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THE GLOBAL L^2 STABILITY OF SOLUTIONS TO THREE DIMENSIONAL MHD EQUATIONS*

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Abstract In this paper, we mainly study the global L^2 stability for large solutions to the MHD equations in three-dimensional bounded or unbounded domains. Under suitable conditions of the large solutions, it is shown that the large solutions are stable. And we obtain the equivalent condition of this stability condition. Moreover, the global existence and the stability of two-dimensional MHD equations under three-dimensional perturbations are also established.

Key words MHD equations; strong solutions; stability

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1 Introduction

Magneto-hydrodynamic (MHD) equations are closely related to many branches of physics, chemistry, metallurgy, nuclear energy and space flight. In this paper, we mainly consider the following MHD equations:

$$\begin{cases} u_t - \varepsilon\triangle u + (u\cdot\bigtriangledown)u - (B\cdot\bigtriangledown)B + \nabla(\frac{1}{2}|B|^2) + \bigtriangledown p = f, \ (x,t)\in\Omega\times(0,\infty), \\ B_t - \mu\triangle B + (u\cdot\bigtriangledown)B - (B\cdot\bigtriangledown)u = 0, \qquad (x,t)\in\Omega\times(0,\infty), \\ \nabla\cdot u = 0, \nabla\cdot B = 0 \\ B|_{t=0} = \theta_0, \qquad u|_{t=0} = u_0, \\ B|_{\partial\Omega} = 0, \qquad u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a domain in R^3, unknown vector function $u = u(x,t)$ is the velocity of fluid, $B = B(x,t)$ and $p = p(x,t)$ are magnetic field and pressure function respectively. $f = f(x,t)$ is the

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given forcing function, $u_0 = u_0(x)$, $B_0 = B_0(x)$ are the initial velocity and the initial magnetic field, $\varepsilon = \frac{1}{Re}$, $\mu = \frac{1}{Rm}$, Re and Rm are Reynolds of fluid and magnetic field, respectively.

The MHD equations are similar to the Navier-Stokes equations in the structure. In particular, if $B = 0$, then equations (1.1) transform to the impressible Navier-Stokes equations. So we can use the similar method of Navier-Stokes equations to solve the problem of MHD equations. But there are unknown magnetic function B, more nonlinear terms and strong coupled terms in the MHD equations. Therefore, it is more difficult to study the MHD equations. In 1994, Ponce, Racke, Sideris and Titi [5] proved the global stability of large solutions to the 3D Navier-Stokes equations. They proved the global stability of the strong solutions to Navier-Stokes on bounded or unbounded domains under suitable conditions. Moreover, they also established the global existence and the stability of two-dimensional MHD equations under three-dimensional perturbations. Decaying qualities of solutions about the Navier-Stokes equations can be seen in [1–3, 6, 12–14, 23], and references therein. Existence, uniqueness and regularity of solutions about Navier-Stokes are related to [4, 7, 8] and references therein.

Many mathematical studies were made on the MHD equations. Duvant and Lions [21] proved the existence and uniqueness of the global strong solutions of two dimensional MHD equations with initial boundary problem. They also proved the existence of the global weak solutions and existence and uniqueness of the locally strong solutions of three dimensional MHD equations with $\varepsilon = 1, \mu = 1$ in (1.1). Based on above arguments, Sermange and Teman [22] studied the large time behaviors of solutions of MHD equations. Kozono [23] proved the existence of the global weak solutions to the two dimensional MHD equations on $\Omega_t \times (0, T)$ (Ω is a bounded domain with smooth boundary in R^2) for $\varepsilon = 1, \mu = 1$. He also proved the existence of the local classical solutions in Hölder space. Wu [24] considered the convergence rate of a class of MHD equations by using the method of vanishing viscosity. He and Xin [16] got the regularity conditions of MHD equations ($\varepsilon = 0, \mu = 0$). Miao, Yuan and Zhang [25] studied the well-posedness problem of the MHD equations in the BMO^{-1} space. Some results about the MHD equations can be seen in [15, 17, 19, 20] and references therein.

We will extend the arguments of global stability of strong solutions to Navier-Stokes equations studied in [5] to that of MHD equations.

Before we state the main results, we give some notations. Throughout this paper we will always assume that $\Omega \subset R^3$ is a domain with boundary $\partial\Omega$ uniformly of class C^3. H denotes a complete space of $\{u \in C_0^\infty(\Omega), \nabla \cdot u = 0\}$ in $L^2(\Omega)$. Let $P : L^2(\Omega)| \to H$ be the Helmholtz projection operator. After applying P on the either sides of the equations in (1.1), we obtain (for simplicity, we choose $\varepsilon = \mu = 1$ and $\|\cdot\| = \|\cdot\|_{L^2}$)

$$\begin{cases} u_t + Au + P(u\cdot\bigtriangledown)u - P(B\cdot\bigtriangledown)B = Pf,\ (x,t) \in \Omega \times (0,\infty), \\ B_t + AB + P(u\cdot\bigtriangledown)B - P(B\cdot\bigtriangledown)u = 0,\ \ (x,t) \in \Omega \times (0,\infty), \\ u = Pu, B = PB \\ B|_{t=0} = B_0, \quad u|_{t=0} = u_0, \\ B|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $A = -P\triangle$ is the Stokes operator with domain $D(A) = H^2(\Omega) \cap V$, $V = H_0^1 \cap H$, where $H^2(\Omega)$ and $H_0^1(\Omega)$ denote the usual Sobolev spaces defined over Ω. In this paper, for Ω bounded or unbounded, we assume that the following Poincare inequality is satisfied

$$\|g\| \le C\|\nabla g\| \tag{1.3}$$

for all $g \in H_0^1(\Omega)$, where C is a constant.

The main results will be given in Section 2 and the corresponding proof will be presented in Section 3.

2 The Main Results

The main results of this paper are as follows:

Theorem 1 Let $v \in L^\infty_{\rm loc}([0,\infty),V)\cap L^2_{\rm loc}([0,\infty),D(A))$, $N \in L^\infty_{\rm loc}([0,\infty),V)\cap L^2_{\rm loc}([0,\infty),D(A))$ be strong solutions of equations (1.1) with data $v(\cdot,0)=v_0 \in V$, $N(\cdot,0)=N_0$, $Pf_1(\cdot,t)\in H$. Suppose that v, N satisfy

$$\int_0^\infty (\|\nabla v(t)\|^4 + \|\nabla N(t)\|^4)\mathrm{d}t < \infty. \tag{2.1}$$

Then we have

(i) Let Ω be a domain which satisfies (1.3) and $Pf_1 \in L^2([0,\infty),H)$. If there is a $\delta>0$, such that $u_0 \in V, B_0 \in V, Pf \in L^2([0,\infty),H)$ and

$$\|\nabla u_0 - \nabla v_0\| + \|\nabla B_0 - \nabla N_{10}\| + \int_0^\infty \|Pf - Pf_1\|^2 \mathrm{d}t < \delta, \tag{2.2}$$

then there exists a unique global strong solution of (1.1) with the initial data (u_0, B_0) and Pf. Moreover, there is an $M=M(\delta)$ with $M(\delta)\to 0$ as $\delta\to 0$ such that

$$\sup_{t\ge 0}(\|\nabla u(t) - \nabla v(t)\| + \|\nabla B(t) - \nabla N(t)\|) \le M(\delta), \tag{2.3}$$

and, if $\|\nabla v(t)\|, \|\nabla N(t)\|$ and $\|Pf(t) - Pf_1(t)\|$ decay to zero exponentially, then $\|\nabla u(t)\|$, $\|\nabla B(t)\|$ decay to zero exponentially as $t\to\infty$.

(ii) Let $\Omega \subset R^3$ be a general domain and $Pf_1 \in L^1\bigcap L^2([0,\infty),H)$. If there exists a $\delta>0$ such that $u_0\in V, B_0\in V$ and $Pf \in L^1\bigcap L^2([0,\infty),H)$ and

$$\|u_0 - v_0\|_{H^1} + \|B_0 - N_0\|_{H^1} + \int_0^\infty (\|Pf - Pf_1\| + \|Pf - Pf_1\|^2)\mathrm{d}t < \delta, \tag{2.4}$$

then there exists a unique global strong solution of (1.1) with the initial data (u_0, B_0) and Pf, and an $M=M(\delta)$, satisfying as $\delta\to 0$, $M(\delta)\to 0$ such that

$$\sup_{t\ge 0}(\|u(t) - v(t)\|_{H^1} + \|B(t) - N(t)\|_{H^1}) \le M(\delta). \tag{2.5}$$

Theorem 2 Assume that Ω is R^3 or a three dimensional domain for which (1.3) holds, with $f_1 \in L^2([0,\infty),L^\infty(\Omega))$. For global strong solutions (v,N) of (1.1) in the class described in Theorem 1, the condition

$$\int_0^\infty (\|\nabla v(t)\|^4 + \|\nabla N(t)\|^4)\mathrm{d}t < \infty$$

is equivalent to $v \in L^q([0,\infty), L^p(\Omega))$, where $\frac{2}{q}+\frac{3}{p}=1, 3<p\leq\infty$.

Theorem 3 Let $v_0=(v_{01},v_{02})\in L^1(R^2)\cap H^1(R^2), N_0=(N_{01},N_{02})\in L^1(R^2)\cap H^1(R^2)$ with $\nabla\cdot v_0=0, \nabla\cdot N_0=0$ and $Pf\in L^2([0,\infty),L^2(R^3))$. There is a $\delta>0$ such that if $w_0\in V(R^3), E_0\in H_0^1(R^3)$, and

$$\|w_0\|^2_{H^1(R^3)}+\|E_0\|^2_{H^1(R^3)}+\int_0^\infty \|Pf(t)-P\tilde{f}_1(t)\|^2\mathrm{d}t<\delta,$$

then there exists a unique global strong solution of (1.1) with the initial data $(u_0,B_0)=(\tilde{v}_0+w_0,\tilde{N}_0+E_0)$, where

$$\begin{aligned}&\tilde{v}_0(x_1,x_2,x_3)=(v_{01}(x_1,x_2),v_{02}(x_1,x_2),0),\\&\tilde{N}_0(x_1,x_2,x_3)=(N_{01}(x_1,x_2),N_{02}(x_1,x_2),0),\\&\tilde{f}_1(x_1,x_2,x_3)(t)=(f_{11}(x_1,x_2)(t),f_{12}(x_1,x_2)(t),0).\end{aligned}$$

3 Proof of Theorems

In this section we will give the proofs of Theorems 1–3. Before we give the usual Young inequality as follows : for $a>0, b>0$ and $\varepsilon>0$, one has

$$ab\leq\varepsilon a^p+C_\varepsilon b^q, \tag{3.1}$$

where $C_\varepsilon=(\varepsilon p)^{-q/p}q^{-1}$.

Proof of Theorem 1 Under the assumptions on u_0, N_0 and Pf, there exists a local strong solution (u,B) of (1.1) for some $T=T(\|\nabla u_0\|,\|\nabla B_0\|)>0$, where $u\in L^\infty((0,T),V)\cap L^2((0,T),D(A)), B\in L^\infty((0,T),V)\cap L^2((0,T),D(A))$. This observation shows that in order to extend solutions globally it suffices to control $\|\nabla u(t)\|$ and $\|\nabla B(t)\|$ uniformly on the interval of local existence.

To this end, let $w:=u-v, E:=B-N$. Then w, E satisfy

$$\begin{aligned}&w_t+Aw+P[(w\cdot\nabla)w+(w\cdot\nabla)v+(v\cdot\nabla)w]\\&-P[(E\cdot\nabla)E+(E\cdot\nabla)N+(N\cdot\nabla)E]=Pf-Pf_1,\end{aligned} \tag{3.2}$$

$$\begin{aligned}&E_t-AE+P[(w\cdot\nabla)E+(w\cdot\nabla)N+(v\cdot\nabla)E]\\&-P[(E\cdot\nabla)w+(E\cdot\nabla)v+(N\cdot\nabla)w]=0\end{aligned} \tag{3.3}$$

with initial value $w_0=u_0-v_0, E_0=B_0-N_0$.

Multiplying Aw on the both sides of equation (3.2), and integrating on Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\|Aw\|^2+\int_\Omega P((w\cdot\nabla)w)Aw+\int_\Omega P((w\cdot\nabla)v)Aw\\&+\int_\Omega P((v\cdot\nabla)w)Aw-\int_\Omega P((E\cdot\nabla)E)Aw-\int_\Omega P((E\cdot\nabla)N)Aw\\&-\int_\Omega P((N\cdot\nabla)E)Aw=\int_\Omega(Pf-Pf_1)Aw.\end{aligned} \tag{3.4}$$

Multiplying $AE(t)$ on the both sides of (3.3), and integrating on Ω, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla E\|^2+\|AE\|^2+\int_\Omega P((w\cdot\nabla)E)\cdot AE+\int_\Omega P((w\cdot\nabla)N)\cdot AE\\&+\int_\Omega P((v\cdot\nabla)E)\cdot AE-\int_\Omega P((E\cdot\nabla)w)\cdot AE-\int_\Omega P((E\cdot\nabla)v)\cdot AE\\&-\int_\Omega P((N\cdot\nabla)w)\cdot AE=0.\end{aligned}\tag{3.5}$$

Adding (3.4) to (3.5), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w\|^2+\|\nabla E\|^2)+(\|Aw\|^2+\|AE\|^2)\\&+\underbrace{\int_\Omega P((w\cdot\nabla)w)\cdot Aw}_{I_1}+\underbrace{\int_\Omega P((w\cdot\nabla)v)\cdot Aw}_{I_2}+\underbrace{\int_\Omega P((v\cdot\nabla)w)\cdot Aw}_{I_3}\\&-\underbrace{\int_\Omega P((E\cdot\nabla)E)\cdot Aw}_{I_4}-\underbrace{\int_\Omega P((E\cdot\nabla)N)\cdot Aw}_{I_5}-\underbrace{\int_\Omega P((N\cdot\nabla)E)\cdot Aw}_{I_6}\\&-\underbrace{\int_\Omega P((w\cdot\nabla)E)\cdot AE}_{I_7}-\underbrace{\int_\Omega P((w\cdot\nabla)N)\cdot AE}_{I_8}-\underbrace{\int_\Omega P((v\cdot\nabla)E)\cdot AE}_{I_9}\\&-\underbrace{\int_\Omega P((E\cdot\nabla)w)\cdot AE}_{I_{10}}-\underbrace{\int_\Omega P((E\cdot\nabla)v)\cdot AE}_{I_{11}}-\underbrace{\int_\Omega P((N\cdot\nabla)w)\cdot AE}_{I_{12}}\\=&\underbrace{\int_\Omega(Pf-Pf_1)Aw}_{I_{13}}.\end{aligned}\tag{3.6}$$

In order to estimate I_1-I_{13}, we need to collect a few interpolation inequalities for functions $g\in D(A)$. First, under the assumption that $\partial\Omega$ is uniformly C^3, it was shown in Lemma 1 of [9] that $\|\partial^2_{ij}g\|\leq C(\|Ag\|+\|\nabla g\|)$. Since $D(A)\subset PL^2(\Omega)$, we may integrate by parts and use Cauchy-Schwarz inequality to get

$$\|\nabla g\|^2=-\int_\Omega\triangle gg=-\int_\Omega P\triangle gg\leq\|Ag\|\|g\|\leq\|Ag\|^2+\|g\|^2.\tag{3.7}$$

Therefore, in general, we have the following inequality

$$\|\partial^2_{ij}g\|\leq C(\|Ag\|+\|g\|).\tag{3.8}$$

Now in the case where (1.3) holds, it is clear from the argument of (3.7) that

$$\|g\|+\|\nabla g\|\leq C\|Ag\|.\tag{3.9}$$

So (3.8) can be improved when Ω satisfies (1.3) to

$$\|\partial^2_{ij}g\|\leq C\|Ag\|.\tag{3.10}$$

Using Fourier analysis, we easily know this inequality is true for R^3 as well. If $g\in D(A)$, one has [12]

$$\|g\|_{L^6}\leq C\|\nabla g\|,\tag{3.11}$$

which holds for any $\Omega \subset R^3$. For the domain satisfying (1.3), using Gagliardo-Nirenberg inequality, (3.10) and (3.11), we obtain

$$\|g\|_{L^\infty} \le C\|g\|_{L^6}^{\frac{1}{2}}\|\partial_{ij}^2 g\|^{\frac{1}{2}} \le C\|g\|_{L^6}^{\frac{1}{2}}\|Ag\|^{\frac{1}{2}} \le C\|\nabla g\|^{\frac{1}{2}}\|Ag\|^{\frac{1}{2}}. \tag{3.12}$$

Thus, using Calderon extension theorem [10] and then (3.8), we obtain for all of the domain under consideration the following inequality is true

$$\|g\|_{L^\infty} \le C\|g\|_{L^6}^{\frac{1}{2}}\|g\|_{H^2}^{\frac{1}{2}} \le C\|g\|_{L^6}^{\frac{1}{2}}(\|Ag\|^{\frac{1}{2}} + \|g\|^{\frac{1}{2}}) \le C\|\nabla g\|^{\frac{1}{2}}(\|Ag\|^{\frac{1}{2}} + \|g\|^{\frac{1}{2}}). \tag{3.13}$$

Moreover, in the case $\Omega = R^3$, we obtain from Gagliardo-Nirenberg inequality and (3.10)

$$\|\nabla g\|_{L^3} \le C\|\nabla g\|^{\frac{1}{2}}\|\partial_{ij}^2 g\|^{\frac{1}{2}} \le C\|\nabla g\|^{\frac{1}{2}}\|Ag\|^{\frac{1}{2}}. \tag{3.14}$$

In the general case, we can apply the Calderon extension theorem and (3.8) to get

$$\|\nabla g\|_{L^3} \le C\|\nabla g\|^{\frac{1}{2}}(\|Ag\|^{\frac{1}{2}} + \|g\|^{\frac{1}{2}}), \tag{3.15}$$

which, exactly as above, improves to (3.14) in the case (1.3) is available.

Now we first consider case (i) where (1.3) is valid. Hence Ω is such that (3.12) and (3.14) hold. $I_1 - I_{13}$ are estimated as follows:

$$\begin{aligned}
|I_1| &= \left|\int_\Omega (w\cdot\nabla)w\cdot Aw\right| \le C\|w\|_{L^6}\|\nabla w\|_{L^3}\|Aw\| \\
&\le C\|w\|_{L^6}(\|\nabla w\|^{\frac{1}{2}}\|Aw\|^{\frac{1}{2}})\|Aw\| \le C\|\nabla w\|^{\frac{3}{2}}\|Aw\|^{\frac{3}{2}} \\
&\le C_\varepsilon\|\nabla w\|^6 + \epsilon\|Aw\|^2 \quad (\varepsilon > 0 \ \text{ arbitrary}),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
|I_2| &= \left|\int_\Omega (w\cdot\nabla)v\cdot Aw\right| \le C\|w\|_{L^\infty}\|\nabla v\|\|Aw\| \\
&\le C\|\nabla v\|\|\nabla w\|^{\frac{1}{2}}\|Aw\|^{\frac{3}{2}} \le C_\epsilon\|\nabla v\|^4\|\nabla w\|^2 + \varepsilon\|Aw\|^2,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
|I_3| &= \left|\int_\Omega (v\cdot\nabla)w\cdot Aw\right| \le C\|v\|_{L^6}\|\nabla w\|_{L^3}\|Aw\| \\
&\le C\|\nabla v\|\|\nabla w\|^{\frac{1}{2}}\|Aw\|^{\frac{3}{2}} \le C_\varepsilon\|\nabla v\|^4\|\nabla w\|^2 + \varepsilon\|Aw\|^2,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
|I_4| &= \left|\int_\Omega (E\cdot\nabla)E\cdot Aw\right| \le C\|E\|_{L^\infty}\|\nabla E\|\|AE\| \\
&\le C\|\nabla E\|^{\frac{3}{2}}\|AE\|^{\frac{1}{2}}\|Aw\| \le C_\varepsilon\|\nabla E\|^3\||AE| + \varepsilon\||Aw|^2 \\
&\le C_\varepsilon\|\nabla E\|^6 + \frac{1}{4}\|AE\|^2 + \varepsilon\|Aw\|^2,
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
|I_5| &= \left|\int_\Omega (E\cdot\nabla)N\cdot Aw\right| \le C\|E\|_{L^\infty}\|\nabla N\|\|Aw\| \\
&\le C\|\nabla E\|^{\frac{1}{2}}\|AE\|^{\frac{1}{2}}\|\nabla N\|\|Aw\| \le C_\varepsilon\|\nabla N\|^2\|\nabla E\|\|AE\| + \varepsilon\||Aw|^2 \\
&\le C_\varepsilon\|\nabla N\|^4\|\nabla E\|^2 + \frac{1}{4}\|AE\|^2 + \varepsilon\|Aw\|^2,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
|I_6| &= \left|\int_\Omega (N\cdot\nabla E)Aw\right| \le C\|N\|_{L^6}\|\nabla E\|_{L^3}\|Aw\| \\
&\le C\|\nabla N\|\|\nabla E\|^{\frac{1}{2}}\|AE\|^{\frac{1}{2}}\|Aw\| \le C_\varepsilon\|\nabla N\|^2\|\nabla E\|\|AE\| + \varepsilon\|Aw\|^2
\end{aligned}$$

$$\leq C_\varepsilon\|\nabla N\|^4\|\nabla E\|^2+\frac{1}{4}\|AE\|^2+\varepsilon\|Aw\|^2, \tag{3.21}$$

$$\begin{aligned}|I_7|&=\left|\int_\Omega(w\cdot\nabla E)AE\right|\leq C\|w\|_{L^6}\|\nabla E\|_{L^3}\|AE\|\\&\leq C\|\nabla w\|\|\nabla E\|^{\frac12}\|AE\|^{\frac32}\leq C_\varepsilon\|\nabla w\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2,\end{aligned} \tag{3.22}$$

$$\begin{aligned}|I_8|&=\left|\int_\Omega(w\cdot\nabla N)AE\right|\leq C\|w\|_{L^\infty}\|\nabla N\|\|AE\|\\&\leq C\|\nabla w\|^{\frac12}\|Aw\|^{\frac12}\|\nabla N\|\|AE\|\leq C_\varepsilon\|\nabla N\|^2\|\nabla w\|\|Aw\|+\varepsilon\|AE\|^2\\&\leq C_\varepsilon\|\nabla v\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2,\end{aligned} \tag{3.23}$$

$$\begin{aligned}|I_9|&=\left|\int_\Omega(v\cdot\nabla)E\cdot AE\right|\leq C\|v\|_{L^6}\|\nabla E\|_{L^3}\|AE\|\\&\leq C\|\nabla v\|\|\nabla w\|^{\frac12}\|AE\|^{\frac32}\leq C_\varepsilon\|\nabla v\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2,\end{aligned} \tag{3.24}$$

$$\begin{aligned}|I_{10}|&=\left|\int_\Omega(E\cdot\nabla)w\cdot AE\right|\leq C\|E\|_{L^6}\|\nabla w\|_{L^3}\|AE\|\\&\leq C\|\nabla E\|\|\nabla w\|^{\frac12}\|AE\|^{\frac12}\|AE\|\leq C_\varepsilon\|\nabla E\|^2\|\nabla w\|\|Aw\|+\varepsilon\|AE\|^2\\&\leq C_\varepsilon\|\nabla E\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2,\end{aligned} \tag{3.25}$$

$$\begin{aligned}|I_{11}|&=\left|\int_\Omega(E\cdot\nabla)v\cdot AE\right|\leq C\|E\|_{L^\infty}\|\nabla v\|\|AE\|\\&\leq C\|\nabla v\|\|\nabla E\|^{\frac12}\|AE\|^{\frac32}\leq C_\epsilon\|\nabla v\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2,\end{aligned} \tag{3.26}$$

$$\begin{aligned}|I_{12}|&=\left|\int_\Omega(N\cdot\nabla)w\cdot AE\right|\leq C\|N\|_{L^6}\|\nabla w\|_{L^3}\|AE\|\\&\leq C\|\nabla N\|\|\nabla w\|^{\frac12}\|Aw\|^{\frac12}\|AE\|\leq C_\varepsilon\|\nabla N\|^2\|\nabla w\|\|Aw\|+\varepsilon\|AE\|^2\\&\leq C_\varepsilon\|\nabla N\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2,\end{aligned} \tag{3.27}$$

$$|I_{13}|=\left|\int_\Omega(Pf-Pf_1))\cdot Aw\right|\leq C_\varepsilon\|Pf-Pf_1\|^2+\varepsilon\|Aw\|^2. \tag{3.28}$$

Choosing ε sufficiently small (for example $\varepsilon=0.01$), substituting (3.16)–(3.28) into (3.6), we obtain

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}(\|\nabla w\|^2+\|\nabla E\|^2)+\tilde C_0(\|Aw\|^2+\|AE\|^2)\\&\leq C[\|\nabla w\|^6+\|\nabla v\|^4\|\nabla w\|^2+\|\nabla E\|^6+\|\nabla N\|^4\|\nabla E\|^2+\|\nabla N\|^4\|\nabla w\|^2\\&\quad+\|\nabla v\|^4\|\nabla E\|^2+\|\nabla E\|^4\|\nabla w\|^2+\|Pf-Pf_1\|^2]\\&\leq C[(\|\nabla w\|^2+\|\nabla E\|^2)^3+(\|\nabla v\|^4+\|\nabla E\|^4)\cdot(\|\nabla w\|^2+\|\nabla E\|^2)+\|Pf-Pf_1\|^2].\end{aligned} \tag{3.29}$$

Defining $h(t)=:\|\nabla w(t)\|^2+\|\nabla E(t)\|^2$, we have proved with help of (3.6), (3.16)–(3.28) that

$$h'(t)+C_0h(t)\leq C[h^3(t)+(\|\nabla v(t)\|^4+\|\nabla N(t)\|^4)h(t)+\|Pf(t)-Pf_1(t)\|^2] \tag{3.30}$$

holds with positive constants C, C_0. Let

$$\lambda:=C\sup_{s\geq0}\exp\left(-C_0\frac{s}{2}\right)\int_0^s\exp\left(C_0\frac{\tau}{2}\right)\|Pf(\tau)-Pf_1(\tau)\|^2\mathrm d\tau.$$

Note that

$$\lambda \le C\int_0^\infty \|Pf(\tau)-Pf_1(\tau)\|^2\mathrm{d}\tau$$

holds. If now

$$\|\nabla w_0\|^2+\|\nabla E_0\|^2+\lambda \le \frac{1}{2\max\{1,\exp(C\int_0^\infty \|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4\mathrm{d}\tau)\}}\Big(\frac{C_0}{2C}\Big)^{\frac12}\le \delta \quad (3.31)$$

is true, then we infer that

$$\|\nabla w(s)\|^2+\|\nabla E(s)\|^2 \le \Big(\frac{C_0}{2C}\Big)^{\frac12} \quad (3.32)$$

holds for $0\le s\le t_1$, and some $t_1>0$. Thus, (3.30) implies that for $s\le t_1$, the following inequality holds

$$h'(s)+\frac{C_0}{2}h(s)\le C[(\|\nabla v(s)\|^4+\|\nabla N(s)\|^4)h(s)+\|Pf(s)-Pf_1(s)\|^2]. \quad (3.33)$$

Consequently, by a generalized Gronwall inequality, we obtain

$$h(s)\le \exp\Big(-\frac{C_0 s}{2}\Big)\Big(h(0)+C\int_0^s\|Pf-Pf_1\|^2\Big)\exp\Big(C\int_0^\infty \|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4\mathrm{d}\tau\Big) \quad (3.34)$$

for any $0\le s\le t_1$. Furthermore, by (3.31) we deduce that $h(s)\le \frac12\big(\frac{C_0}{2C}\big)^{\frac12}$ is true for all $0\le s\le t_1$. Thus (3.32) holds on the domain of definition of $h(t)$, so $\|\nabla w(t)\|+\|\nabla E(t)\|$ is uniformly bounded in the interval where solutions exist.

On the other hand, the equations with the solution (v,N) as follows:

$$\begin{cases} v_t+Av+P((v\cdot\bigtriangledown)v)-P((N\cdot\bigtriangledown)N)=Pf_1,\ (x,t)\in\Omega\times(0,\infty),\\ N_t+AN+P((v\cdot\bigtriangledown)N)-P((N\cdot\bigtriangledown)v)=0,\quad (x,t)\in\Omega\times(0,\infty),\\ v=Pv, N=PN,\\ N|_{t=0}=N_0,\qquad v|_{t=0}=v_0,\\ N|_{\partial\Omega}=0,\qquad v|_{\partial\Omega}=0. \end{cases} \quad (3.35)$$

Multiplying Av on the both sides of the first equation of (3.35), integrating on Ω, we obtain

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2+\|Av\|^2+\int_\Omega P((v\cdot\nabla)v)\cdot Av-\int_\Omega P((N\cdot\nabla)N)\cdot Av=\int_\Omega Pf_1\cdot Av. \quad (3.36)$$

Multiplying AN on the both sides of the second equation of (3.35), integrating on Ω, we get

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla N\|^2+\|AN\|^2+\int_\Omega P((v\cdot\nabla)N)\cdot AN-\int_\Omega P((N\cdot\nabla)v)\cdot AN=0. \quad (3.37)$$

Adding (3.36) to (3.37) leads to

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla v\|^2+\|\nabla N\|^2)+\|Av\|^2+\|AN\|^2+\underbrace{\int_\Omega P((v\cdot\nabla)v)\cdot Av}_{J_1}-\underbrace{\int_\Omega P((N\cdot\nabla)N)\cdot Av}_{J_2}$$

$$-\underbrace{\int_\Omega P((v\cdot\nabla)N)\cdot AN}_{J_3}-\underbrace{\int_\Omega P((N\cdot\nabla)v)\cdot AN}_{J_4}=\underbrace{\int_\Omega Pf_1\cdot Av}_{J_5}.$$

$J_1 - J_3$ are estimated as follows:

$$|J_1| = \left|\int_\Omega P((v\cdot\nabla)v)\cdot Av\right| = \left|\int_\Omega (v\cdot\nabla)v\cdot Av\right|$$
$$\le C\|v\|_{L^6}\|\nabla v\|_{L^3}\|Av\| \le C\|\nabla v\|^{\frac{3}{2}}\|Av\|^{\frac{3}{2}} \le C_\varepsilon\|\nabla v\|^6 + \varepsilon\|Av\|^2,$$

$$|J_2| = \left|\int_\Omega P((N\cdot\nabla)N)\cdot Av\right| \le C\|N\|_{L^6}\|\nabla N\|_{L^3}\|Av\|$$
$$\le C\|\nabla N\|^{\frac{3}{2}}\|AN\|^{\frac{1}{2}}\|Av\| \le \ C_\varepsilon\|\nabla N\|^3\|AN\| + \varepsilon\|Av\|^2$$
$$\le C_\varepsilon\|\nabla N\|^6 + \frac{1}{4}\|AN\|^2 + \varepsilon\|Av\|^2,$$

$$|J_3| = \left|\int_\Omega P((v\cdot\nabla)N)\cdot AN\right| = \left|\int_\Omega (v\cdot\nabla)N\cdot AN\right|$$
$$\le C\|v\|_{L^6}\|\nabla N\|_{L^3}\|AN\| \le C\|\nabla v\|\|\nabla N\|^{\frac{1}{2}}\|AN\|^{\frac{3}{2}}$$
$$\le C_\varepsilon\|\nabla v\|^4\|\nabla N\|^2 + \varepsilon\|AN\|^2,$$

$$|J_4| = \left|\int_\Omega P((N\cdot\nabla)v)\cdot AN\right| = \left|\int_\Omega (N\cdot\nabla)v\cdot AN\right|$$
$$\le C\|N\|_{L^6}\|\nabla v\|_{L^3}\|AN\| \le C\|\nabla N\|\|\nabla v\|^{\frac{1}{2}}\|Av\|^{\frac{1}{2}}\|AN\|$$
$$\le C_\varepsilon\|\nabla N\|^2\|\nabla v\|\|Av\| + \varepsilon\|AN\|^2 \le C_\varepsilon\|\nabla N\|^4\|\nabla v\|^2 + \varepsilon\|AN\|^2,$$

$$|J_5| = \left|\int_\Omega Pf_1\cdot Av\right| \le C_\varepsilon\|Pf_1\|^2 + \varepsilon\|Av\|^2.$$

Choosing ε sufficiently small (for example $\varepsilon = 0.01$), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla v\|^2 + \|\nabla N\|^2) + \tilde{C_0}(\|Av\|^2 + \|AN\|^2)$$
$$\le C(\|\nabla v\|^4 + \|\nabla N\|^4)(\|\nabla v\|^2 + \|\nabla N\|^2) + C\|Pf_1\|^2. \tag{3.38}$$

By Gronwall inequality, we have

$$\|\nabla v\|^2 + \|\nabla N\|^2$$
$$\le \exp\left(C\int_0^t \|\nabla v(\tau)\|^4 + \|\nabla N(\tau)\|^4 \mathrm{d}\tau\right)\left(\|\nabla v_0\|^2 + \|\nabla N_0\|^2 + C\int_0^\infty \|Pf_1\|^2\mathrm{d}\tau\right). \tag{3.39}$$

Therefore, $\|\nabla v(t)\| + \|\nabla N(t)\|$ is uniformly bounded. Since

$$\|\nabla u(t)\| + \|\nabla B(t)\| = \|\nabla u(t) - \nabla v(t) + \nabla v(t)\| + \|\nabla B(t) - \nabla N(t) + \nabla N(t)\|$$
$$\le \|\nabla w(t)\| + \|\nabla E(t)\| + \|\nabla v(t)\| + \|\nabla N(t)\|, \tag{3.40}$$

it concludes that $\|\nabla u(t)\| + \|\nabla B(t)\|$ is uniformly bounded on the domain of definition. Thanks to continuity method, the solution (u, B) of (1.1) exists globally on $(0, \infty)$ for t. The remaining statements in Theorem 1, (i) can be deduced by estimates (3.34), (3.39) and (3.40). Note that δ in (2.2) is chosen according to (3.31).

Now, we turn to the general case (ii), where the domain Ω satisfy (3.14), (3.12). Multiplying w on the both sides of (3.2), integrating on Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 + \|\nabla w\|^2 + \int_\Omega P((w\cdot\nabla)w)\cdot w + \int_\Omega P((w\cdot\nabla)v)\cdot w + \int_\Omega P((v\cdot\nabla)w)\cdot w$$

$$
\begin{aligned}
&-\int_\Omega P((E\cdot\nabla)E)\cdot w-\int_\Omega P((E\cdot\nabla)N)\cdot w-\int_\Omega P((N\cdot\nabla)E)\cdot w\\
&=\int_\Omega (Pf-pf_1)\cdot w,
\end{aligned}\tag{3.41}
$$

where the following equality holds

$$\int_\Omega P((w\cdot\nabla)w)\cdot w=\int_\Omega P((v\cdot\nabla)w)\cdot w=0.\tag{3.42}$$

Multiplying w on the both sides of (3.3), integrating on Ω, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|E\|^2+\|\nabla E\|^2+\int_\Omega P((w\cdot\nabla)E)\cdot E+\int_\Omega P((w\cdot\nabla)N)\cdot E\\
&+\int_\Omega P((v\cdot\nabla)E)\cdot E-\int_\Omega P((E\cdot\nabla)w)\cdot E-\int_\Omega P((E\cdot\nabla)v)\cdot E\\
&-\int_\Omega P((N\cdot\nabla)w)\cdot E=0,
\end{aligned}\tag{3.43}
$$

where the following equality holds

$$\int_\Omega P((w\cdot\nabla)E)\cdot E=\int_\Omega P((v\cdot\nabla)E)\cdot E=0.\tag{3.44}$$

Since

$$
\begin{aligned}
&(E\cdot\nabla)w\cdot w+(E\cdot\nabla)w\cdot E+(E\cdot\nabla)E\cdot w+(E\cdot\nabla)E\cdot E=(E\cdot\nabla)(w+E)\cdot(w+E),\\
&(N\cdot\nabla)w\cdot w+(N\cdot\nabla)w\cdot E+(N\cdot\nabla)E\cdot w+(N\cdot\nabla)E\cdot E=(N\cdot\nabla)(w+E)\cdot(w+E).
\end{aligned}
$$

Similar to (3.42) and (3.44), we can obtain

$$
\begin{aligned}
&\int_\Omega P((E\cdot\nabla)w)\cdot E+\int_\Omega P((E\cdot\nabla)E)\cdot w=0,\\
&\int_\Omega P((N\cdot\nabla)w)\cdot E+\int_\Omega P((N\cdot\nabla)E)\cdot w=0.
\end{aligned}
$$

Adding (3.41) to (3.43), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2+\|E\|^2)+\|\nabla w\|^2+\|\nabla E\|^2\\
&+\underbrace{\int_\Omega P((w\cdot\nabla)v)\cdot w}_{K_1}+\underbrace{\int_\Omega P((w\cdot\nabla)N)\cdot E}_{K_2}-\underbrace{\int_\Omega P((E\cdot\nabla)N)\cdot w}_{K_3}-\underbrace{\int_\Omega P((E\cdot\nabla)v)\cdot E}_{K_4}\\
&=\underbrace{\int_\Omega (pf-Pf_1))\cdot w}_{K_5}.
\end{aligned}\tag{3.45}
$$

K_1-K_5 are estimated as follows:

$$
\begin{aligned}
|K_1|&=\left|\int_\Omega (w\cdot\nabla)v\cdot w\right|\le C\|w\|_{L^4}^2\|\nabla v\|\le C\|w\|^{\frac{1}{2}}\|w\|_{L^6}^{\frac{3}{2}}\|\nabla v\|\\
&\le C\|w\|^{\frac{1}{2}}\|\nabla w\|^{\frac{3}{2}}\|\nabla v\|\le C_\varepsilon\|\nabla v\|^4\|w\|^2+\varepsilon\|\nabla w\|^2,
\end{aligned}\tag{3.46}
$$

$$\begin{aligned}|K_2| &= \left|\int_\Omega (w\cdot\nabla)N\cdot E\right| \le C\|w\|_{L^4}\|\nabla N\|\|E\|_{L^4}\\ &\le C\|w\|^{\frac14}\|\nabla w\|^{\frac34}\|\nabla N\|\|E\|^{\frac14}\|\nabla E\|^{\frac34}\\ &\le C\|w\|^{\frac12}\|\nabla w\|^{\frac32}\|\nabla N\| + C\|E\|^{\frac12}\|\nabla E\|^{\frac32}\|\nabla N\|\\ &\le C_\varepsilon\|\nabla N\|^4\|w\|^2 + \varepsilon\|\nabla w\|^2 + C_\varepsilon\|\nabla N\|^4\|E\|^2 + \varepsilon\|\nabla E\|^2, \end{aligned} \tag{3.47}$$

$$\begin{aligned}|K_3| &= \left|\int_\Omega (E\cdot\nabla)N\cdot w\right|\\ &\le C_\varepsilon\|\nabla N\|^4\|w\|^2 + \varepsilon\|\nabla w\|^2 + C_\varepsilon\|\nabla N\|^4\|E\|^2 + \varepsilon\|\nabla E\|^2, \end{aligned} \tag{3.48}$$

$$|K_4| = \left|\int_\Omega (w\cdot\nabla)v\cdot w\right| \le C\|w\|^{\frac12}\|\nabla w\|^{\frac32}\|\nabla v\| \le C_\varepsilon\|\nabla v\|^4\|w\|^2 + \varepsilon\|\nabla w\|^2, \tag{3.49}$$

$$|K_5| = \left|\int_\Omega (Pf - Pf_1))\cdot w\right| \le C\|Pf - Pf_1\|\|w\|. \tag{3.50}$$

Choosing ε sufficiently small (for example $\varepsilon = 0.01$), substituting (3.47)–(3.51) into (3.46), we obtain

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}(\|w\|^2 + \|E\|^2) + C_0(\|\nabla w\|^2 + \|\nabla E\|^2)\\ \le\ & C(\|\nabla v\|^4 + \|\nabla N\|^4)(\|w\|^2 + \|E\|^2) + \|f - f_1\|_{L^\infty}\sqrt{\|w\|^2 + \|E\|^2}. \end{aligned} \tag{3.51}$$

Consequently,

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}\sqrt{\|w(t)\|^2 + \|E(t)\|^2}\\ \le\ & C(\|\nabla v(t)\|^4 + \|\nabla N(t)\|^4)\sqrt{\|w(t)\|^2 + \|E(t)\|^2} + \|f(t) - f_1(t)\|_{L^\infty}. \end{aligned} \tag{3.52}$$

By Gronwall inequality, we obtain

$$\begin{aligned}\sqrt{\|w(t)\|^2 + \|E(t)\|^2} &\le \exp\left(C\int_0^\infty \|\nabla v(\tau)\|^4 + \|\nabla N(\tau)\|^4\mathrm d\tau\right)\\ &\quad\times\left(\|w_0\| + \|E_0\| + \int_0^\infty \|f(\tau) - f_1(\tau)\|_{L^\infty}\mathrm d\tau\right) \end{aligned} \tag{3.53}$$

for all $t > 0$.

Concerning (3.6), when Ω is the general domain, $I_1 - I_{13}$ can be estimated as follows:

$$\begin{aligned}|I_1| &= \left|\int_\Omega P((w\cdot\nabla)w)\cdot Aw\right| = \left|\int_\Omega (w\cdot\nabla)w\cdot Aw\right|\\ &\le C\|w\|_{L^6}[\|\nabla w\|^{\frac12}(\|Aw\|^{\frac12} + \|w\|^{\frac12})]\|Aw\|\\ &\le C_\varepsilon\|\nabla w\|^6 + \varepsilon\|Aw\|^2 + C_\varepsilon\|\nabla w\|^3\|w\| + \varepsilon\|Aw\|^2\\ &\le C_\varepsilon\|\nabla w\|^6 + \frac{C_\varepsilon}{2}(\|\nabla w\|^6 + \|w\|^2) + 2\varepsilon\|Aw\|^2\\ &\le C(\|\nabla w\|^6 + \|w\|^2) + 2\varepsilon\|Aw\|^2, \end{aligned} \tag{3.54}$$

$$\begin{aligned}|I_2| &= \left|\int_\Omega P((w\cdot\nabla)v)\cdot Aw\right| = \left|\int_\Omega (w\cdot\nabla)v\cdot Aw\right|\\ &\le C\|w\|_{L^\infty}\|\nabla v\|\|Aw\| \le C\|\nabla v\|[\|\nabla w\|^{\frac12}(\|Aw\|^{\frac12} + \|w\|^{\frac12})]\|Aw\| \end{aligned}$$

$$
\begin{aligned}
&\le C_\varepsilon\|\nabla v\|^4\|\nabla w\|^2+\varepsilon\|Aw\|^2+C_\varepsilon\|\nabla v\|^2\|\nabla w\|\|w\|+\varepsilon\|Aw\|^2\\
&\le C_\varepsilon\|\nabla v\|^4\|\nabla w\|^2+\frac{C_\varepsilon}{2}(\|\nabla v\|^4\|\nabla w\|^2+\|w\|^2)+2\varepsilon\|Aw\|^2\\
&\le C(\|\nabla v\|^4\|\nabla w\|^2+\|w\|^2)+2\varepsilon\|Aw\|^2, \qquad (3.55)
\end{aligned}
$$

$$
\begin{aligned}
|I_3|&=\left|\int_\Omega P((v\cdot\nabla)w)\cdot Aw\right|=\left|\int_\Omega (v\cdot\nabla)w\cdot Aw\right|\\
&\le \|v\|_{L^6}\|\nabla w\|_{L^3}\|Aw\|\le C\|\nabla v\|[\|\nabla w\|^{\frac12}(\|Aw\|^{\frac12}+\|w\|^{\frac12})]\|Aw\|\\
&\le C(\|\nabla v\|^4\|\nabla w\|^2+\|w\|^2)+2\varepsilon\|Aw\|^2, \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
|I_4|&=\left|\int_\Omega (E\cdot\nabla)E\cdot Aw\right|\le C\|E\|_{L^\infty}\|\nabla E\|\|Aw\|\\
&\le C[\|\nabla E\|^{\frac12}(\|AE\|^{\frac12}+\|E\|^{\frac12})]\|\nabla E\|\|Aw\|\\
&= C\|\nabla E\|^{\frac32}\|AE\|^{\frac12}\|Aw\|+C\|\nabla E\|^{\frac32}\|E\|^{\frac12}\|Aw\|\\
&\le C_\varepsilon\|\nabla E\|^6+\frac14\|AE\|^2+\varepsilon\|Aw\|^2+\frac{C_\varepsilon}{2}(\|\nabla E\|^6+\|E\|^2)+\varepsilon\|Aw\|^2\\
&\le C\|\nabla E\|^6+\frac14\|AE\|^2+2\varepsilon|Aw\|^2, \qquad (3.57)
\end{aligned}
$$

$$
\begin{aligned}
|I_5|&=\left|\int_\Omega (E\cdot\nabla)N\cdot Aw\right|\le C\|E\|_{L^\infty}\|\nabla N\|\|Aw\|\\
&\le C[\|\nabla E\|^{\frac12}(\|AE\|^{\frac12}+\|E\|^{\frac12})]\|\nabla N\|\|Aw\|\\
&= C\|\nabla E\|^{\frac12}\|AE\|^{\frac12}\|\nabla N\|\|Aw\|+C\|\nabla E\|^{\frac12}\|E\|^{\frac12}\|\nabla N\|\|Aw\|\\
&\le C_\varepsilon\|\nabla N\|^2\|\nabla E\|\|AE\|+\varepsilon\|Aw\|^2+C_\varepsilon\|\nabla N\|^2\|\nabla E\|\|E\|+\varepsilon\|Aw\|^2\\
&\le C_\varepsilon\|\nabla N\|^4\|\nabla E\|^2+\frac14\|AE\|^2+\varepsilon\|Aw\|^2+\frac{C_\varepsilon}{2}(\|\nabla N\|^4\|\nabla E\|^2+\|E\|^2)+\varepsilon\|Aw\|^2\\
&\le C\|\nabla N\|^4\|\nabla E\|^2+\frac14\|AE\|^2+C\|E\|^2+2\varepsilon\|Aw\|^2, \qquad (3.58)
\end{aligned}
$$

$$
\begin{aligned}
|I_6|&=\left|\int_\Omega P((N\cdot\nabla)E)\cdot Aw\right|=\left|\int_\Omega (N\cdot\nabla)E\cdot Aw\right|\\
&\le C\|N\|_{L^6}[\|\nabla E\|^{\frac12}(\|AE\|^{\frac12}+\|E\|^{\frac12})]\|Aw\|\\
&\le C_\varepsilon\|\nabla N\|^2\|\nabla E\|\|AE\|+\varepsilon\|Aw\|^2+C_\varepsilon\|\nabla N\|^2\|\nabla E\|\|E\|+\varepsilon\|Aw\|^2\\
&\le C_\varepsilon\|\nabla N\|^4\|\nabla E\|^2+\frac14\|AE\|^2+\varepsilon\|Aw\|^2+\frac{C_\varepsilon}{2}(\|\nabla N\|^4\|\nabla E\|^2+\|E\|^2)+\varepsilon\|Aw\|^2\\
&\le C(\|\nabla N\|^4\|\nabla E\|^2+\|E\|^2)+\frac14\|AE\|^2+2\varepsilon\|Aw\|^2, \qquad (3.59)
\end{aligned}
$$

$$
\begin{aligned}
|I_7|&=\left|\int_\Omega P((w\cdot\nabla)E)\cdot AE\right|=\left|\int_\Omega (w\cdot\nabla)E\cdot AE\right|\\
&\le C\|w\|_{L^6}[\|\nabla E\|^{\frac12}(\|AE\|^{\frac12}+\|E\|^{\frac12})]\|AE\|\\
&\le C_\varepsilon\|\nabla w\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2+C_\varepsilon\|\nabla w\|^2\|\nabla E\|\|E\|+\varepsilon\|AE\|^2\\
&\le C_\varepsilon\|\nabla w\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2+\frac{C_\varepsilon}{2}(\|\nabla w\|^4\|\nabla E\|^2+\|E\|^2)+\varepsilon\|AE\|^2\\
&\le C(\|\nabla w\|^4\|\nabla E\|^2+\|E\|^2)+2\varepsilon\|AE\|^2, \qquad (3.60)
\end{aligned}
$$

$$
\begin{aligned}
|I_8| &= \left|\int_\Omega (w\cdot\nabla)N\cdot AE\right| \le C\|w\|_{L^\infty}\|\nabla N\|\|AE\| \\
&\le C[\|\nabla w\|^{\frac12}(\|Aw\|^{\frac12}+\|w\|^{\frac12})]\|\nabla\theta_1\|\|\triangle E\| \\
&\le C_\varepsilon\|\nabla N\|^2\|\nabla w\|\|Aw\|+\varepsilon\|AE\|^2+C_\varepsilon\|\nabla N\|^2\|\nabla w\|\|w\|+\varepsilon\|AE\|^2 \\
&\le C_\varepsilon\|\nabla N\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\frac{C_\varepsilon}{2}(\|\nabla N\|^4\|\nabla w\|^2+\|w\|^2)+2\varepsilon\|AE\|^2 \\
&\le C(\|\nabla N\|^4\|\nabla w\|^2+\|w\|^2)+\frac14\|Aw\|^2+2\varepsilon\|AE\|^2, \qquad (3.61)
\end{aligned}
$$

$$
\begin{aligned}
|I_9| &= \left|\int_\Omega P((v\cdot\nabla)E)\cdot AE\right| = \left|\int_\Omega (v\cdot\nabla)E\cdot AE\right| \\
&\le C\|v\|_{L^6}[\|\nabla E\|^{\frac12}(\|AE\|^{\frac12}+\|E\|^{\frac12})]\|AE\| \\
&\le C_\varepsilon\|\nabla v\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2+C_\varepsilon\|\nabla v\|^2\|\nabla E\|\|E\|+\varepsilon\|AE\|^2 \\
&\le C_\varepsilon\|\nabla v\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2+\frac{C_\varepsilon}{2}(\|\nabla v\|^4\|\nabla E\|^2+\|E\|^2)+\varepsilon\|AE\|^2 \\
&\le C(\|\nabla v\|^4\|\nabla E\|^2+\|E\|^2)+2\varepsilon\|AE\|^2, \qquad (3.62)
\end{aligned}
$$

$$
\begin{aligned}
|I_{10}| &= \left|\int_\Omega P((E\cdot\nabla)w)\cdot AE\right| = \left|\int_\Omega (E\cdot\nabla)w\cdot AE\right| \\
&\le C\|E\|_{L^6}[\|\nabla w\|^{\frac12}(\|Aw\|^{\frac12}+\|w\|^{\frac12})]\|AE\| \\
&\le C_\varepsilon\|\nabla E\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2+C_\varepsilon\|\nabla E\|^2\|\nabla w\|\|w\|+\varepsilon\|AE\|^2 \\
&\le C_\varepsilon\|\nabla E\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2+\frac{C_\varepsilon}{2}(\|\nabla E\|^4\|\nabla w\|^2+\|w\|^2)+\varepsilon\|AE\|^2 \\
&\le C(\|\nabla E\|^4\|\nabla w\|^2+\|w\|^2)+\frac14\|Aw\|^2+2\varepsilon\|AE\|^2, \qquad (3.63)
\end{aligned}
$$

$$
\begin{aligned}
|I_{11}| &= \left|\int_\Omega P((E\cdot\nabla)v)\cdot AE\right| = \left|\int_\Omega (E\cdot\nabla)v\cdot AE\right| \\
&\le C[\|\nabla E\|^{\frac12}(\|AE\|^{\frac12}+\|w\|^{\frac12})]\cdot\|\nabla v\|\|AE\| \\
&\le C_\varepsilon\|\nabla v\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2+C_\varepsilon\|\nabla v\|^2\|\nabla E\|\|E\|+\varepsilon\|AE\|^2 \\
&\le C_\varepsilon\|\nabla v\|^4\|\nabla E\|^2+\varepsilon\|AE\|^2+\frac{C_\varepsilon}{2}(\|\nabla v\|^4\|\nabla E\|^2+\|E\|^2)+\varepsilon\|AE\|^2 \\
&\le C(\|\nabla v\|^4\|\nabla E\|^2+\|E\|^2)+2\varepsilon\|AE\|^2, \qquad (3.64)
\end{aligned}
$$

$$
\begin{aligned}
|I_{12}| &= \left|\int_\Omega P((N\cdot\nabla)w)\cdot AE\right| = \left|\int_\Omega (N\cdot\nabla)w\cdot AE\right| \\
&\le C\|N\|_{L^6}[\|\nabla w\|^{\frac12}(\|Aw\|^{\frac12}+\|w\|^{\frac12})]\|AE\| \\
&\le C_\varepsilon\|\nabla N\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2+C_\varepsilon\|\nabla N\|^2\|\nabla w\|\|w\|+\varepsilon\|AE\|^2 \\
&\le C_\varepsilon\|\nabla N\|^4\|\nabla w\|^2+\frac14\|Aw\|^2+\varepsilon\|AE\|^2+\frac{C_\varepsilon}{2}(\|\nabla N\|^4\|\nabla w\|^2+\|w\|^2)+\varepsilon\|AE\|^2 \\
&\le C(\|\nabla N\|^4\|\nabla w\|^2+\|w\|^2)+\frac14\|Aw\|^2+2\varepsilon\|AE\|^2, \qquad (3.65)
\end{aligned}
$$

$$
|I_{13}| = \left|\int_\Omega (Pf-Pf_1)\cdot Aw\right| \le C_\varepsilon\|Pf-Pf_1\|^2+\varepsilon\|Aw\|^2. \qquad (3.66)
$$

Choosing ε sufficient small (for example $\varepsilon = 0.01$) and substituting (3.54)–(3.66) into (3.6) reach

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w\|^2+\|\nabla E\|^2)+C_0(\|Aw\|^2+\|AE\|^2)\\
&\le C[\|\nabla w\|^6+\|\nabla v\|^4\|\nabla w\|^2+\|\nabla E\|^4\|\nabla w\|^2+\|\nabla N\|^4\|\nabla w\|^2\\
&\quad+\|\nabla v\|^4\|\nabla E\|^2+\|\nabla N\|^4\|\nabla E\|^2+\|w\|^2+\|E\|^2+\|Pf-Pf_1\|^2]\\
&\le C[(\|\nabla w\|^2+\|\nabla E\|^2)^3+(\|\nabla v\|^4+\|\nabla N\|^4)(\|\nabla w\|^2+\|\nabla E\|^2)\\
&\quad+\|w\|^2+\|E\|^2+\|Pf-Pf_1\|^2]. \qquad (3.67)
\end{aligned}$$

Since

$$\|\nabla w\|^2+\|\nabla E\|^2\le C(\|Aw\|^2+\|w\|^2+\|AE\|^2+\|E\|^2), \qquad (3.68)$$

we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w\|^2+\|\nabla E\|^2)+C_0(\|\nabla w\|^2+\|\nabla E\|^2)\\
&\le C[(\|\nabla w\|^2+\|\nabla E\|^2)^3+(\|\nabla v\|^4+\|\nabla N\|^4)(\|\nabla w\|^2+\|\nabla E\|^2)\\
&\quad+\|w\|^2+\|E\|^2+\|Pf-Pf_1\|^2]. \qquad (3.69)
\end{aligned}$$

Defining

$$h(t):=\|\nabla w(t)\|^2+\|\nabla E(t)\|^2,$$

and

$$M:=\exp\left(C\int_0^\infty \|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4\|\mathrm{d}\tau\right),$$

using (2.4), (3.69), we obtain

$$h'(t)+\tilde{C}_0h(t)\le C[h^3(t)+(\|\nabla v(t)\|^4+\|\nabla N(t)\|^4)h(t)+\|Pf(t)-Pf_1(t)\|^2+(\delta M)^2]. \quad (3.70)$$

Choosing δ small enough and using the similar method as Theorem 1 (i), we can get the proof of Theorem 1 (ii). This completes the proof of Theorem 1. □

Proof of Theorem 2 Now we turn to the proof of Theorem 2. we assume that $\int_0^\infty \|\nabla v(t)\|^4+\|\nabla N(t)\|^4\mathrm{d}t<\infty$. The same argument used to Theorem 1, leads to (3.38). Integrating (3.38) from 0 to t, we obtain

$$\begin{aligned}
&(\|\nabla v(t)\|^2+\|\nabla N(t)\|^2)-(\|\nabla v_0\|^2+\|\nabla N_0\|^2)+\tilde{C}_0\int_0^t \|Av(\tau)\|^2+\|AN(\tau)\|^2\mathrm{d}\tau\\
&\le C\int_0^t(\|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4)(\|\nabla v(\tau)\|^2+\|\nabla N(\tau)\|^2)\mathrm{d}\tau+\int_0^t\|f_1(\tau)\|^2\mathrm{d}\tau\\
&\le C\sup_{t\ge0}(\|\nabla v(t)\|^2+\|\nabla N(t)\|^2)\int_0^\infty \|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4\mathrm{d}\tau+\int_0^\infty\|f_1(\tau)\|^2\mathrm{d}\tau. \quad (3.71)
\end{aligned}$$

Consequently

$$\begin{aligned}
&\int_0^\infty \|Av(\tau)\|^2+\|AN(\tau)\|^2\mathrm{d}\tau\\
&\le C\sup_{t\ge0}(\|\nabla v(t)\|^2+\|\nabla N(t)\|^2)\int_0^\infty \|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4\mathrm{d}\tau+\int_0^\infty\|f_1(\tau)\|^2\mathrm{d}\tau\\
&<\infty. \qquad (3.72)
\end{aligned}$$

Noticing that (v,N) satisfies (1.2) and by the usual energy estimate, we know that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|v(t)\|^2+\|N(t)\|^2)+2(\|\nabla v(t)\|^2+\|\nabla N(t)\|^2)\le 2C_\varepsilon\|f_1\|_{L^\infty}^2+2\varepsilon\|\nabla v\|^2.$$

Taking $\varepsilon=1/4$ and integrating from 0 to t, we get

$$\begin{aligned}&\|v(t)\|^2+\|N(t)\|^2+C\int_0^t\|\nabla v(\tau)\|^2+\|\nabla N(\tau)\|^2\mathrm{d}\tau\\ \le&\int_0^\infty\|f_1(\tau)\|_{L^\infty}^2\mathrm{d}\tau+(\|v_0\|^2+\|N_0\|^2).\end{aligned}\tag{3.73}$$

For $p\ge 6$, $\frac{2}{q}+\frac{3}{p}=1$, thanks to Gagliardo-Nirenberg inequality, (3.11) and Hölder inequality, we obtain

$$\|v(\tau)\|_{L^p}\le C\|\partial_{ij}^2v(\tau)\|^{\frac12-\frac3p}\|v(\tau)\|_{L^6}^{\frac12+\frac3p}\le C\|\nabla v(\tau)\|^{\frac12+\frac3p}\|Av(\tau)\|^{\frac12-\frac3p},$$

and

$$\begin{aligned}\int_0^\infty\|v(\tau)\|_{L^p}^q\mathrm{d}\tau&\le C\int_0^\infty\|\nabla v(\tau)\|^{q(\frac12+\frac3p)}\|Av(\tau)\|^{q(\frac12-\frac3p)}\mathrm{d}\tau\\ &\le C\Big(\int_0^\infty\|\nabla v(\tau)\|^{\alpha q(\frac12+\frac3p)}\mathrm{d}\tau\Big)^{\frac1\alpha}\Big(\int_0^\infty\|Av(\tau)\|^2\mathrm{d}\tau\Big)^{\frac1\beta},\end{aligned}\tag{3.74}$$

where $\alpha=\frac{4}{q}$, therefore $2\le\alpha q(\frac12+\frac3p)\le 4$. Using (3.73) and the condition of Theorem 2, we have

$$\int_0^\infty\|\nabla v(\tau)\|^2\mathrm{d}\tau<\infty,\quad \int_0^\infty\|\nabla v(\tau)\|^4\mathrm{d}\tau<\infty.$$

Furthermore, by the inner interpolation inequality, we have

$$\int_0^\infty\|\nabla v(\tau)\|^{\alpha q(\frac12+\frac3p)}\mathrm{d}\tau<\infty.$$

Using (3.72), we obtain

$$\int_0^\infty\|v(\tau)\|_{L^p}^q\mathrm{d}\tau<\infty,\tag{3.75}$$

which completes the proof for $p\ge 6$. For $p<6$, $\frac2q+\frac3p=1$, using (3.71) and (3.73), we have

$$\begin{aligned}\int_0^\infty\|v(\tau)\|_{L^p}^q\mathrm{d}\tau&\le C\int_0^\infty\|v(\tau)\|^{\frac q2-2}\|\nabla v(\tau)\|^{\frac q2+2}\mathrm{d}\tau\\ &\le C(\sup_{t\ge0}\|v(t)\|)^{\frac q2-2}(\sup_{t\ge0}\|\nabla v(t)\|)^{\frac q2-2}\int_0^\infty\|\nabla v(\tau)\|^4\mathrm{d}\tau\\ &<\infty.\end{aligned}\tag{3.76}$$

Similarly, we have

$$\int_0^\infty\|N(\tau)\|_{L^p}^q\mathrm{d}\tau<\infty.$$

Conversely, assume $v\in L^q([0,\infty),L^p(\Omega))$ for some p,q satisfying $\frac2q+\frac3p=1$. We will prove that

$$\int_0^\infty\|\nabla v(\tau)\|^4+\|\nabla N(\tau)\|^4\mathrm{d}\tau<\infty.\tag{3.77}$$

Firstly, applying the derivatives x_i on both sides of the following equation

$$v_t - \triangle v + (v\cdot\bigtriangledown)v - (N\cdot\bigtriangledown)N + \nabla(\frac{1}{2}|N|^2) + \bigtriangledown p = f_1, \tag{3.78}$$

then applying Helmholtz projection operator P, multiplying $\partial_i v$ on both sides of the equation and integrating on Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i v\|^2 + \|\nabla\partial_i v\|^2 = &-\int_\Omega(\partial_i v\cdot\nabla)v\cdot\partial_i v + \int_\Omega(\partial_i N\cdot\nabla)N\cdot\partial_i v\\ &+\int_\Omega(N\cdot\nabla)\partial_i N\cdot\partial_i v + \int_\Omega P(\partial_i(f_1))\cdot\partial_i v.\end{aligned} \tag{3.79}$$

Similarly, we have

$$N_t - \triangle N + (v\cdot\bigtriangledown)N - (N\cdot\bigtriangledown)v = 0,$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i N\|^2 + \|\nabla\partial_i N\|^2\\ =&-\int_\Omega(\partial_i v\cdot\nabla)N\cdot\partial_i N + \int_\Omega(\partial_i N\cdot\nabla)v\cdot\partial_i N + \int_\Omega(N\cdot\nabla)\partial_i v\cdot\partial_i N.\end{aligned} \tag{3.80}$$

Adding (3.79) to (3.80) reaches to

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_i v\|^2 + \|\partial_i N\|^2) + \|\nabla\partial_i v\|^2 + \|\nabla\partial_i N\|^2\\ =&\underbrace{-\int_\Omega(\partial_i v\cdot\nabla)v\cdot\partial_i v}_{L_1}\underbrace{+\int_\Omega(\partial_i N\cdot\nabla)N\cdot\partial_i v}_{L_2}\underbrace{-\int_\Omega(\partial_i v\cdot\nabla)N\cdot\partial_i N}_{L_3}\\ &\underbrace{+\int_\Omega(\partial_i N\cdot\nabla)v\cdot\partial_i N}_{L_4}\underbrace{+\int_\Omega P(\partial_i(f_1))\cdot\partial_i v}_{L_5}.\end{aligned} \tag{3.81}$$

Integrating by parts and Hölder inequality, we have

$$\begin{aligned}|L_1| &= \left|-\int_\Omega(\partial_i v\cdot\nabla)v\cdot\partial_i v\right| \le \left|\int_\Omega(v\cdot\nabla)\partial_i v\cdot\partial_i v\right| + \left|\int_\Omega(v\cdot\nabla)v\cdot\partial_i\partial_i v\right|\\ &\le C\|v\|_{L^p}\|\nabla v\|_{L^{\frac{2p}{p-2}}}\|D^2 v\|.\end{aligned} \tag{3.82}$$

By Gargliardö-Nirenberg inequality $\|\nabla v\|_{L^{\frac{2p}{p-2}}} \le C\|D^2 v\|^{\frac{3}{p}}\|\nabla v\|^{1-\frac{3}{p}}$ and Young inequality, we obtain

$$|L_1| \le \varepsilon\|D^2 v\|^2 + C_\varepsilon\|v\|_{L^p}^q\|\nabla v\|^2. \tag{3.83}$$

Similarly, we have

$$|L_2| = \left|\int_\Omega(\partial_i N\cdot\nabla)N\cdot\partial_i v\right| \le \varepsilon\|D^2 N\|^2 + C_\varepsilon\|v\|_{L^p}^q\|\nabla N\|^2, \tag{3.84}$$

$$|L_3| = \left|\int_\Omega(\partial_i v\cdot\nabla)N\cdot\partial_i N\right| \le \varepsilon\|D^2 N\|^2 + C_\varepsilon\|v\|_{L^p}^q\|\nabla N\|^2, \tag{3.85}$$

$$|L_4| = \left|\int_\Omega(\partial_i N\cdot\nabla)v\cdot\partial_i N\right| \le \varepsilon\|D^2 N\|^2 + C_\varepsilon\|v\|_{L^p}^q\|\nabla N\|^2, \tag{3.86}$$

and

$$|L_5| = \left|\int_\Omega P(\partial_i(f_1))\cdot\partial_i v\right| = \left|\int_\Omega (\partial_i(f_1))\cdot\partial_i v\right| = \left|\int_\Omega f_1\cdot\partial_i\partial_i v\right|$$
$$\le C\|f_1\|_{L^\infty}\|D^2 v\| \le C_\varepsilon\|f_1\|_{L^\infty}^2 + \varepsilon\|D^2 v\|^2.$$

Choosing ε sufficiently small, substituting the estimates above into (3.81) then adding them from 1 to 3 about i, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla v(t)\|^2 + \|\nabla N(t)\|^2) + C_0(\|D^2 v\|^2 + \|D^2 N\|^2)$$
$$\le C\|v\|_{L^p}^q(\|\nabla v\|^2 + \|\nabla N\|^2) + C\|f_1\|_{L^\infty}^2. \tag{3.87}$$

By Gronwall inequality, we obtain

$$\|\nabla v(t)\|^2 + \|\nabla N(t)\|^2 \le C(\|\nabla v_0\|^2 + \|\nabla N_0\|^2 + \int_0^t \|f_1(\tau)\mathrm{d}\tau\|_{L^\infty}^2)e^{\int_0^t \|v(\tau)\|_{L^p}^q \mathrm{d}\tau}. \tag{3.88}$$

Therefore,

$$\int_0^\infty \|\nabla v(\tau)\|^4 + \|\nabla N(\tau)\|^4 \mathrm{d}\tau$$
$$\le \sup_{t\ge 0}\|\nabla v(t)\|^2 \int_0^\infty \|\nabla v(\tau)\|^2\mathrm{d}\tau + \sup_{t\ge 0}\|\nabla N(t)\|^2 \int_0^\infty \|\nabla N(\tau)\|^2 \mathrm{d}\tau$$
$$< \infty. \tag{3.89}$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Let $v = (v_1, v_2)$, N be a strong solution of the 2D MHD equations with data

$$v_0 = (v_{01}, v_{02}) \in V(R^2)\cap L^1(R^2), N_0 \in H_0^1(R^2)\cap L^1(R^2)$$

and external forcing $\tilde{f}_1(x_1,x_2,x_3)(t) = (f_{11}(x_1,x_2)(t), f_{12}(x_1,x_2)(t), 0) \in L^2(R^2)$. Then, we have

$$v \in L^\infty([0,\infty); V)\cap L^2_{\mathrm{loc}}([0,\infty); D(A)),$$
$$N \in L^\infty([0,\infty); H_0^1(R^2))\cap L^2_{\mathrm{loc}}([0,\infty); H^2(R^2)).$$

By the smoothing properties of the MHD, we know that $\triangle v(t)\in L^2(R^2), \triangle N(t)\in L^2(R^2)$ for all $t>0$. we also know $\int_0^\infty \|\triangle v(\tau)\|^2\mathrm{d}\tau < \infty, \int_0^\infty \|\triangle N(\tau)\|^2\mathrm{d}\tau < \infty$. Moreover, using this estimate and the decay of solutions in [11],

$$\|\partial_x^j v(t)\|_{L^p(R^2)} \le C(1+t)^{-d},$$
$$\|\partial_x^j N(t)\|_{L^p(R^2)} \le C(1+t)^{-d},$$

where

$$d = \min\left\{1, \frac{1}{p'} - \frac{1}{p} + \frac{j}{2}\right\}, \quad p\in[2,\infty], \quad \frac{1}{p'} + \frac{1}{p} = 1, \quad j = 0, 1,$$

it follows hat

$$\int_0^\infty \|\nabla v(s)\|_{L^4(R^2)}^4 \mathrm{d}s < \infty, \int_0^\infty \|\nabla N(s)\|_{L^4(R^2)}^4 \mathrm{d}s < \infty, \tag{3.90}$$

$$\int_0^\infty \|v(s)\|_{L^\infty(R^2)}^2 \mathrm{d}s < \infty, \int_0^\infty \|N(s)\|_{L^\infty(R^2)}^2 \mathrm{d}s < \infty. \tag{3.91}$$

We are going to construct a global solution of the form $u = \tilde{v} + w, B = \tilde{N} + E$ where $\tilde{v} = (v_1, v_2, 0), \tilde{N} = (N_1, N_2, 0)$, and w is small in $V(R^3)$, E is small in $H_0^1(R^3)$. The equations for w and E are given by

$$\begin{aligned} & w_t + Aw + P[(w\cdot\nabla)w + (w\cdot\nabla)\tilde{v} + (\tilde{v}\cdot\nabla)w] \\ & -P[(E\cdot\nabla)E + (\tilde{N}\cdot\nabla)E + (E\cdot\nabla)\tilde{N}] = P(f - \tilde{f}_1)), \end{aligned} \tag{3.92}$$

$$\begin{aligned} & E_t - \triangle E + (w\cdot\nabla)E + (w\cdot\nabla)\tilde{N} + (\tilde{v}\cdot\nabla)E \\ & -P[(E\cdot\nabla)w + (E\cdot\nabla)\tilde{v} + (\tilde{N}\cdot)w] = 0. \end{aligned} \tag{3.93}$$

We claim that for $w(t) \in V(R^3)$, $(w\cdot\nabla)w, (w\cdot\nabla)\tilde{v}$, $(\tilde{v}\cdot\nabla)w$ and $(f - \tilde{f}_1)$ lie in $L^2(R^3)$. Hence, for positive times in (3.92), it makes sense to apply the three-dimensional Helmholtz projection to them. An application of Hölder's inequality is enough to bound the following terms:

$$\begin{aligned} & \|(\tilde{v}\cdot\nabla)w\| \le \|v\|_{L^\infty(R^2)}\|\nabla w\|, \|(\tilde{v}\cdot\nabla)E\| \le \|v\|_{L^\infty(R^2)}\|\nabla E\|, \\ & \|(\tilde{N}\cdot\nabla)E\| \le \|N\|_{L^\infty(R^2)}\|\nabla E\|, \|(\tilde{N}\cdot\nabla)w\| \le \|N\|_{L^\infty(R^2)}\|\nabla w\|. \end{aligned}$$

The other terms are estimated by considering the variables x_1, x_2 separately from x_3:

$$\begin{aligned} & \|(w\cdot\nabla)\tilde{v}\|^2_{L^2(R^2)} \le \|w(\cdot, x_3)\|^2_{L^4(R^2)}\|\nabla v\|^2_{L^4(R^2)}, \\ & \|(w\cdot\nabla)\tilde{N}\|^2_{L^2(R^2)} \le \|w(\cdot, x_3)\|^2_{L^4(R^2)}\|\nabla N\|^2_{L^4(R^2)}, \\ & \|(E\cdot\nabla)\tilde{v}\|^2_{L^2(R^2)} \le \|E(\cdot, x_3)\|^2_{L^4(R^2)}\|\nabla v\|^2_{L^4(R^2)}, \\ & \|(E\cdot\nabla)\tilde{N}\|^2_{L^2(R^2)} \le \|E(\cdot, x_3)\|^2_{L^4(R^2)}\|\nabla N\|^2_{L^4(R^2)}. \end{aligned}$$

Using the Gagliardo-Nirenberg inequality in two dimensions, we have

$$\begin{aligned} & \|w(\cdot, x_3)\|^2_{L^4(R^2)} \le \|\nabla_{x_1,x_2} w(\cdot, x_3)\|\|w(\cdot, x_3)\|_{L^2(R^2)}, \\ & \|E(\cdot, x_3)\|^2_{L^4(R^2)} \le \|\nabla_{x_1,x_2} E(\cdot, x_3)\|\|E(\cdot, x_3)\|_{L^2(R^2)}. \end{aligned}$$

Integrating with respect to x_3 and use Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} & \|(w\cdot\nabla)\tilde{v}\| \le \|\nabla w\|^{\frac12}\|w\|^{\frac12}\|\nabla v\|_{L^4(R^2)}, \\ & \|(w\cdot\nabla)\tilde{N}\| \le \|\nabla w\|^{\frac12}\|w\|^{\frac12}\|\nabla N\|_{L^4(R^2)}, \\ & \|(E\cdot\nabla)\tilde{v}\| \le \|\nabla E\|^{\frac12}\|E\|^{\frac12}\|\nabla v\|_{L^4(R^2)}, \\ & \|(E\cdot\nabla)\tilde{N}\| \le \|\nabla E\|^{\frac12}\|E\|^{\frac12}\|\nabla N\|_{L^4(R^2)}. \end{aligned}$$

The corresponding estimates of $E(t)$ can be done similarly like that of $w(t)$.

With this observation, equation (3.92) for w can be solved locally in $V(R^3)$ and equation (3.93) for E can be solved locally in $H_0^1(R^3)$ in a straightforward manner employing, for example, their equivalent formulation as an integral equation and standard estimates for the linear heat equation. Thus, it is enough to get a priori bound for $\|w(t)\|_{H^1(R^3)}$ and $\|E(t)\|_{H^1(R^3)}$. This will be done in a fashion similar to the proof of Theorem 1 part (ii). Multiplying (3.92) by w and multiplying (3.93) by E, then integrating on R^3, adding them together lead to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2 + \|E\|^2) + \|\nabla w\|^2 + \|\nabla E\|^2 + \int_{R^3} P((w\cdot\nabla)\tilde{v})\cdot w$$

$$
\begin{aligned}
&+\int_{R^3} P((w\cdot\nabla)\tilde{N})\cdot E-\int_{R^3} P((E\cdot\nabla)\tilde{N})\cdot w-\int_{R^3} P((E\cdot\nabla)\tilde{v})\cdot E \\
&=\int_{R^3}(Pf-P\tilde{f}_1)\cdot w. \qquad (3.94)
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|\int_{R^3} P((w\cdot\nabla)\tilde{v})\cdot w\mathrm{d}x\right| &= \left|\int_{R^3} P((w\cdot\nabla)w)\cdot \tilde{v}\mathrm{d}x\right| \le \|v\|_{L^\infty(R^2)}\|w\|\|\nabla w\| \\
&\le C\|v\|^2_{L^\infty(R^2)}\|w\|^2+\varepsilon\|\nabla w\|^2,
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\left|\int_{R^3}(w\cdot\nabla)\tilde{N}\cdot E\mathrm{d}x\right| \le C\|N\|^2_{L^\infty}\|w\|^2+\varepsilon\|\nabla E\|^2, \\
&\left|\int_{R^3}(E\cdot\nabla)\tilde{N}\cdot E\mathrm{d}x\right| \le C\|N\|^2_{L^\infty}\|E\|^2+\varepsilon\|\nabla E\|^2, \\
&\left|\int_{R^3}(E\cdot\nabla)\tilde{v}\cdot E\mathrm{d}x\right| \le C\|v\|^2_{L^\infty}\|E\|^2+\varepsilon\|\nabla E\|^2, \\
&\left|\int_{R^3}(Pf-P\tilde{f}_1)\cdot w\mathrm{d}x\right| \le \varepsilon(\|w\|^2+\|E\|^2)+C_\varepsilon\|f-\tilde{f}_1\|^2,
\end{aligned}
$$

taking ε sufficiently small (for example $\varepsilon=0.01$), we obtain the differential inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|w\|^2+\|E\|^2)+C_0(\|\nabla w\|^2+\|\nabla E\|^2) \\
&\le C(\|v\|^2_{L^\infty(R^2)}+\|N\|^2_{L^\infty(R^2)})(\|w\|^2+\|E\|^2)+C\|f-\tilde{f}_1\|^2,
\end{aligned}
$$

which immediately gives the bound

$$
\begin{aligned}
\|w(t)\|^2+\|E(t)\|^2 \le &\left(\|w(0)\|^2+\|E(0)\|^2+\int_0^\infty\|f(s)-\tilde{f}_1(s)\|^2\mathrm{d}s\right) \\
&\times\exp\left(C\int_0^\infty\|v(s)\|^2_{L^\infty(R^2)}+\|N(s)\|^2_{L^\infty(R^2)}\mathrm{d}s\right). \qquad (3.95)
\end{aligned}
$$

On the other hand, multiplying (3.92) by $Aw(t)$ and (3.93) by $-\triangle E(t)$ yield

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w\|^2+\|\nabla E\|^2)+\|Aw\|^2+\|\triangle E\|^2 \\
&+\int_{R^3}P((w\cdot\nabla)w)\cdot Aw+\int_{R^3}P((w\cdot\nabla)\tilde{v})\cdot Aw+\int_{R^3}P((\tilde{v}\cdot\nabla)w)\cdot Aw \\
&-\int_{R^3}P((E\cdot\nabla)E)\cdot Aw-\int_{R^3}P((N\cdot\nabla)E)\cdot Aw-\int_{R^3}P((w\cdot\nabla)E)\cdot\triangle E \\
&+\int_{R^3}P((E\cdot\nabla)N)\cdot Aw-\int_{R^3}P((E\cdot\nabla)w)\cdot\triangle E-\int_{R^3}P((E\cdot\nabla)v)\cdot\triangle E \\
&-\int_{R^3}P((N\cdot\nabla)w)\cdot\triangle E-\int_{R^3}P((w\cdot\nabla)\tilde{N})\cdot\triangle E-\int_{R^3}P((\tilde{v}\cdot\nabla)E)\cdot\triangle E \\
&=\int_{R^3}P((f-\tilde{f}_1))\cdot Aw. \qquad (3.96)
\end{aligned}
$$

We have the estimates as follows:

$$
\left|\int_{R^3}P((w\cdot\nabla)w)\cdot Aw\right|\le C\|\nabla w\|^6+\varepsilon\|Aw\|^2, \qquad (3.97)
$$

$$\left|\int_{R^3} P((w\cdot\nabla)\tilde{v})\cdot Aw\right| \le \|\nabla w\|^{\frac12}\|w\|^{\frac12}\|\nabla v\|_{L^4(R^2)}\|Aw\| \\ \le C\|w\|^2 + C\|\nabla v\|^4_{L^4(R^2)}\|\nabla w\|^2 + \varepsilon\|Aw\|^2, \tag{3.98}$$

$$\left|\int_{R^3} P((\tilde{v}\cdot\nabla)w)\cdot Aw\right| \le C\|v\|^2_{L^\infty(R^2)}\|\nabla w\|^2 + \varepsilon\|Aw\|^2, \tag{3.99}$$

$$\left|\int_{R^3} P((E\cdot\nabla)E)\cdot Aw\right| \le C\|\nabla E\|^6 + C\|\nabla E\|^2 + \varepsilon\|Aw\|^2, \tag{3.100}$$

$$\left|\int_{R^3} P((N\cdot\nabla)E)\cdot Aw\right| \le C\|N\|_{L^\infty(R^2)}\|\nabla E\|^2 + \varepsilon\|Aw\|^2, \tag{3.101}$$

$$\left|\int_{R^3} P((E\cdot\nabla)N)\cdot Aw\right| \le C\|\nabla E\|^{\frac12}\|E\|^{\frac12}\|\nabla N\|_{L^4R^2}\|Aw\| \\ \le C\|E\|^2 + C\|\nabla N\|^4_{L^4R^2}\|\nabla E\|^2 + \varepsilon\|Aw\|^2, \tag{3.102}$$

$$\left|\int_{\Omega} (w\cdot\nabla)E\cdot\triangle E\right| \le C\|w\|_{L^\infty}\|\nabla E\|\|\triangle E\| \le C\|\nabla w\|^{\frac12}\|Aw\|^{\frac12}\|\nabla E\|\|\triangle E\| \\ \le C_\varepsilon\|\nabla E\|^2\|\nabla w\|\|Aw\| + \varepsilon\|\triangle E\|^2 \\ \le C_\varepsilon\|\nabla E\|^4\|\nabla w\|^2 + \frac14\|Aw\|^2 + \varepsilon\|\triangle E\|^2, \tag{3.103}$$

$$\left|\int_{R^3} (w\cdot\nabla)\tilde{N}\cdot\triangle E\right| \le \|\nabla w\|^{\frac12}\|w\|^{\frac12}\|\nabla N\|_{L^4(R^2)}\|\triangle E\| \\ \le C\|w\|^2 + C\|\nabla N\|^4_{L^4(R^2)}\|\nabla w\|^2 + \varepsilon\|\triangle E\|^2, \tag{3.104}$$

$$\left|\int_{R^3} (\tilde{v}\cdot\nabla)E\cdot\triangle E\right| \le \|v\|^2_{L^\infty(R^2)}\|\nabla E\|^2 + \varepsilon\|\triangle E\|^2, \tag{3.105}$$

$$\left|\int_{R^3} P((E\cdot\nabla)w)\cdot\triangle E\right| \le C\|\nabla w\|^6 + C\|\nabla E\|^6 + \varepsilon\|\triangle E\|^2, \tag{3.106}$$

$$\left|\int_{R^3} (E\cdot\nabla)\tilde{v}\cdot\triangle E\right| \le \|\nabla E\|^{\frac12}\|E\|^{\frac12}\|\nabla v\|_{L^4(R^2)}\|\triangle E\| \\ \le C\|E\|^2 + C\|\nabla v\|^4_{L^4(R^2)}\|\nabla E\|^2 + \varepsilon\|\triangle E\|^2, \tag{3.107}$$

$$\left|\int_{R^3} P((\tilde{N}\cdot\nabla)w)\cdot\triangle E\right| \le C\|N\|^2_{L^\infty(R^2)}\|\nabla w\|^2 + \varepsilon\|\triangle E\|^2, \tag{3.108}$$

$$\left|\int_{\Omega} (Pf - P\tilde{f}_1)\cdot Aw\right| \le C\|f-\tilde{f}_1\|^2 + \varepsilon\|Aw\|^2. \tag{3.109}$$

Choosing ε sufficiently small (for example $\varepsilon = 0.01$) reaches to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla w\|^2 + \|\nabla E\|^2) + \tilde{C}_0(\|Aw\|^2 + \|\triangle E\|^2) \\ \le C[(\|\nabla w\|^2 + \|\nabla E\|^2)^3 + (\|v\|^2_{L^\infty(R^2)} + \|\nabla v\|^4_{L^4(R^2)} + \|N\|^2_{L^\infty(R^2)} \\ + \|\nabla N\|^4_{L^4(R^2)})(\|\nabla w\|^2 + \|\nabla E\|^2) + \|w\|^2 + \|E\|^2 + \|f-\tilde{f}_1\|^2]. \tag{3.110}$$

Therefore, $\|\nabla w(t)\|^2 + \|\nabla E(t)\|^2$ remains bounded if $\|w_0\|_{H^1} + \|E_0\|_{H^1}$ is sufficiently small as in the proof of Theorem 1 (ii). We complete the proof of Theorem 3. □

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