Theoretical Solution for Dynamic Responses of Saturated Soils

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ABSTRACT: Dynamic responses of soil foundation are difficult to analyze though it is related to kinds of practices. In this paper dynamic responses of soil foundation under dynamic load were analyzed theoretically. Perturbation method was used to obtain the theoretical solution. Controlling equations were about liquid-solid two-phase media. Flow function and potential function were introduced to decouple the controlling equations. Then perturbation expansion was introduced into the equations of flow function and potential function. The responses characteristics are discussed.

INTRODUCTION

More and more dynamic responses of soil foundations are required to be analyzed in practice. For an example, offshore platforms are applied in more and more complicated ocean environments and geological conditions, so dynamic failure becomes the key problem of platforms (Wang et al. 2006; Lu et al. 2005a; Lu et al. 2005; Lu et al. 2005b; Bye et al. 1995). Load of ocean wave or ice-induced vibration can be transferred to the ocean floor by platform foundations (piles, suction caissons etc.) to cause the dynamic responses of ocean floor such as deformation and even liquefaction (Lu et al. 2004).

Up to now, few theoretical methodologies have been proposed to analyze the dynamic responses of soil foundations subjected to dynamic load (Mamoon and Banerjee 1990; Lu et al. 2006). Structural foundations are most embedded in saturated soils. Thus the methodology must be able to consider the percolation in saturated soils and pore pressure increase and the soil strength decrease. The development of pore pressure and soil strength is crucial for understanding the liquefaction mechanism of the soil foundations.

In this paper the dynamic responses of soil foundation in saturated soils under horizontal dynamic load are investigated by using the perturbation expansion method. The behavior of saturated soils is described by poro-elastic two-phase media. By introducing the potential and flow functions, the solving process is simplified. The development of pore pressure is mainly discussed.

PROBLEM AND CONTROLLING EQUATION

Problem is assumed as a half-infinite plane (Fig. 1). Dynamic load is applied at the left boundary. The upper boundary is free, the bottom is fixed and undrained. The soil layer is saturated. The water and soil grains are incompressible. Change of porosity is small and its gradient can be neglected. All equations are linearized with constant coefficients. Density of each phase is constant. Hooke's law suits for soil skeleton.

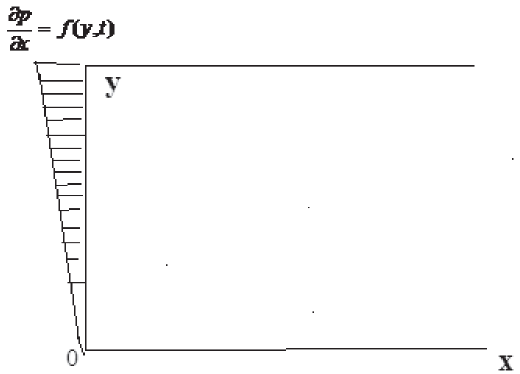


FIG. 1. Sketch of the problem.

MODEL OF THE PROBLEM

Momentum Equations of The Skeleton and the Pore Water

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} - \varepsilon \rho g_i - (1-\varepsilon)\rho_s g_i = (1-\varepsilon)\rho_s \frac{\partial^2 u_i}{\partial t^2} + \varepsilon \rho \frac{\partial v_i}{\partial t}$$

$$-\frac{\partial \varepsilon p}{\partial x_i} + p\frac{\partial \varepsilon}{\partial x_i} - \frac{\varepsilon^2 \mu}{K}(v_i - \frac{\partial u_i}{\partial t}) - \varepsilon \rho g_i = \varepsilon \rho \frac{\partial v_i}{\partial t} \tag{1}$$

To remove the static effective stresses and pore pressure we shall replace henceforth σ_{ij} by $\sigma_{ij} - (1-\varepsilon)(\rho_s - \rho) g y \delta_{ij}$ and p by $p + \rho g y$.

Then the above equations become homogeneous

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} = (1-\varepsilon)\rho_s \frac{\partial^2 u_i}{\partial t^2} + \varepsilon \rho \frac{\partial v_i}{\partial t}$$

$$-\frac{\partial p}{\partial x_i} - \frac{\varepsilon \mu}{K}(v_i - \frac{\partial u_i}{\partial t}) = \rho \frac{\partial v_i}{\partial t} \tag{2}$$

Eliminating p the first equation becomes

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\varepsilon \mu}{K}(v_i - \frac{\partial u_i}{\partial t}) = (1-\varepsilon)(\rho_s \frac{\partial^2 u_i}{\partial t^2} - \rho \frac{\partial v_i}{\partial t}) \tag{3}$$

Mass Conservation Equations of Skeleton and Pore Water

$$\frac{\partial \varepsilon \rho}{\partial t} + \frac{\partial \varepsilon \rho v_i}{\partial x_i} = 0$$
$$\frac{\partial (1-\varepsilon)\rho_s}{\partial t} + \frac{\partial^2 (1-\varepsilon)\rho_s u_i}{\partial t \partial x_i} = 0 \tag{4}$$

For constant density, these two equations combine to form

$$\varepsilon \frac{\partial v_i}{\partial x_i} + (1-\varepsilon)\frac{\partial^2 u_i}{\partial t \partial x_i} = 0 \tag{5}$$

Constitutive Equations–Hooke's Law

$$\sigma_{ij} = G\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\frac{2G\nu}{1-2\nu}\frac{\partial u_k}{\partial x_k} \tag{6}$$

Reduction of Equations

Introduce potential and "flow" functions such that

$$\bar{u} = Grad\varphi_s + Curl\bar{\psi}_s \text{ with } Div\bar{\psi}_s = 0$$
$$\bar{v} = Grad\varphi + Curl\bar{\psi} \text{ with } Div\bar{\psi} = 0 \tag{7}$$

From the mass conservation relation (5)

$$\varepsilon\nabla^2\varphi + (1-\varepsilon)\nabla^2\frac{\partial \varphi_s}{\partial t} = 0 \tag{8}$$

We shall choose

$$\varepsilon\varphi + (1-\varepsilon)\frac{\partial \varphi_s}{\partial t} = 0 \tag{9}$$

From the second of equation (2), we have

$$\nabla^2 p = -\rho\nabla^2\frac{\partial \varphi}{\partial t} - \frac{\varepsilon\mu}{K}\nabla^2\left(\varphi - \frac{\partial \varphi_s}{\partial t}\right) \tag{10}$$

And

$$\frac{\varepsilon\mu}{K}\nabla^2\left(\bar{\psi} - \frac{\partial \bar{\psi}_s}{\partial t}\right) = -\rho\nabla^2\frac{\partial \bar{\psi}}{\partial t} \tag{11}$$

We choose

$$\frac{\varepsilon\mu}{K}\left(\bar{\psi} - \frac{\partial \bar{\psi}_s}{\partial t}\right) = -\rho\frac{\partial \bar{\psi}}{\partial t} \tag{12}$$

Similarly taking the divergence and curl of equation (3) respectively, we have

$$\frac{2G(1-\nu)}{\rho_s(1-2\nu)}\nabla^2\varphi_s = (1-\varepsilon)\left(1 + \frac{1-\varepsilon}{\varepsilon}\frac{\rho}{\rho_s}\right)\frac{\partial^2 \varphi_s}{\partial t^2} + \frac{\mu}{\rho_s K}\frac{\partial \varphi_s}{\partial t} \tag{13}$$

and

$$\frac{G}{\rho_s}\nabla^2\bar{\psi}_s = \left(1 - \varepsilon + \frac{\varepsilon\rho}{\rho_s}\right)\frac{\partial^2 \bar{\psi}_s}{\partial t^2} \tag{14}$$

In this problem, $\eta = \omega\rho_s K/\mu$ is a small parameter, we can give the asymptotic expansions by using of multi-scale method:

$$\begin{cases}\varphi_s = \varphi_s^0(x_i,\tau_1,\tau_2) + \sum_{n=1}^{\infty}\eta^n \varphi_s^{(n)}(x_i,\tau_1,\tau_2) \\ \overline{\psi}_s = \overline{\psi}_s^0(x_i,\tau_1,\tau_2) + \overline{\psi}_s^{(n)}(x_i,\tau_1,\tau_2)\end{cases} \tag{15}$$

in which $\tau_1 = \tau, \tau_2 = \eta\tau$. Institute these expressions into eq.(13) and (14),we can obtain:

$$\begin{cases} D\nabla^2\varphi_s^{(1)} = \dfrac{\partial \varphi_s^{(1)}}{\partial \tau_1} \\ D\nabla^2\varphi_s^{(n)} = \dfrac{\partial \varphi_s^{(n)}}{\partial \tau_1} + \dfrac{\partial \varphi_s^{(n-1)}}{\partial \tau_2} + (1-\varepsilon)(1+\dfrac{1-\varepsilon}{\varepsilon}\dfrac{\rho}{\rho_s})(\dfrac{\partial^2 \varphi_s^{(n-1)}}{\partial \tau_1^2} + \dfrac{\partial^2 \varphi_s^{(n-2)}}{\partial \tau_1 \partial \tau_2} + \dfrac{\partial^2 \varphi_s^{(n-3)}}{\partial \tau_2^2}) \\ \lambda\nabla^2\overline{\psi}_s^{(0)} = \dfrac{\partial^2 \overline{\psi}_s^{(0)}}{\partial \tau_1^2} \\ \lambda\nabla^2\overline{\psi}_s^{(1)} = \dfrac{\partial^2 \overline{\psi}_s^{(1)}}{\partial \tau_1^2} + \dfrac{\partial^2 \overline{\psi}_s^{(0)}}{\partial \tau_1 \partial \tau_2} \\ \lambda\nabla^2\overline{\psi}_s^{(n)} = \dfrac{\partial^2 \overline{\psi}_s^{(n)}}{\partial \tau_1^2} + \dfrac{\partial^2 \overline{\psi}_s^{(n-1)}}{\partial \tau_1 \partial \tau_2} + \dfrac{\partial^2 \overline{\psi}_s^{(n-2)}}{\partial \tau_2^2} \end{cases} \tag{16}$$

in which $D = \dfrac{2GK(1-\nu)}{\mu\omega(1-2\nu)}$, $\varphi_s^{(0)} = 0$, $\lambda = \dfrac{G}{\omega^2[(1-\varepsilon)\rho_s + \varepsilon\rho]}$.

Institute eq.(15) into eq.(10) and neglecting the high order small parameter, we can obtain:

$$\begin{cases} \dfrac{1}{\rho}\nabla^2 p^{(0)} = -\varepsilon\nabla^2\left(\varphi^{(1)} - \dfrac{\partial \varphi_s^{(1)}}{\partial \tau_1}\right) \\ \dfrac{1}{\rho}\nabla^2 p^{(1)} = -\nabla^2\dfrac{\partial \varphi^{(1)}}{\partial \tau_1} + \varepsilon\nabla^2\dfrac{\partial \varphi_s^{(1)}}{\partial \tau_2} \end{cases} \tag{17}$$

Then institute eq. (9) into the above equation:

$$\begin{cases} \dfrac{1}{\rho}\nabla^2 p^{(0)} = \nabla^2\dfrac{\partial \varphi_s^{(1)}}{\partial \tau_1} \\ \dfrac{1}{\rho}\nabla^2 p^{(1)} = -\dfrac{1-\varepsilon}{\varepsilon}\nabla^2\dfrac{\partial^2 \varphi_s^{(1)}}{\partial \tau_1^2} + \varepsilon\nabla^2\dfrac{\partial \varphi_s^{(1)}}{\partial \tau_2} \end{cases} \tag{18}$$

Considering the first one in the above equation, the first one of eq.(16) becomes:

$$D\nabla^2 p^{(0)} = \frac{\partial p^{(0)}}{\partial \tau_1} \tag{19}$$

The boundary and initial conditions for p is as follows:

$$\begin{cases} x = 0; \quad u_{sx} = aye^{i\omega t} \\ x \to \infty; \quad No\ radiation \\ y = 0; \quad \dfrac{\partial p^{(0)}}{\partial y} = 0 \\ y = L; \quad p^{(0)} = 0 \\ t = 0; \quad p^{(0)} = 0 \end{cases} \tag{20}$$

in which L is the depth of the soil layer. Then p can be obtained as follows:

$$p^{(0)}=\sum_{n=0}^{\infty}\frac{2a\left[(-1)^n\beta_n L-1\right]}{\beta_n^2 L\sqrt{\beta_n^2+\frac{i\omega}{D}}}\cos\beta_n y e^{-\sqrt{\beta_n^2+\frac{i\omega}{D}}x}e^{i\omega t} \tag{21}$$

in which $\beta_n=\frac{(2n+1)\pi}{2L}$.

By the first of eq.(18), we have

$$\varphi^{(1)}-\frac{\partial\varphi_s^{(1)}}{\partial\tau_1}=-\sum_{n=0}^{\infty}\frac{2a\left[(-1)^n\beta_n L-1\right]}{\rho\varepsilon\beta_n^2 L\sqrt{\beta_n^2+\frac{i\omega}{D}}}\cos\beta_n y e^{-\sqrt{\beta_n^2+\frac{i\omega}{D}}x}e^{i\omega t} \tag{22}$$

So

$$\frac{\partial\varphi_s^{(1)}}{\partial\tau_1}=\sum_{n=0}^{\infty}\frac{2a\left[(-1)^n\beta_n L-1\right]}{\rho\beta_n^2 L\sqrt{\beta_n^2+\frac{i\omega}{D}}}\cos\beta_n y e^{-\sqrt{\beta_n^2+\frac{i\omega}{D}}x}e^{i\omega t} \tag{23}$$

and

$$\varphi_s^{(1)}=-\sum_{n=0}^{\infty}\frac{2ia\left[(-1)^n\beta_n L-1\right]}{\omega\rho\beta_n^2 L\sqrt{\beta_n^2+\frac{i\omega}{D}}}\cos\beta_n y e^{-\sqrt{\beta_n^2+\frac{i\omega}{D}}x}e^{i\omega t} \tag{24}$$

We first seek for the zero order solution

$$\lambda\nabla^2\overline{\psi}_s^{(0)}=\frac{\partial^2\overline{\psi}_s^{(0)}}{\partial\tau_1^2} \tag{25}$$

Boundary conditions: y=0, $\frac{\partial\overline{\psi}_s^0}{\partial x}=0$,

y=L, $\frac{\partial^2\overline{\psi}_s^0}{\partial x^2}=\frac{\partial^2\overline{\psi}_s^0}{\partial y^2}$

x=0, $\frac{\partial\overline{\psi}_s^0}{\partial y}=aye^{i\omega t}$

$$\text{x}=\infty,\quad \frac{\partial\overline{\psi}_s^0}{\partial x}=0 \tag{26}$$

$$\psi_s^{(0)}=\sum_0^{\infty}\frac{(-1)^{n+1}}{n}\frac{2aL}{\pi}e^{(-\sqrt{\omega^2-\left(\frac{n\pi}{L}\right)^2})x}\sin\frac{n\pi}{L}ye^{i\omega t} \tag{27}$$

By the fourth of eq.(14), $\overline{\psi}_s^{(1)}$ can be obtained in the following way.
Boundary conditions:

$$x=0,\frac{\partial\overline{\psi}_s^{(1)}}{\partial y}=-\sum_{n=0}^{\infty}a_n\cos\beta_n y\sqrt{\beta_n^2+\frac{i\omega}{D}}e^{i\omega\tau_1} \tag{28}$$

$$x\to\infty,\overline{\psi}_s^{(1)}\to 0 \tag{29}$$

$$y = 0, \frac{\partial \overline{\psi}_s^{(1)}}{\partial y} = 0 \tag{30}$$

$$y = L, \frac{\partial^2 \overline{\psi}_s^{(1)}}{\partial y^2} - \frac{\partial^2 \overline{\psi}_s^{(1)}}{\partial x^2} = 2\sum_{n=0}^{\infty} a_n \beta_n \sin \beta_n y \sqrt{\beta_n^2 + \frac{i\omega}{D}} e^{-\sqrt{\beta_n^2 + \frac{i\omega}{D}} x} e^{i\omega \tau_1} \tag{31}$$

Initial conditions:

$$t = 0, \overline{\psi}_s^{(1)} = 0 \tag{32}$$

$\overline{\psi}_s$ can be expressed as:

$$\overline{\psi}_s^{(1)} = e^{i\omega \tau_1} \left[\sum_{n=0}^{\infty} \sin \gamma_n y \left(d_n e^{-\sqrt{r^n - \omega^2} x} + V_v e^{-\sqrt{\beta^2 + i\omega/D} x} \right) + \sum_{i=0}^{\infty} C_n \sin \beta_n y e^{-\sqrt{\beta^2 + i\omega/D} x} \right] \tag{33}$$

in which $C_n = -\frac{2a_{n\beta_n}}{2\beta_n^2 + i\omega/D} \sqrt{\beta_n^2 + i\omega/D}$

$$V_n = \frac{(\omega^2 + i\omega/D)C_n}{\lambda(\beta_n^2 - \gamma_n^2 + i\omega/D + \omega^2)} \frac{8n}{(4n+1)\pi}$$

$$\gamma_n = \frac{n\pi}{L}$$

$$u_n = -\frac{a_n \sqrt{\beta_n^2 + i\omega/D} + \beta_n C_n}{\gamma_n} \frac{8n+4}{(4n+1)\pi}$$

$$d_n = u_n - V_n$$

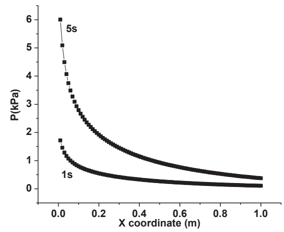
By the second of eq.(18), $p^{(1)}$ can be obtained:

$$p^{(1)} = -\frac{\rho(1-\varepsilon)}{\varepsilon} \sum_{n=0}^{\infty} \frac{2\omega i a[(-1)^n \beta_n L - 1]}{\rho \beta_n^2 L \sqrt{\beta_n^2 + \frac{i\omega}{D}}} \cos \beta_n y e^{-\sqrt{\beta_n^2 + \frac{i\omega}{D}} x} e^{i\omega t} \tag{34}$$

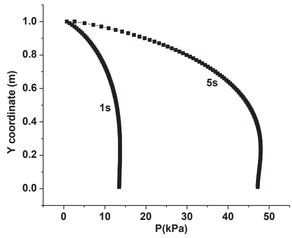
The solution of the problem (eqs.(21), (27),(32) and (34)) is formed by stable terms and decaying terms in x direction, which means that the seepage and deformation decay gradually away from the load side.

Responses of soil layer can be obtained by computation with the above equations. Basic parameters are as follows: depth of soil layer L=1, porosity ε=0.3, density of water ρ_w=1000kg/m^3, density of skeleton ρ_s=2650kg/m^3, Poisson ratio μ=0.3, shear modulus G=10^{5}Pa, viscosity of water ν=0.001, physical permeability 10^{-10}m^2, coefficient a=10^5, frequency ω=1.0.

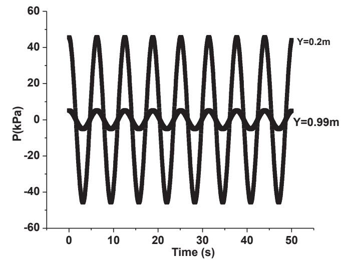
The numerical solutions show that the response of the soil decays gradually in horizontal direction. Pore pressure decreases fast in the range of 20% of the total length in horizontal direction near the load end (Fig.2a). In vertical direction, the pore pressure increases from the top to the bottom. Pore pressure increases fast in the upper part about 50% of the total depth (Fig.2b). At any location, pore pressure fluctuations with time (Fig.2c). It is shown that the numerical results are agreement well with the experimental results (Lu et al. 2007) (Fig.2d).



(a) In x direction(y=0.99m)

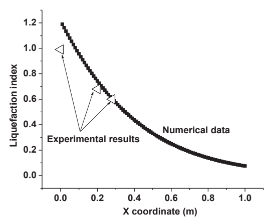


(b) In y direction (x=0.01m)



Development with time

(x=0.1, y=0.99)



(d)Comparison with experimental results (Fig. 14 in literature [3])

FIG.2. Development of pore pressure.

CONCLUSIONS

Dynamic responses of soil foundation under horizontal distributed dynamic load were analyzed theoretically. A method to obtain the direct analytical solution is presented. Based on liquid-solid media theory, two dimensional two-phase controlling equations are obtained. Flow function and potential function are introduced to decouple the controlling equations, which makes the solving of the equations much simple.

The response is formed by stable part and decaying part, which means that the response becomes to stable gradually. At about 1m from the load end, about equals to the length of the soil layer, the response disappears. So there is a maximum affected zone in the soil layer under dynamic load.

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NOMENCLATURE

ρ density of fluid
ρ_s density of solid
u_i displacements of solid in three directions
v_i velocities of fluid in three directions
ε porosity
p fluid pressure
σ_{ij} effective stress
g gravitational acceleration
G shear modulus
ν Poisson ratio
φ_s Potential functions of solid
ψ_s flow functions of solid
φ Potential functions of water
ψ flow functions of water
K the physical permeability $(= \rho_w g/(\mu k))$
ρ_w the density of the water
k the Darcy’s permeability
g the gravity acceleration