for the single-truss defect mode and the double-truss defect mode while the effective Poisson's ratio increases gradually for the triple-truss defect mode and the cell defect mode, see Fig. 7. We therefore could conclude that not only the effective Young's and shear moduli but also the effective Poisson's ratio of triangular lattice materials depend on the number density of fractured cell walls and their spatial arrangements.

The proposed homogenization method provides a new vehicle for analytical estimation of lattice materials containing microstructural defects. Extension of the proposed micromechanical method to damaged lattice materials with other types of cell shapes, such as 2D square and hexagonal honeycombs and 3D octet-truss materials is possible, and will be investigated in detail in a forthcoming paper by the authors. Additionally, Day et al. (1992) applied computer simulation techniques to obtain the elastic moduli of an infinite triangular spring network containing circular holes. They found that the Poisson's ratio goes to a fixed value of $1/3$ for random arrangements. The same Poisson ratio was also predicted using effective medium theories (Jun and Jasiuk, 1993). The above conclusions are different from the dilute solutions of the present model. The reason for this difference is that each circular hole composes of many cell defects, which interact strongly with each other. In this case, the condition of dilute distribution of cell defects is not satisfied, and the obtained effective moduli need to be further improved by taking into account the effect of micro-defect interactions.

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