不可压缩 N-S 方程的 FVC 紧致格式

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摘要 本文给出一种求解不可压缩 Navier-Stokes 方程的四阶精度交错网格紧致格式：FVC 格式。动量方程用紧致差分格式，连续方程用有限体积法。在时间方向，我们用 Runge-Kutta 方法。Runge-Kutta 法中间层边界处理我们采用一种比传统方式高一阶精度的方法。

关键词 紧致差分，差分格式，有限体积法，不可压缩 Navier-Stokes 方程，Runge-Kutta 法

1 引 言

紧致差分格式在差分点不仅用到速度、压力的函数值，而且用到导数值[16,17,14,6]，精度高，差分点少，稳定性好，对高频波的分辨率高，边界差分点少。对复杂网格，即使精度不能达到四阶，也可能有必要采用这样的紧致差分格式。紧致格式的一个应用是湍流直接数值模拟[13,14]。

非定常粘性不可压缩 Navier-Stokes 方程为

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{A}(\boldsymbol{V}) + \boldsymbol{\nabla} p = \boldsymbol{0}, \qquad (1.1)$$

其中 $\boldsymbol{A}(\boldsymbol{V}) = (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V} - \nu\nabla^2\boldsymbol{V}$

不可压缩连续方程：

$$\mathrm{div}\,\boldsymbol{V} = 0, \qquad (1.2)$$

(1.1)(1.2)的显示离散格式：

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^n}{\Delta t} + \boldsymbol{\nabla}_h p^{n+1} = 0 \qquad (1.3)$$

$$\mathrm{div}_h \boldsymbol{V}^{n+1} = 0, \qquad (1.4)$$

其中

$$\boldsymbol{V}^* = \boldsymbol{V}^n - \Delta t \boldsymbol{A}_h(\boldsymbol{V}^n)$$

这里时间方向的精度是一阶的。我们在第四部分给出对时间方向的 Runger-Kutta 方法。

2 连续方程的离散格式

这部分考虑二维不可压缩问题。在

e=((i-1)△x,i△x)× ((j-1)△y,j△y) 上积分(1.2)得到

$$\int_{\partial e} \boldsymbol{V}\cdot\boldsymbol{n}\,\mathrm{d}\gamma = 0, \qquad (2.1)$$

其中 **V**=(u,v)，设

$$\bar{u}_{i,j-\frac{1}{2}} = \frac{1}{\Delta y}\int_{(j-1)\Delta y}^{j\Delta y} u(i\Delta x, y)\mathrm{d}y$$
$$\bar{v}_{i-\frac{1}{2},j} = \frac{1}{\Delta x}\int_{(i-1)\Delta x}^{i\Delta x} v(x, j\Delta y)\mathrm{d}x \qquad (2.2)$$

则(2.1)可以写为

$$\frac{\bar{u}_{i,j-\frac{1}{2}} - \bar{u}_{i-1,j-\frac{1}{2}}}{\Delta x} + \frac{\bar{v}_{i-\frac{1}{2},j} - \bar{v}_{i-\frac{1}{2},j-1}}{\Delta y} = 0,$$
$$1 \le i \le N, \quad 1 \le j \le M \qquad (2.3)$$

以下近似是四阶的

$$\bar{u}_{i,j-\frac{1}{2}} = u_{i,j-\frac{1}{2}} + \frac{1}{24}(u_{i,j-\frac{3}{2}} - 2u_{i,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}})$$
$$1 \le i \le N-1, \quad 2 \le j \le M-1 \qquad (2.4)_1$$

$$\bar{v}_{i-\frac{1}{2},j} = v_{i-\frac{1}{2},j} + \frac{1}{24}(v_{i-\frac{3}{2},j} - 2v_{i-\frac{1}{2},j} + v_{i+\frac{1}{2},j})$$

$$\begin{matrix} 2 < i < N-1 \\ 1 \le j \le M-1 \end{matrix} \qquad (2.4)_2$$

其中

$$u_{i,j-\frac{1}{2}} = u(i\Delta x, (j-\tfrac{1}{2})\Delta y)$$

$$v_{i-\frac{1}{2},j} = v((i-\tfrac{1}{2})\Delta x, j\Delta y)$$

对离散格式用同样的符号则(2.3)(2.4)为连续方程(1.2)的四阶离散格式。
边界上三阶离散格式为

$$\bar{u}_{i,\frac{1}{2}} = u_{i,\frac{1}{2}} + \frac{1}{18}(2u^{\Gamma}_{i,0} - 3u_{i,\frac{1}{2}} + u_{i,\frac{3}{2}}) \quad (1 \le i \le N) \qquad (2.5)_1$$

$$\bar{u}_{i,M-\frac{1}{2}} = u_{i,M-\frac{1}{2}} + \frac{1}{18}(2u^{\Gamma}_{i,M} - 3u_{i,M-\frac{1}{2}} + u_{i,M-\frac{3}{2}}) \quad (1 \le i \le N) \qquad (2.5)_2$$

$$\bar{v}_{\frac{1}{2},j} = v_{\frac{1}{2},j} + \frac{1}{18}(2v^{\Gamma}_{0,j} - 3v_{\frac{1}{2},j} + v_{\frac{3}{2},j}) \quad (1 \le j \le M) \qquad (2.5)_3$$

$$\bar{v}_{N-\frac{1}{2},j} = v_{N-\frac{1}{2},j} + \frac{1}{18}(2v^{\Gamma}_{N,j} - 3v_{N-\frac{1}{2},j} + v_{N-\frac{3}{2},j}) \quad (1 \le j \le M) \qquad (2.5)_4$$

$$u^{\Gamma}_{i,0} = u^{\Gamma}(i\Delta x, 0), u^{\Gamma}_{i,M} = u^{\Gamma}(i\Delta x, L^y),$$

$$v^{\Gamma}_{0,j} = v^{\Gamma}(0, j\Delta y), v^{\Gamma}_{N,j} = v^{\Gamma}(L^x, j\Delta y)$$

$\bar{u}, \bar{v}$ 是积分平均值，例如

$$\bar{u}_{0,j-\frac{1}{2}} = \frac{1}{\Delta y}\int_{(j-1)\Delta y}^{j\Delta y} u(0,y)\mathrm{d}y \qquad (2.6)$$

3 交错网格紧致差分格式

考虑二维非定常粘性不可压缩 Navier-Stokes 方程(1.1)(1.2)中的

$$\boldsymbol{A}(\boldsymbol{V}) = (uu_x + vu_y - \nu(u_{xx} + u_{yy}), uv_x + vv_y - \nu(v_{xx} + v_{yy}))^T \qquad (3.1)$$

$$\boldsymbol{\nabla} p = (p_x, p_y)^T, \mathrm{div}\boldsymbol{V} = u_x + v_y \qquad (3.2)$$

我们考虑(1.1)(1.2)(3.1)(3.2)的差分格式

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^n}{\Delta t} + \boldsymbol{A}_h(\boldsymbol{V}^n) + \boldsymbol{\nabla}_h p^{n+1} = 0, \qquad (3.3)$$

$$\mathrm{div}_h \boldsymbol{V}^{n+1} = 0, \qquad (3.4)$$

其中

$$\boldsymbol{V}^n = (u^n, v^n)^T,$$

（1） $\boldsymbol{A}_h(\boldsymbol{V})$ 中的一阶导数 u_x, u_y 的差分离散为

$$\frac{u'_{i-1,j-\frac{1}{2}} + 4u'_{i,j-\frac{1}{2}} + u'_{i+1,j-\frac{1}{2}}}{6} = \frac{u_{i+1,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{2\Delta x}, \quad \begin{matrix} 1 \le i \le N-1 \\ 1 \le j \le M \end{matrix} \qquad (3.5)_1$$

$$\frac{u'_{i,j-\frac{3}{2}} + 4u'_{i,j-\frac{1}{2}} + u'_{i,j+\frac{1}{2}}}{6} = \frac{u_{i,j+\frac{1}{2}} - u_{i,j-\frac{3}{2}}}{2\Delta y}, \quad \begin{matrix} 1 \le i \le N-1 \\ 2 \le j \le M-1 \end{matrix} \qquad (3.5)_2$$

（2） $\boldsymbol{A}_h(\boldsymbol{V})$ 中的二阶导数 u_{xx}, u_{yy} 的差分离散为

$$\frac{u''_{i-1,j-\frac{1}{2}} + 10u''_{i,j-\frac{1}{2}} + u''_{i+1,j-\frac{1}{2}}}{12} =$$

$$\frac{u_{i-1,j-\frac{1}{2}} - 2u_{i,j-\frac{1}{2}} + u_{i+1,j-\frac{1}{2}}}{(\Delta x)^2}, \quad (3.6)_1$$

$$(1 \le i \le N-1, \quad 1 \le j \le M)$$

$$\frac{u''_{i,j-\frac{3}{2}} + 10u''_{i,j-\frac{1}{2}} + u''_{i,j+\frac{1}{2}}}{12} =$$

$$\frac{u_{i,j-\frac{3}{2}} - 2u_{i,j-\frac{1}{2}} + u_{i,j+\frac{1}{2}}}{(\Delta y)^2}, \quad \begin{matrix} 1 \le i \le N-1 \\ 2 \le j \le M-1 \end{matrix} \quad (3.6)_2$$

（3）压力的一阶导数的差分离散

$$\frac{p'_{i-1,j-\frac{1}{2}} + 22p'_{i,j-\frac{1}{2}} + p'_{i+1,j-\frac{1}{2}}}{24} =$$

$$\frac{p_{i+\frac{1}{2},j-\frac{1}{2}} - p_{i-\frac{1}{2},j-\frac{1}{2}}}{\Delta x}, \quad \begin{matrix} 2 \le i \le N-2 \\ 1 \le j \le M \end{matrix} \quad (3.7)$$

（4）连续方程中速度一阶导数的差分离散（本文主要采用另外的离散格式，见第二部分）

$$\frac{u'_{i-\frac{3}{2},j-\frac{1}{2}} + 22u'_{i-\frac{1}{2},j-\frac{1}{2}} + u'_{i+\frac{1}{2},j-\frac{1}{2}}}{24} =$$

$$\frac{u_{i,j-\frac{1}{2}} - u_{i-1,j-\frac{1}{2}}}{\Delta x}, \quad \begin{matrix} 2 \le i \le N-1 \\ 1 \le j \le M \end{matrix} \quad (3.8)$$

（5）非线性项中的 uv_x 里的 u 用以下插值公式

$$u_{i-\frac{1}{2},j} =$$

$$\frac{\tilde{u}_{i-1,j-\frac{1}{2}} + \tilde{u}_{i,j-\frac{1}{2}} + \tilde{u}_{i-1,j+\frac{1}{2}} + \tilde{u}_{i,j+\frac{1}{2}}}{4}$$

$$, \quad \begin{matrix} 1 \le i \le N \\ 1 \le j \le M-1 \end{matrix}, \quad (3.9)$$

其中二阶导数的近似可用(3.6)(3.12)(3.13)的结果。

下面给出边界上的差分逼近。我们主要给出x=0 上的差分方程：

$(1)_1$

$$(u_x)_{0,j-\frac{1}{2}} = (-v_y^\Gamma)_{0,j-\frac{1}{2}}$$
$$(u_x)_{N,j-\frac{1}{2}} = (-v_y^\Gamma)_{N,j-\frac{1}{2}},$$
$$(1 \le j \le M) \quad (3.10)$$

$(1)_2$ u_y at $y = \frac{3}{4}\Delta y$ and $y = L^y - \frac{3}{4}\Delta y$,

$$\frac{3u'_{i,\frac{1}{2}} + u'_{i,\frac{3}{2}}}{4} = \frac{u_{i,\frac{3}{2}} - u_{i,0}}{\frac{3}{2}\Delta y}$$

$$\frac{u'_{i,M-\frac{3}{2}} + 3u'_{i,M-\frac{1}{2}}}{4} = \frac{u_{i,M} - u_{i,M-\frac{3}{2}}}{\frac{3}{2}\Delta y},$$

$$(1 \le i \le N-1) \quad (3.11)$$

here $u_{i,0} = u^\Gamma(i\Delta x, 0)$,
$u_{i,M} = u^\Gamma(i\Delta x, L^y)$;

$(2)_1$ u_{xx} at $x = \frac{2}{3}\Delta x$:

$$\frac{u''_{0,j-\frac{1}{2}} + 2u''_{1,j-\frac{1}{2}}}{3} =$$

$$\frac{1}{\Delta x}\left(\frac{u_{2,j-\frac{1}{2}} - u_{0,j-\frac{1}{2}}}{2\Delta x} - u'_{0,j-\frac{1}{2}}\right),$$

$$(1 \le j \le M) \quad (3.12)$$

其中

$$u'_{0,j-\frac{1}{2}} = -(v_y^\Gamma)_{0,j-\frac{1}{2}}$$
$$= -v_y^\Gamma(0, (j-\tfrac{1}{2})\Delta y)$$

$(2)_2$ u_{yy} at $y = \frac{2}{3}\Delta y$:

$$\frac{5u''_{i,\frac{1}{2}} + u''_{i,\frac{3}{2}}}{6} - \frac{1}{48}(u''_{i,\frac{1}{2}} - 2u''_{i,\frac{3}{2}} + u''_{i,\frac{5}{2}})$$

$$= \frac{4}{3} \cdot \frac{2u_{i,0} - 3u_{i,\frac{1}{2}} + u_{i,\frac{3}{2}}}{(\Delta y)^2}, \quad (3.13)$$

其中 $u_{i,0} = u^{\Gamma}_{i,0}$

(3) p_x at $x = \Delta x$:

$$p'_{1,j-\frac{1}{2}} + \frac{p'_{1,j-\frac{1}{2}} - 2p'_{2,j-\frac{1}{2}} + p'_{3,j-\frac{1}{2}}}{24} = \frac{p_{\frac{3}{2},j-\frac{1}{2}} - p_{\frac{1}{2},j-\frac{1}{2}}}{\Delta x}, \quad (1 \le j \le M) \quad (3.14)$$

(4) u_x in div$\boldsymbol{V}$ at $x = \frac{1}{2}\Delta x$, (this is for (3.8)):

$$\frac{2u'_{0,j-\frac{1}{2}} + 15u'_{\frac{1}{2},j-\frac{1}{2}} + u'_{\frac{3}{2},j-\frac{1}{2}}}{18} = \frac{u_{1,j-\frac{1}{2}} - u_{0,j-\frac{1}{2}}}{\Delta x}, (1 \le j \le M), \quad (3.15)$$

其中

$$u'_{0,j-\frac{1}{2}} = -(v^{\Gamma}_y)_{0,j-\frac{1}{2}}$$

(5) boundary values of u_{xx}, u_{yy} in $(3.9)_2$: for $j = 1, 2, ..., M$,

$$(u_{xx})_{0,j-\frac{1}{2}} = 2(u_{xx})_{\frac{1}{2},j-\frac{1}{2}} - (u_{xx})_{1,j-\frac{1}{2}}, \quad (3.16)_1$$

$$(u_{xx})_{\frac{1}{2},j-\frac{1}{2}} = ((u_x)_{1,j-\frac{1}{2}} - (u_x)_{0,j-\frac{1}{2}})/\Delta x, \quad (3.16)_2$$

$$(u_{yy})_{0,j-\frac{1}{2}} = (u^{\Gamma}_{yy})_{0,j-\frac{1}{2}}, \quad (3.16)_3$$

其中 $(u_x)_{1,j-\frac{1}{2}}$ 可以由(3.5)1,(3.10)和 $(u_x)_{0,j-\frac{1}{2}} = -(v^{\Gamma}_y)_{0,j-\frac{1}{2}}$ 得到。

以上差分格式中的 u,v,p 的右上角的撇表示导数对应的差分方程中的未知量。

4 非定常问题时间方向离散的 Runger-Kutta 方法

定义 $\boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{A}_h(\boldsymbol{V}) + \boldsymbol{\nabla}_h p$,

其中 $p = p(\boldsymbol{V})$ **满足**

$$\text{div}_h(\boldsymbol{A}_h(\boldsymbol{V}) + \boldsymbol{\nabla}_h p) = 0.$$

这样 f 是 **V** 的函数。

求解 $\boldsymbol{V}_t + \boldsymbol{f}(\boldsymbol{V}) = \boldsymbol{0}$ 的四阶 Ruger-Kutta 方法为：

$$\frac{\boldsymbol{V}^{n+1} - \boldsymbol{V}^n}{\Delta t} + \frac{\boldsymbol{f}(\boldsymbol{V}^n) + 2\boldsymbol{f}(\boldsymbol{V}^{n+\frac{1}{2}}) + 2\boldsymbol{f}(\bar{\boldsymbol{V}}^{n+\frac{1}{2}}) + \boldsymbol{f}(\bar{\boldsymbol{V}}^{n+1})}{6} = \boldsymbol{0}, \quad (4.1)$$

其中

$$\boldsymbol{V}^{n+\frac{1}{2}} = \boldsymbol{V}^n - \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{V}^n)$$
$$\bar{\boldsymbol{V}}^{n+\frac{1}{2}} = \boldsymbol{V}^n - \frac{\Delta t}{2}\boldsymbol{f}(\boldsymbol{V}^{n+\frac{1}{2}})$$
$$\bar{\boldsymbol{V}}^{n+1} = \boldsymbol{V}^n - \Delta t\boldsymbol{f}(\bar{\boldsymbol{V}}^{n+\frac{1}{2}})$$

边界上常用

$$\boldsymbol{V}^{n+\frac{1}{2}} = \boldsymbol{V}|_{t=(n+\frac{1}{2})\Delta t}$$
$$\bar{\boldsymbol{V}}^{n+\frac{1}{2}} = \boldsymbol{V}|_{t=(n+\frac{1}{2})\Delta t} \quad (4.2)$$
$$\bar{\boldsymbol{V}}^{n+1} = \boldsymbol{V}|_{t=(n+1)\Delta t}$$

（4.2）在中间层（n+0.5 时间层）精度只有一阶。我们给出如下改进，精度比(4.2)提高一阶

$$\boldsymbol{V}^{n+\frac{1}{2}} = \boldsymbol{V}^n + \frac{\Delta t}{2}(\frac{\partial \boldsymbol{V}}{\partial t}|_{t=n\Delta t}), \quad (4.3)_1$$

$$\bar{\boldsymbol{V}}^{n+\frac{1}{2}} = 2(\boldsymbol{V}|_{t=(n+\frac{1}{2})\Delta t}) - \boldsymbol{V}^{n+\frac{1}{2}}, \quad (4.3)_2$$

$$\bar{\boldsymbol{V}}^{n+1} = \boldsymbol{V}|_{t=(n+1)\Delta t}, \qquad (4.3)_3$$

5 方腔问题数值计算

考虑在顶部受剪切力驱动的二维方腔流动。控制方程为(1.1)(1.2)(3.1)(3.1),计算区域为 0<x<1, 0<y<1,边界上的速度：

上边界上(u,v)=(1,0),其他边界上速度均为零。

$$\Delta x = 1/N, \Delta y = 1/M,$$
$$M = N,\ \mathrm{Re}=1/\nu$$

以下**格式 CD-V** 与**格式 CD** 差别是前者连续方程用第二部分的离散公式，后者用（3.8）.
格式 CD-V 即 FVC 格式：（3.3）-（3.7）（2.3）-（2.6）（3.9）-（3.14）(3.16).

下表给出用 Re=1000 计算到定常时中线上 u（在 x=0.5），v（在 y=0.5）的极值与位置本文方法与前人方法对比，

	$u_{\min}^{(x=0.5)}$	$y_{\min}$	$v_{\max}^{(y=0.5)}$	$x_{\max}$	$v_{\min}^{(y=0.5)}$	$x_{\min}$
Ghia [2]	−0.3829	0.171875	0.37095	0.15625	−0.51550	0.90625
Zhang [7]	−0.39009	0.16992	0.37847	0.15820	−0.52839	0.90820
Bruneau [1]	−0.3764	0.1602	0.3665	0.1523	−0.5208	0.9102
CD-V,N=256	−0.3885729	0.1716965	0.3769494	0.1578361	−0.5270795	0.9092451
CD-V,N=128	−0.3885091	0.1717298	0.3768988	0.1578476	−0.5269636	0.9092524
CD-V,N=64	−0.3874597	0.1722462	0.3759533	0.1580876	−0.5250540	0.9090746
CD-V,N=32	−0.3818458	0.1773832	0.3709188	0.1603107	−0.5114055	0.9051549
CD, N=256	−0.3840607	0.1724296	0.3721873	0.1586994	−0.5213952	0.9089328
CD, N=128	0.3795204	0.1732002	0.3674140	0.1595922	0.5156458	0.9086300
CD, N=64	−0.3705614	0.1751717	0.3585948	0.1614637	−0.5043853	0.9078775
CD, N=32	−0.3519381	0.1822162	0.3401250	0.1667745	−0.4761842	0.9032001

上表和下表中文献[2],[7],[1]的极值是在格点上。

下表是用 Re=1000 计算到定常时流函数在极值点的数值和位置本文方法与前人方法对比，

	Primary vortex $\psi_{\min}$ (location x, y)	Secondary vortex (BL) $\psi_{\max}$ (location x, y)	Secondary vortex (BR) $\psi_{\max}$ (location x, y)
Ghia [2]	−.117929 (.5313,.5625)	.000231129 (.0859,.0781)	.00175102 (.8594,.1094)
Zhang [7]	−.1193 (.5313,.5664)	.000235 (.0820,.0781)	.00174 (.8633,.1133)
Bruneau[1]	−.1163 (.5313,.5586)	.000325 (.0859,.0820)	.00191 (.8711,.1094)
CD-V,256	−.118938(.530789,.56524)	.0002335(.08327,.078095)	.0017297(.86404,.11181)
CD-V,128	−.118925(.530785,.56526)	.0002333(.08325,.078105)	.0017294(.86345,.11149)
CD-V, 64	−.118691(.530796,.56552)	.0002309(.08319,.077927)	.0017238(.86414,.11208)
CD-V, 32	−.117567(.530532,.56810)	.0002108(.08318,.076787)	.0017243(.86208,.11363)
CD,N=256	−.117741(.531154,.56559)	.0002234(.08298,.077663)	.0016898(.86449,.11183)
CD,N=128	−.116541(.531521,.56596)	.0002135(.08265,.077228)	.0016501(.86495,.11185)
CD, N=64	−.114019(.532291,.56689)	.0001930(.08182,.076336)	.0015680(.86600,.11210)
CD, N=32	−.109233(.533577,.57032)	.0001483(.08038,.073383)	.0014351(.86615,.11340)

数值结果证实了 FVC 格式的高精度。

(3.13)和(3.14)左端第二项有和没有差一阶精度，但数值结果差别较小。整体精度不受影响的一个原因是由于特定的边界条件。

部分详细内容见：

http://www.cfdchina.com/3rd/yuxin.pdf
http://www.yuxin.net/FVC

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