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A Constrained-Transport E-CUSP scheme for Ideal Magnetohydrodynamic Equations

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Abstract

The constrained transport algorithm combined with an E-CUSP scheme is developed to solve the ideal magnetohydrodynamic equations. The algorithm can preserve the divergence-free condition for the magnetic field and maintain the advantages of simplicity and low diffusion of the E-CUSP scheme. The numerical results demonstrate the robustness and efficiency of this new algorithm.

1 Introduction

Many hypersonic aerodynamics and astrophysics problems need to solve the solutions of ideal magnetohydrodynamics(MHD) equations. Since the ideal MHD equations have a wave-like structure analogous to that of the hydrodynamics equations, various numerical schemes for hydrodynamics equations have been extended to solve the MHD equations in the past two decades. The approximate Riemann solvers, which are based on eigenvalue and eigenvector analysis, are widely used for high speed flow as well as for high speed MHD applications. Beginning with the work of Brio and Wu[1], the numerical methods for MHD equations based on approximate Riemann solvers have been extensively studied and developed. For example, Roe's Riemann solvers are developed by Brio and Wu[1], Dai and Woodward[2], Zachary and Collelaz[3], Roe and Balsara[4], and Cargo and Gallice[5]. HLL(Harten-Lax-van Leer)-type schemes are developed by Janhunen[6] and Honkkila and Janhunen[7], Gurski[8], Li[9], Miyoshi and Kusano[10], Balsara et al.[11]. Flux vector splitting methods are developed by MacCormack[12], Jiang and Wu[13]. The equations of magnetohydrodynamics are not homogeneous of degree one with respect to the state vector and hence can not directly perform flux vector splitting. To overcome this difficulty MacCormack introduces an extra variable $\tilde{a}$ in Ref. [12]. The flux splitting schemes based on eigenvalues and eigenvectors system are generally very complicated. In our study, we noticed that, in the eigensystem of Roe and Balsara[4], the eigenvalues of the Alfven waves do not affect the flux. In other words, any values can be used for the eigenvalues of the Alfven waves and the flux will be the same. This makes the flux splitting based on Roe's approximate Riemann solver uncertain.

The low disspation high order filter schemes developed by Yee and Sjogreen[14] for MHD systems involve a dissipative portion of higher order Lax-Friedrichs scheme or an approximate Riemann solver. Moreover, Balbas[15] developed a central differencing scheme based on the evolution of cell averages over staggered grids. Gaitonde[16] developed a compact difference method for MHD with a local filter switching procedure to change the higher order filter to a second order filter locally for shock capturing. The central differencing

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scheme and the compact difference scheme do not need a detailed knowledge of the eigenstructure of the Jacobian matrices. However, the central differencing schemes have difficulty in capturing the shock waves.

In recent years, the convective upwind and split pressure (CUSP) family schemes, which simultaneously consider the convective upwind characteristics and avoid the complex eigen-decomposition process, have achieved great success in gasdynamics. The CUSP schemes can be basically categorized to two types, the H-CUSP and E-CUSP[17, 18, 19]. The H-CUSP schemes have the total enthalpy from the energy equation in their convective vector, whereas the E-CUSP schemes use the total energy in the convective vector. The Liou's AUSM family schemes[20, 21, 22, 23, 24], Van Leer-Hänel scheme[25], and Edwards's LDFSS schemes[26, 27] belong to the H-CUSP group. The schemes developed by Zha, et al.[28, 29, 30, 31, 32] belong to the E-CUSP group.

Most of the CUSP schemes mentioned above are low diffusive. However, as discussed in [33], the low diffusion scheme combined with high-order reconstruction is much more probable to yield numerical oscillations in a shock wave. Agarwal et al. [34] applied the original AUSM method with first-order spatial accuracy to one-dimensional MHD cases. Han et al. [33] developed a AUSMPW+/M-AUSMPW+ schemes combined with the MLP interpolation method to achieve the higher order accuracy for MHD equations.

Recently, Shen el at developed an E-CUSP scheme for MHD system[35], which maintains the advantages of simplicity and low diffusion of the E-CUSP scheme. This scheme avoids the complication of deriving the eigenvalues and eigenvector system when the MHD equations are incorporated. In [35], the new E-CUSP scheme is used with a high order WENO reconstruction for the magnetohydrodynamics equations. The numerical experiments in [35] have demonstrated the new scheme's accuracy and robustness.

For numerical simulation of the magnetohydrodynamic (MHD) equations, a crucial issue is to preserve the divergence-free condition $\nabla \cdot \mathbf{B} = 0$ for the magnetic field $\mathbf{B}$. There are several approaches to deal with this problem. Powell et al.[36] added a source term that is proportional to $\nabla \cdot \mathbf{B}$ to the original set of MHD equations, and present a set of characteristic system. However, this system may generate some uncertainty[35]. The projection method proposed in[37] has been widely used. The projection involves the solution of a Poisson equation and also restricts the choice of boundary conditions. The constrained transport (CT) method by Evans and Hawley[38] is another approach to keep $\nabla \cdot \mathbf{B}$ to the accuracy of machine round-off error. Toth[39] found the flux constrained transport method was one of the most accurate second schemes that he tested. This approach has been combined with various shock-capturing schemes by many authors[2, 40, 41, 42, 43, 44, 45, 46, 47].

The purpose of this paper is to develop the constrained transport algorithm combined with an E-CUSP scheme to solve the ideal magnetohydrodynamic equations.

2 Numerical Method

2.1 Governing Equations

The ideal MHD equations for inviscid flow can be expressed in vector form as [48]

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = 0, \tag{1}$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{V} \\ \rho e \\ \mathbf{B} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho\mathbf{V} \\ \rho\mathbf{V}\mathbf{V} + p_t\mathbf{I} - \mathbf{B}\mathbf{B} \\ (\rho e + p_t)\mathbf{V} - \mathbf{B}(\mathbf{V}\cdot\mathbf{B}) \\ \mathbf{V}\mathbf{B} - \mathbf{B}\mathbf{V} \end{pmatrix},$$

$$p_t = p + \frac{1}{2}B^2, \quad \rho e = \frac{1}{2}\rho\mathbf{V}^2 + \frac{1}{2}\mathbf{B}^2 + \frac{p}{(\gamma - 1)},$$

Subject to the constraint

$$\nabla \cdot \mathbf{B} = 0. \tag{2}$$

Where ρ is the flow density, $\mathbf{V}$ is the velocity vector, ρe is the energy, p is the pressure, $\mathbf{B}$ is the magnetic field.

The governing equations. [Eq. (1)] can be written in the Cartesian coordinate as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0 \tag{3}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e \\ B_x \\ B_y \\ B_z \end{bmatrix}, \mathbf{E} = \begin{bmatrix} \rho u \\ \rho u^2 + p_t - B_x^2 \\ \rho uv - B_x B_y \\ \rho uw - B_x B_z \\ (\rho e + p_t)u - B_x(uB_x + vB_y + wB_z) \\ uB_x - uB_x \\ uB_y - vB_x \\ uB_z - wB_x \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} \rho v \\ \rho uv - B_y B_x \\ \rho v^2 + p_t - B_y^2 \\ \rho vw - B_y B_z \\ (\rho e + p_t)v - B_y(uB_x + vB_y + wB_z) \\ vB_x - uB_y \\ vB_y - vB_y \\ vB_z - wB_y \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \rho w \\ \rho uw - B_z B_x \\ \rho vw - B_z B_y \\ \rho w^2 + p_t - B_z^2 \\ (\rho e + p_t)w - B_z(uB_x + vB_y + wB_z) \\ wB_x - uB_z \\ wB_y - vB_z \\ wB_z - wB_z \end{bmatrix}$$

At x-direction, the speed of sound is

$$c = \sqrt{\frac{\gamma p}{\rho}},$$

the Alfven speed is

$$c_a = \frac{|B_x|}{\sqrt{\rho}},$$

and the fast and slow speeds are given by

$$c_{f,s} = \sqrt{\frac{1}{2}\left[c^2 + b^2 \pm \sqrt{(c^2 + b^2)^2 - 4c^2 c_a^2}\right]},$$

where $b^2 = \frac{B_x^2 + B_y^2 + B_z^2}{\rho}$.

In the generalized computational coordinates, Eq.(3) can be written as:

$$\frac{\partial \mathbf{U}'}{\partial t} + \frac{\partial \mathbf{E}'}{\partial \xi} + \frac{\partial \mathbf{F}'}{\partial \eta} + \frac{\partial \mathbf{G}'}{\partial \zeta} = 0, \tag{4}$$

where

$$\mathbf{U}' = \frac{1}{J}\mathbf{U},$$

$$\mathbf{E}' = \frac{1}{J}(\xi_x \mathbf{E} + \xi_y \mathbf{F} + \xi_z \mathbf{G}),$$

$$\mathbf{F}' = \frac{1}{J}(\eta_x \mathbf{E} + \eta_y \mathbf{F} + \eta_z \mathbf{G}),$$

$$\mathbf{G}' = \frac{1}{J}(\zeta_x \mathbf{E} + \zeta_y \mathbf{F} + \zeta_z \mathbf{G}).$$

At ξ-direction, the eigenvalues of the Jacobian matrix $A = \frac{\partial E'}{\partial U'}$ in system (4) are

$$U - \bar{C}_f,\ U - \bar{C}_a,\ U - \bar{C}_s,\ U,\ U, U + \bar{C}_s,\ U + \bar{C}_a,\ U + \bar{C}_f,$$

where

$$U = \xi_x u + \xi_y v + \xi_z w,$$

$$\bar{C}_a = \frac{|\bar{B}_x|}{\sqrt{\rho}},$$

$$\bar{C}_{f,s} = \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}\sqrt{\frac{1}{2}\left[c^2 + b^2 \pm \sqrt{(c^2 + b^2)^2 - 4c^2\frac{\bar{C}_a^2}{\xi_x^2 + \xi_y^2 + \xi_z^2}}\right]},$$

where $\bar{B}_x = \xi_x B_x + \xi_y B_y + \xi_z B_z$, and b^2 and c are same as in the Cartesian system. $C = c\sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2}$ will be used in the next section.

2.2 E-CUSP scheme for MHD equations

The semi-discretized conservative one-dimensional MHD equations can be written as

$$\frac{d\mathbf{U}'}{dt} + \frac{1}{\Delta x}\left(\mathbf{E'}_{i+1/2} - \mathbf{E'}_{i-1/2}\right) = 0. \tag{5}$$

Following the E-CUSP scheme in[32], the flux $\mathbf{E}'$ may be decomposed to convective and wave flux as the following,

$$\mathbf{E}' = \mathbf{f}U + \mathbf{P} + \psi U, \tag{6}$$

where

$$\mathbf{f} = \left\{\begin{array}{l} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e \\ B_x \\ B_y \\ B_z \end{array}\right\},\ \mathbf{P} = \left\{\begin{array}{l} 0 \\ \xi_x p_t - B_x \bar{B}_x \\ \xi_y p_t - B_y \bar{B}_x \\ \xi_z p_t - B_z \bar{B}_x \\ -\bar{B}_x(wB_x + vB_y + wB_z) \\ -u\bar{B}_x \\ -v\bar{B}_x \\ -w\bar{B}_x \end{array}\right\},\ \psi = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ p_t \\ 0 \\ 0 \\ 0 \end{array}\right\}.$$

Similar to the pressure term pU that is separated from the enthalpy term ρHU in the E-CUSP scheme, the term $p_t U$ (ψ) is also separated.

The numerical flux of the E-CUSP scheme is constructed based on the one given in[32] as the following,

$$\mathbf{E'}_{1/2} = a_{1/2}\left[C^+\mathbf{f}_L + C^-\mathbf{f}_R\right] + \left[D_L^+\mathbf{P}_L + D_R^-\mathbf{P}_R\right] + \psi_{1/2}, \tag{7}$$

where

$$
\begin{aligned}
& M_{L,R} = \frac{U_{L,R}}{a_{1/2}}, \\
& C^{+} = \alpha_L^{+}(1+\beta_L)M_L - \frac{1}{4}\beta_L(M_L+1)^2, \\
& C^{-} = \alpha_R^{-}(1+\beta_R)M_R + \frac{1}{4}\beta_R(M_R-1)^2, \\
& \alpha_{L,R}^{\pm} = \frac{1}{2}\left[1 \pm sign(M_{L,R})\right], \\
& \beta_{L,R} = -max\left[0, 1-int(|M_{L,R}|)\right], \\
& D_{L,R}^{\pm} = \alpha_{L,R}^{\pm}(1+\beta_{L,R}) - \frac{1}{2}\beta_{L,R}\left(1 \pm M_{L,R}\right),
\end{aligned}
\tag{8}
$$

and

$$
\psi_{1/2} = a_{1/2}(C^{+}+C^{-})(D^{+}\phi_L + D^{-}\phi_R). \tag{9}
$$

Note that, in [33], the speed of a fast magnetosonic wave is used to define the Mach number $M = \frac{u}{c_f}$, which means $M_{L,R}$ is defined as $M_{L,R} = \frac{U_{L,R}}{C_{f_{1/2}}}$. In the present study, we find that using $M_{L,R} = \frac{U_{L,R}}{C_f + C_{1/2}}$ is smoother and more accurate for the compound wave of Brio-Wu's shock tube, and there is almost no difference in other regions. Hence, the Mach number is defined as

$$
M_{L,R} = \frac{U_{L,R}}{a_{1/2}}, \tag{10}
$$

where

$$
a_{1/2} = \frac{1}{2}(C_{f\,L} + C_L + C_{f\,R} + C_R)
$$

is adopted.

2.3 Constrained transport[42]

In this paper, the constrained transport method proposed by Balsara and Spicer[42] is applied. For completeness, the important formulae are given as follows. First, in Eq.(1), Faraday's equation can be written as

$$
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{11}
$$

For ideal MHD the electric field $\mathbf{E}$ is given by

$$
\mathbf{E} = -\mathbf{v} \times \mathbf{B} \tag{12}
$$

In the constrained transport method, the magnetic field $\mathbf{B}$ in Eq. (11) is to be treated as an area-weighted average $\bar{\mathbf{B}}$ on the zone face[42]. The electric fields $\mathbf{E}$ are collocated at zone edges. Then the line integral of the electric field over a zone edge gives the electromotive force over that edge. Hence, the semi-discrete form of Eq. (11) is given by

$$
\begin{aligned}
\frac{d}{dt}\bar{B}_{x\,i-1/2,j,k} &= -\frac{E_{z\,i-1/2,j+1/2,k} - E_{z\,i-1/2,j-1/2,k}}{\delta y} + \frac{E_{y\,i-1/2,j,k+1/2} - E_{y\,i-1/2,j,k-1/2}}{\delta z} \\
\frac{d}{dt}\bar{B}_{y\,i,j-1/2,k} &= \frac{E_{z\,i+1/2,j-1/2,k} - E_{z\,i-1/2,j-1/2,k}}{\delta x} - \frac{E_{x\,i,j-1/2,k+1/2} - E_{x\,i,j-1/2,k-1/2}}{\delta z} \\
\frac{d}{dt}\bar{B}_{z\,i,j,k-1/2} &= -\frac{E_{y\,i+1/2,j,k-1/2} - E_{y\,i-1/2,j,k-1/2}}{\delta x} + \frac{E_{x\,i,j+1/2,k-1/2} - E_{x\,i,j-1/2,k-1/2}}{\delta y}
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
E_{x i,j+1/2,k+1/2} &= \frac{1}{4}\left(G_{7i,j,k+1/2} + G_{7i,j+1,k+1/2} - F_{8i,j+1/2,k} - F_{8i,j+1/2,k+1}\right) \\
E_{y i+1/2,j,k+1/2} &= \frac{1}{4}\left(E_{8i+1/2,j,k} + E_{8i+1/2,j,k+1} - G_{6i,j,k+1/2} - G_{6i+1,j,k+1/2}\right) \\
E_{z i+1/2,j+1/2,k} &= \frac{1}{4}\left(F_{6i,j+1/2,k} + F_{6i+1,j+1/2,k} - E_{7i+1/2,j,k} - E_{7i+1/2,j+1,k}\right)
\end{aligned}
\tag{14}
$$

where E_n, F_n, G_n $(n = 6, 7, 8)$ is the n-th flux of the E-CUSP scheme from above subsection.

For the 2D case, Eq. (14) reduce to

$$
\begin{aligned}
E_{x i,j+1/2} &= -F_{8i,j+1/2} \\
E_{y i+1/2,j} &= E_{8i+1/2,j} \\
E_{z i+1/2,j+1/2} &= \frac{1}{4}\left(F_{6i,j+1/2} + F_{6i+1,j+1/2} - E_{7i+1/2,j} - E_{7i+1/2,j+1}\right)
\end{aligned}
\tag{15}
$$

The magnetic fields stored on the faces $\bar{\mathbf{B}}$ are averaged to the zone center value $\mathbf{B}$. They are then used to correct the energy density for the new magnetic field[42].

2.4 Time Runge-Kutta method

The 3rd-order TVD Runge-Kutta method developed by Shu and Osher[49] is used in this paper. To solve the equation

$$\frac{du}{dt} = L(u), \tag{16}$$

the 3rd-order TVD Runge-Kutta method is

$$
\left\{
\begin{aligned}
u^{(1)} &= u^{(0)} + \Delta t L(u^{(0)}) \\
u^{(2)} &= \frac{3}{4}u^{(0)} + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\
u^{(3)} &= \frac{1}{3}u^{(0)} + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}).
\end{aligned}
\right.
\tag{17}
$$

3 Numerical examples

3.1 Brio-Wu shock tube problem

The initial left and right values have been suggested by Brio and Wu[1] and are commonly used to test numerical schemes for one-dimensional ideal MHD. Note that the hydrodynamics data used here are identical to those in Sod's shock tube Riemann problem.

$$
(\rho, u, v, w, B_y, B_z, p) = \left\{
\begin{array}{ll}
(1.0, \quad 0, 0, 0, +1, 0, 1.0), & for\ x < 0 \\
(0.125, 0, 0, 0, -1, 0, 0.1), & for\ x > 0
\end{array}
\right.
$$

with $B_x = 0.75$, $\gamma = 2$.

The numerical example involves a compound wave, which is a typical feature of the solutions of MHD systems. For each quantity, the solution contains five constant states separated by a fast rarefaction wave, a slow compound wave, a slow shock, and a fast rarefaction. The density presents a sixth constant state because this variable is discontinuous across the contact discontinuity[1].

Fig. 1 shows the solution with 800 points at $t = 0.2$ by using 3rd-order WENO reconstruction. It can be seen that the present method resolves well all the complex waves. Since the constraint $\nabla \cdot \mathbf{B} = 0$ is automatically statisfied for 1D cases, this case only show that the E-CUSP scheme for MHD eqautions is efficient and accurate.

3.2 Orszag-Tang MHD turbulence problem

Since the Orszag-Tang MHD turbulence problem[50] has many significant characteristics of MHD turbulence, such as interactions of multiple shock waves generated as the vortex evolves, it is considered as one of the standard models to validate a MHD numerical method[51, 13, 52, 15, 33].

The initial conditions are given by

$$\begin{array}{lll} \rho(x,y,0)=\gamma^2, & u(x,y,0)=-\sin(y), & v(x,y,0)=\sin(x), \\ p(x,y,0)=\gamma, & B_x(x,y,0)=-\sin(y), & B_y(x,y,0)=\sin(2x), \end{array}$$

where $\gamma = 5/3$. As in [51, 13, 52], the computational domain is $[0, 2\pi] \times [0, 2\pi]$ with a uniform mesh of 192×192 grid points. Periodic boundary conditions are imposed in both x- and y-directions. Figs. 2-3 show the numerical results at times $t = 2$, and 3, where 20 contours are plotted. The 3rd-order WENO reconstruction is used for this case. It can be seen that, without constrained transport (CT) method, there are perturbations in both the pressure field and magenetic field, and these perturbations become larger with time increasing. The new algorithm coupled with the constrained transport method improves the solution greatly.

3.3 2D rotor problem

The rotor problem has been suggested by Balsara and Spicer[42] as a test to check the propagation of torsional Alfven waves, and then it has been widely used as a standard test model. The initial condition consists of a rapidly rotating cylinder of dense gas embedded in a lighter fluid at rest. The system is threaded by a uniform magnetic field along the x-axis and the problem is defined on the 2D Cartesian domain $(x, y) \in [-0.5, 0.5]^2$

$$\begin{array}{l} \rho = 1 + 9f(r), \\ p = 1, \\ \begin{cases} u = -2f(r)y/0.1, & v = 2f(r)x/0.1, & w = 0, & if r < 0.1 \\ u = -2f(r)y/r, & v = 2f(r)x/r, & w = 0, & if r \geq 0.1 \end{cases} \\ B_x = 5/\sqrt{4\pi}, \quad B_y = B_z = 0, \end{array} \tag{18}$$

where, $r = \sqrt{x^2 + y^2}$,

$$f(r) = \begin{cases} 1 & if r < 0.1 \\ \frac{200}{3}(0.115 - r) & if 0.1 \leq r \leq 0.115 \\ 0 & if r > 0.115 \end{cases}$$

The adiabatic index $\gamma = 1.4$. The contour plots of pressure p and x-component of magnetic field B_x with grid of 401×401 are shown in Fig. 4 at the final time $t = 0.154$. For this case, only the results of the 1st upwind reconstruct is obtained. It can be seen that, without constrained transport algorithm, there is obvious noise in the magnetic field.

We tested the 3rd WENO reconstruction, and found the same problem as in[53], that is: the reconstructed densities and pressures may not be positive, and even modified these variables to guarantee their positivity, but they still can not guarantte the update ones. Study of this issue is currently underway and will be reported at an upcoming paper.

4 Conclusions

The constrained transport algorithm is successfully combined with the E-CUSP scheme and employed to solve the MHD problems. The algorithm can preserve the divergence-free condition for the magnetic field, and maintains the advantages of simplicity and low diffusion of the E-CUSP scheme. The numerical results show the robustness and efficiency of this new algorithm.

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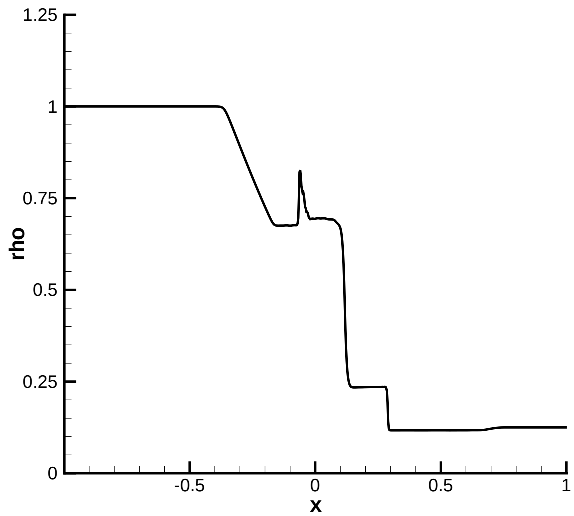
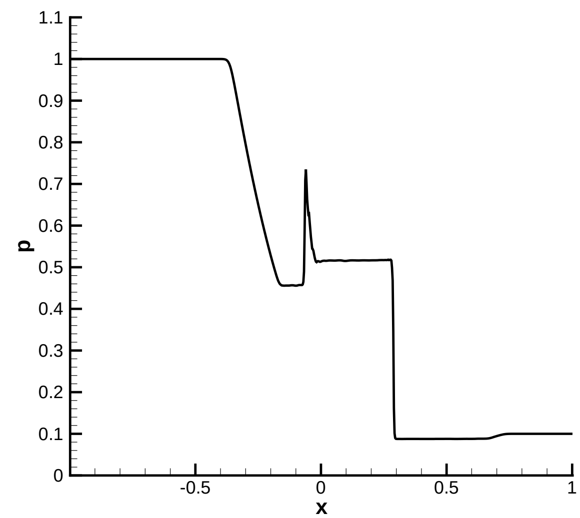
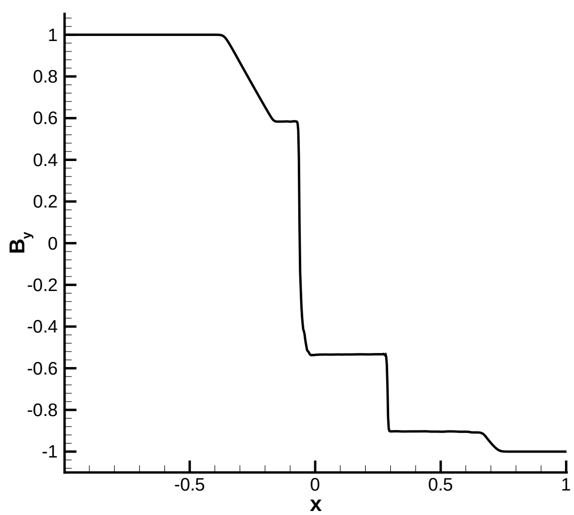


Density



Pressure



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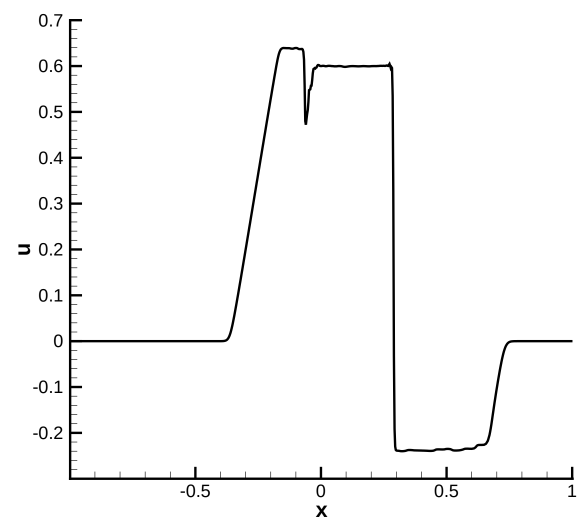
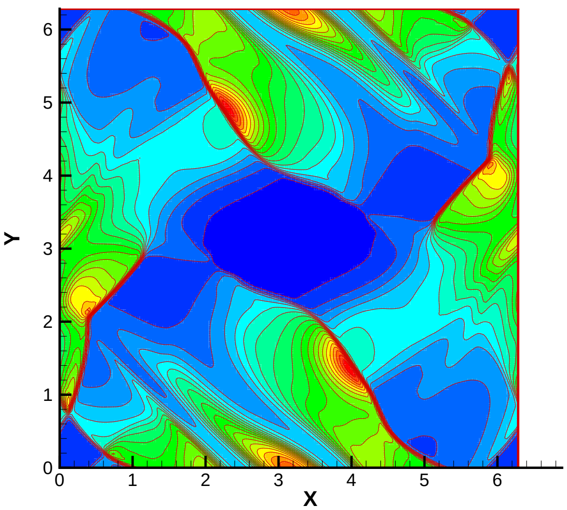
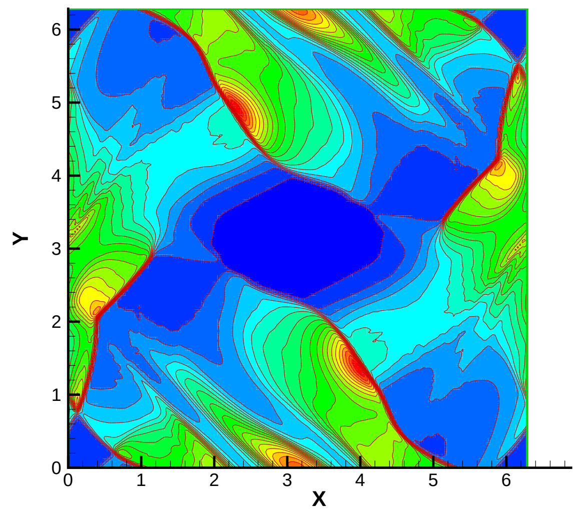


Velocity, u

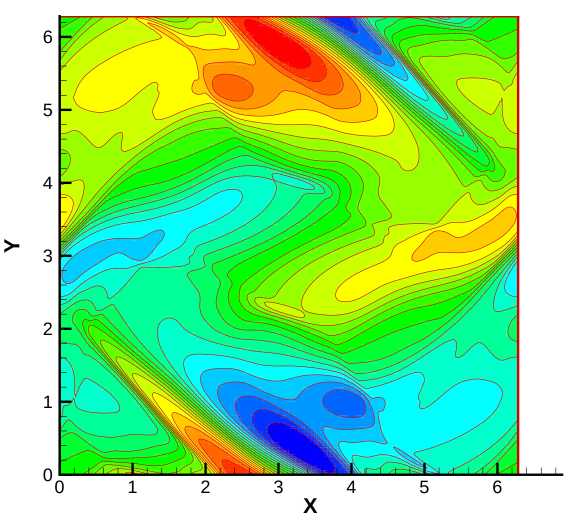
Figure 1: Brio-Wu shock tube problem



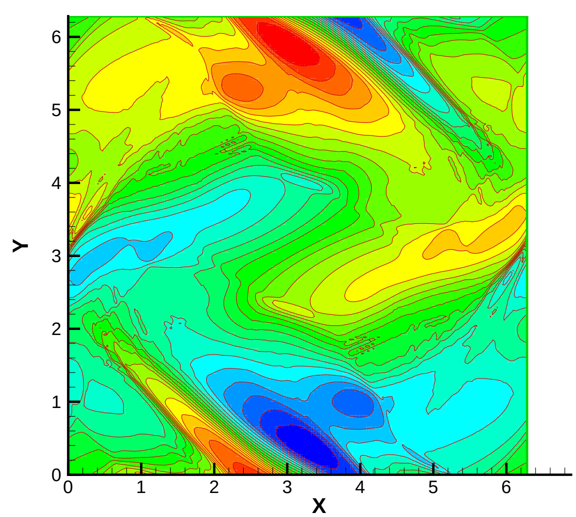
Pressure contours, CT



Pressure contours, without CT

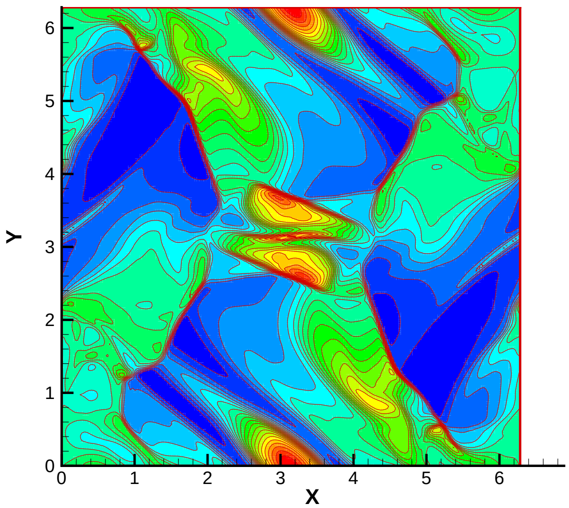


B_x, CT



B_x, without CT

Figure 2: Orszag-Tang MHD turbulence problem, $t = 2.0$



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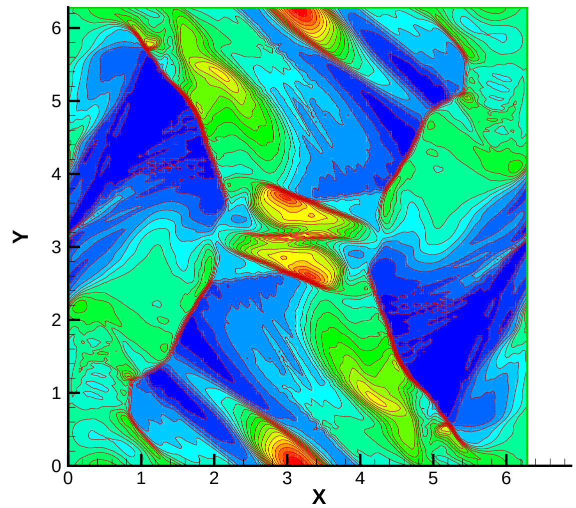
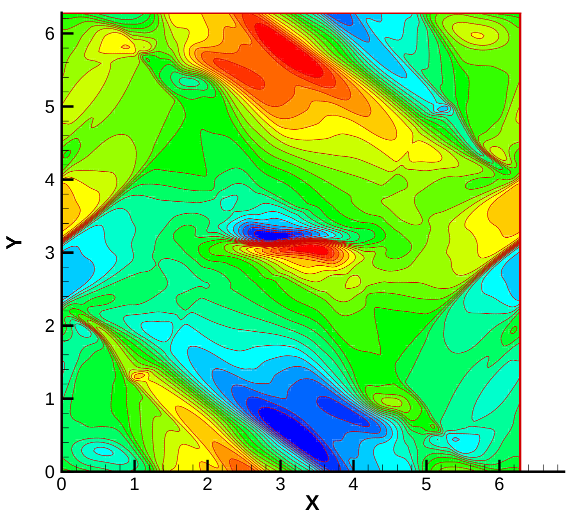
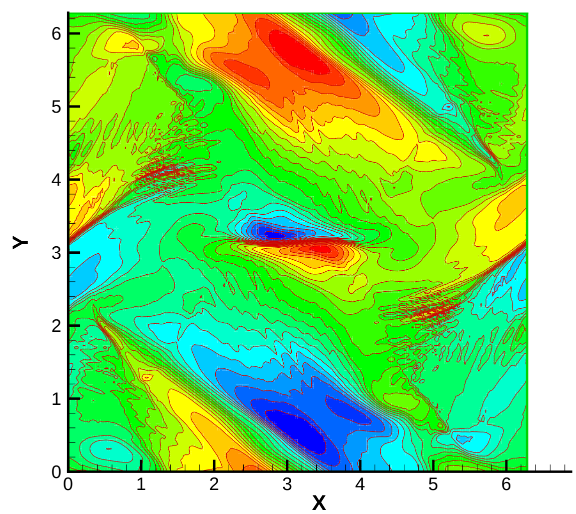


Pressure contours, without CT

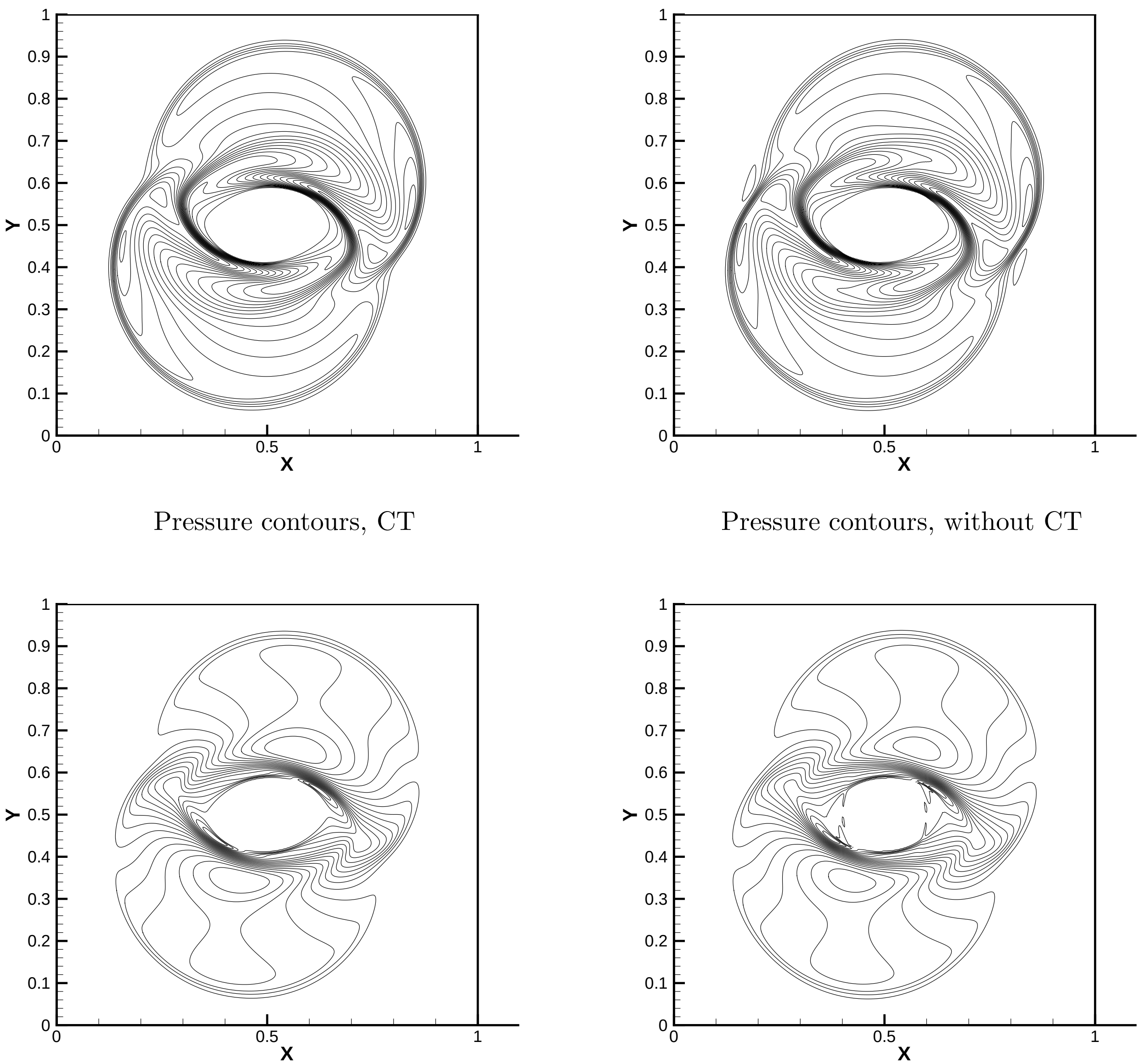


B_x, CT



B_x, without CT

Figure 3: Orszag-Tang MHD turbulence problem, $t = 3.0$



Pressure contours, CT

Pressure contours, without CT

B_x, CT

B_x, without CT

Figure 4: 2D rotor problem, $t = 0.15$