



# Thermocapillary migration of a droplet with a thermal source at large Reynolds and Marangoni numbers



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## ABSTRACT

The *unsteady* process for thermocapillary droplet migration at large Reynolds and Marangoni numbers has been previously reported by identifying a nonconservative integral thermal flux across the surface in the *steady* thermocapillary droplet migration (Wu and Hu, 2013) [15]. Here we add a thermal source in the droplet to preserve the integral thermal flux across the surface as conservative, so that thermocapillary droplet migration at large Reynolds and Marangoni numbers can reach a *quasi-steady* process. Under assumptions of *quasi-steady* state and non-deformation of the droplet, we make an analytical result for the *steady* thermocapillary migration of droplet with the thermal source at large Reynolds and Marangoni numbers. The result shows that the thermocapillary droplet migration speed slowly increases with the increase of Marangoni number.

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## 1. Introduction

Thermocapillary migration of a droplet or bubble at a uniform temperature gradient is a very interesting topic in both microgravity and microfluidics [1,2]. On the one hand, due to the development of space exploration, the studies of the physical mechanism of the droplet/bubble migration phenomena in the microgravity environment become more and more important. The first investigation of thermocapillary droplet migration was completed by Young et al (YGB) in 1959, who gave an analytical prediction for the migration speed at zero limit Reynolds ( $Re$ ) and Marangoni ( $Ma$ ) numbers [3]. Subramanian [4] proposed the quasi-steady state assumption for thermocapillary bubble migration and obtained analytical results with high order expansion at small  $Ma$  numbers. Balasubramaniam and Chai [5] derived an exact result for thermocapillary droplet migration at small  $Ma$  numbers. The experimental results for the droplet migration speed at small  $Re$  numbers obtained by Braun et al [6] are in agreement with those of the YGB model. On the other hand, the manipulation and actuation of droplets in microfluidic devices have an extensive application on the chemical industry and biological engineering. Many interest problems, such as static contact and dynamic control of the droplets on solid substrates, were theoretically analyzed and experimentally observed [7–10].

Although the thermocapillary droplet migration processes at small  $Ma$  numbers under the microgravity environment are understood very well in the series of theoretical analyses and experimental investigations, the physical behaviors at large  $Ma$  numbers are rather complicated due to the momentum and energy transfer through the interface of two-phase fluids. In large  $Re$  and  $Ma$  number area, it was reported [11] that the migration speed of a droplet increases with the increase of  $Ma$  number, as is in qualitative agreement with the corresponding numerical simulation [12]. The above theoretical analysis and numerical simulation are based on assumptions of the quasi-steady state and non-deformation of the droplet. The experimental investigation carried out by Hadland et al. [13] and Xie et al. [14] showed that the droplet migration speed non-denationalized by the YGB velocity decreases with the increase of  $Ma$  number. The results are not in qualitative agreement with the above theoretical and numerical ones. Moreover, a nonconservative integral thermal flux across the surface in the steady thermocapillary droplet migration at large  $Ma$  ( $Re$ ) numbers was identified [15]. It was indicated that the thermocapillary droplet migration at large  $Ma$  ( $Re$ ) numbers is an unsteady process. Therefore, the thermocapillary droplet migration at large  $Ma$  ( $Re$ ) numbers remains a topic to be studied with respect to its physical mechanism.

In this paper, first, under the assumption of the quasi-steady process of thermocapillary droplet migration at large  $Re$  and  $Ma$  numbers, a thermal source is added in the droplet to change the integral thermal flux across the surface from nonconservative to conservative. Then, we make an analytical result for the steady

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thermocapillary migration of droplet with the thermal source at large Re and Ma numbers and elucidate effects of the adding thermal source in the droplet on thermocapillary migration process.

**2. Problem formulation**

Consider the thermocapillary migration of a spherical droplet of radius  $R_0$ , density  $\gamma\rho$ , dynamic viscosity  $\alpha\mu$ , thermal conductivity  $\beta k$ , and thermal diffusivity  $\lambda\kappa$  in a continuous phase fluid of infinite extent with density  $\rho$ , dynamic viscosity  $\mu$ , thermal conductivity  $k$ , and thermal diffusivity  $\kappa$  under a uniform temperature gradient  $G$ . The rate of change of the interfacial tension between the droplet and the continuous phase fluid with temperature is denoted by  $\sigma_T$ . Fig. 1 displays a schematic diagram of thermocapillary migration of a droplet with a thermal source in a laboratory coordinate system denoted by a bar. Axisymmetric energy equations for the continuous phase and the fluid within the droplet are written as follows

$$\begin{aligned} \frac{\partial \bar{T}}{\partial t} + \bar{\mathbf{v}} \nabla \bar{T} &= \kappa \Delta \bar{T}, \\ \frac{\partial \bar{T}'}{\partial t} + \bar{\mathbf{v}}' \nabla \bar{T}' &= \lambda \kappa \Delta \bar{T}' + \Omega, \end{aligned} \tag{1}$$

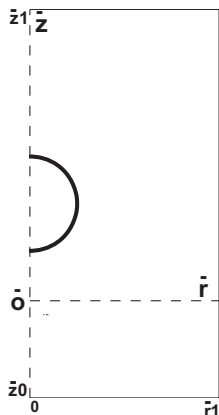
where  $\bar{\mathbf{v}}$  and  $\bar{T}$  are velocity and temperature, a prime and  $\Omega$  denote quantities and a thermal source in the droplet. By going over an unstable migration process from beginning, the droplet migration can reach a steady state, i.e., with the constant migration speed  $V_\infty$ . Using the coordinate transformation from the laboratory coordinate system to a coordinate system moving with the droplet velocity  $V_\infty$

$$\begin{aligned} \bar{\mathbf{r}} &= \mathbf{r} + V_\infty t \mathbf{k}, \quad \bar{\mathbf{v}}(\bar{\mathbf{r}}, t) = \mathbf{v}(\mathbf{r}) + V_\infty \mathbf{k}, \quad \bar{T}(\bar{\mathbf{r}}, t) = T(\mathbf{r}) + GV_\infty t, \\ \bar{\mathbf{v}}'(\bar{\mathbf{r}}, t) &= \mathbf{v}'(\mathbf{r}) + V_\infty \mathbf{k}, \quad \bar{T}'(\bar{\mathbf{r}}, t) = T'(\mathbf{r}) + GV_\infty t, \end{aligned} \tag{2}$$

the problem (1) can be formulated as

$$\begin{aligned} GV_\infty + \mathbf{v} \nabla T &= \kappa \Delta T, \\ GV_\infty + \mathbf{v}' \nabla T' &= \lambda \kappa \Delta T' + \Omega. \end{aligned} \tag{3}$$

By taking the radius of the droplet  $R_0$ , the YGB model velocity  $v_o = -\sigma_T GR_0 / \mu$  and  $GR_0$  as reference quantities to make coordinates, velocity and temperature dimensionless, the energy Eq. (3) can be written in the following dimensionless form in the spherical coordinate system  $(r, \theta)$  as follows



**Fig. 1.** A schematic diagram of thermocapillary migration of droplet with a thermal source in a laboratory coordinate system.

$$1 + u \frac{\partial T}{\partial r} + \frac{v}{r} \frac{\partial T}{\partial \theta} = \epsilon^2 \Delta T, \tag{4}$$

$$1 + u' \frac{\partial T'}{\partial r} + \frac{v'}{r} \frac{\partial T'}{\partial \theta} = \lambda \epsilon^2 \Delta T' + \frac{\Omega}{G v_o V_\infty}, \tag{5}$$

where the small parameter  $\epsilon$  and Ma number are defined, respectively, as

$$\epsilon = \frac{1}{\sqrt{Ma V_\infty}} \tag{6}$$

and

$$Ma = \frac{v_o R_0}{\kappa}. \tag{7}$$

In the space experiments [13,14], Prandtl numbers ( $Pr = \frac{\mu}{\rho\kappa}$ ) are constants in terms of the fixed fluid media, but Ma numbers are changed duo to the variable droplet sizes. It is common to adopt Re numbers ( $Re = \frac{\rho v_o R_0}{\mu} = \frac{Ma}{Pr}$ ) instead of Pr numbers in order to capture velocity fields with the changes of Re numbers. In general, the deformation of droplet usually depends on the Weber numbers [ $We = \frac{\rho v_o^2 R}{\sigma_o} = Ca Re$ ,  $Ca (= \frac{v_o \mu}{\sigma_o})$  is the Capillary number], it is not included in this study due to the small We numbers in the space experiments. For example, We is in  $O(10^{-3}) - O(10^{-1})$  when  $Ma < 1000$  [14,16]. Thus, the solutions of Eqs. (4) and (5) have to satisfy the following boundary conditions at the interface of two phase fluids ( $r = 1$ )

$$T(1, \theta) = T'(1, \theta), \tag{8}$$

$$\frac{\partial T}{\partial r}(1, \theta) = \beta \frac{\partial T'}{\partial r}(1, \theta) \tag{9}$$

and at places far away from the droplet

$$T \rightarrow r \cos \theta, \quad \text{as } r \rightarrow \infty. \tag{10}$$

Besides the above temperature boundary conditions, the stress boundary conditions at the interface are also applied for the steady droplet migration. However, the normal stress boundary condition is removed in terms of the non-deformation assumption. The shear stress boundary condition is expressed by

$$\tau_{r\theta} - \alpha \tau'_{r\theta} = \frac{1}{V_\infty r} \frac{\partial T}{\partial \theta}, \tag{11}$$

where

$$\tau_{r\theta} = r \frac{\partial}{\partial r} \left( \frac{v}{r} \right) + \frac{1}{r} \frac{\partial u}{\partial \theta}. \tag{12}$$

For large Re numbers, the inner and outer momentum boundary layers at  $r$ -direction are introduced near the surface of droplet. In the flow fields outside the momentum boundary layers, there are still the potential flows. The scaled inviscid velocity field in the continuous phase and Hill's spherical vortex within the droplet can be written as, respectively [17]

$$U = -\cos \theta \left( 1 - \frac{1}{r^3} \right), \tag{13}$$

$$V = \sin \theta \left( 1 + \frac{1}{2r^3} \right)$$

and

$$U' = \frac{3}{2} \cos \theta (1 - r^2), \tag{14}$$

$$V' = -\frac{3}{2} \sin \theta (1 - 2r^2).$$

By introducing the perturbation velocity fields  $\hat{\mathbf{v}} = \mathbf{v} - \mathbf{U}$ ,  $\hat{\mathbf{v}}' = \mathbf{v}' - \mathbf{U}'$  and the depth of boundary layer  $\delta x = r - 1$  ( $\delta = \sqrt{\frac{1}{Re}}$ )

into the boundary condition (11) and the momentum equation for the continuous fluid, they are derived and truncated in the leading orders  $O(1/\delta)$  and  $O(\delta)$  as, respectively [17]

$$\frac{\partial \hat{v}}{\partial x} = \frac{3}{2}(2 + 3\alpha) \sin \theta + \frac{1}{V_\infty} \frac{\partial T}{\partial \theta}. \quad (15)$$

and

$$\frac{3}{2} \frac{\partial}{\partial \theta} (\hat{v} \sin \theta) - 3x \cos \theta \frac{\partial \hat{v}}{\partial x} = \frac{\partial^2 \hat{v}}{\partial x^2}, \quad (16)$$

where  $\frac{\partial}{\partial r} = O(\frac{1}{\delta})$ ,  $U/U' = O(\delta)/O(\delta)$ ,  $V/V' = O(1)/O(1)$ ,  $\hat{u}/\hat{u}' = O(\delta^2)/O(\delta^2)$  and  $\hat{v}/\hat{v}' = O(\delta)/O(\delta)$  are in the momentum boundary layers. Following the derivation for the steady migration speed of a bubble at large Re numbers [18], the migration speed of the droplet is written as follows

$$\begin{aligned} V_\infty &= -\frac{1}{2(2 + 3\alpha)} \int_0^\pi \sin^2 \theta \frac{\partial T}{\partial \theta}(1, \theta) d\theta \\ &= \frac{1}{2 + 3\alpha} \int_0^\pi \sin \theta \cos \theta T(1, \theta) d\theta. \end{aligned} \quad (17)$$

Thus, to determine  $V_\infty$ , the temperature  $T(1, \theta)$  on the surface of droplet is required. However, before the energy equations are solved, the self-consistency of temperature fields for the steady droplet migration system will be determined. It means that the solutions of steady energy equations will satisfy the conservative integral thermal flux across the surface of droplet. In this case, the temperature field at infinity in Eq. (10) was derived by using the asymptotic expansion method and expressed as [15]

$$T \approx r \cos \theta - \frac{1}{2r^2} \cos \theta + o(1). \quad (18)$$

Integrating Eqs. (4) and (5) in the continuous phase domain ( $r \in [1, r_\infty]$ ,  $\theta \in [0, \pi]$ ) with boundary condition (18) and within the droplet region ( $r \in [0, 1]$ ,  $\theta \in [0, \pi]$ ), respectively, we obtain

$$\int_0^\pi \frac{\partial T}{\partial r}(1, \theta) \sin \theta d\theta = -\frac{1}{3\epsilon^2} \quad (19)$$

and

$$\int_0^\pi \frac{\partial T'}{\partial r}(1, \theta) \sin \theta d\theta = \frac{2}{3\lambda\epsilon^2} \left(1 - \frac{\Omega}{Gv_0V_\infty}\right). \quad (20)$$

From Eqs. (19) and (20), we have

$$\begin{aligned} \beta \int_0^\pi \frac{\partial T'}{\partial r}(1, \theta) \sin \theta d\theta - \int_0^\pi \frac{\partial T}{\partial r}(1, \theta) \sin \theta d\theta \\ = \int_0^\pi \left[ \beta \frac{\partial T'}{\partial r}(1, \theta) - \frac{\partial T}{\partial r}(1, \theta) \right] \sin \theta d\theta \\ = \frac{1}{3\epsilon^2} \left(1 + \frac{2\beta}{\lambda}\right) - \frac{2\beta\Omega}{3\lambda Gv_0}. \end{aligned} \quad (21)$$

For large Ma numbers and finite  $V_\infty$ , Eqs. (19) and (20) should satisfy the thermal flux boundary condition (9), i.e., the right side of Eq. (21) will be zero. So, we have

$$\Omega = \frac{\lambda Gv_0V_\infty}{2\beta} \left(1 + \frac{2\beta}{\lambda}\right) = Gv_0V_\infty \left(1 + \frac{\lambda}{2\beta}\right). \quad (22)$$

Using Eq. (22), Eq.(5) is rewritten as

$$u' \frac{\partial T'}{\partial r} + \frac{v'}{r} \frac{\partial T'}{\partial \theta} = \lambda\epsilon^2 \Delta T' + \frac{\lambda}{2\beta}. \quad (23)$$

In following, we will focus on the steady thermocapillary migration of droplet with the thermal source under a uniform temperature gradient and determine the dependence of the migration speed on large Ma numbers.

### 3. Analysis and results

#### 3.1. Outer temperature field in the continuous phase

By using an outer expansion for the scaled temperature field in the continuous phase

$$T = T_0 + T_1\epsilon + O(\epsilon^2), \quad (24)$$

the energy equation for the outer temperature field in its leading order can be obtained from Eq. (4) as follows

$$1 + U \frac{\partial T_0}{\partial r} + \frac{V}{r} \frac{\partial T_0}{\partial \theta} = 0. \quad (25)$$

By using the coordinate transformation from  $(r, \theta)$  to  $(\psi_0, \theta)$  to solve Eq. (25), its solution can be written as

$$T_0(r, \theta) = G(\psi_0) - \int \frac{2r^4}{2r^3 + 1} \frac{d\theta}{\sin \theta}, \quad (26)$$

where  $G(\psi_0)$  is a function of  $\psi_0$  (the streamfunction in the continuous phase). Following [19], the solution near  $r = 1$  is simplified as

$$\begin{aligned} T_0(r, \theta) &= \left(1 + \frac{\pi}{6\sqrt{3}} - \frac{1}{6} \ln 432\right) \\ &\quad - \frac{1}{18} \left(\frac{\pi}{\sqrt{3}} + \ln 432\right) \left(r^2 - \frac{1}{r}\right) \sin^2 \theta + \frac{1}{3} \\ &\quad \times \ln \left(r^2 - \frac{1}{r}\right) + \frac{2}{3} \ln(1 + \cos \theta) + \frac{2}{9} \left(r^2 - \frac{1}{r}\right) \cos \theta \\ &\quad + \frac{1}{9} \left(r^2 - \frac{1}{r}\right) \ln \left(r^2 - \frac{1}{r}\right) \sin^2 \theta \\ &\quad + \frac{2}{9} \left(r^2 - \frac{1}{r}\right) \sin^2 \theta \ln(1 + \cos \theta). \end{aligned} \quad (27)$$

By using the boundary layer approximation

$$x = \frac{r-1}{\epsilon}, \quad (28)$$

the temperature field near interface can be expressed as

$$\begin{aligned} t(x, \theta) &= 1 + \frac{\pi}{6\sqrt{3}} - \frac{1}{6} \ln 48 + \frac{2}{3} \ln \left(\frac{1 + \cos \theta}{\sin \theta}\right) \\ &\quad + \frac{1}{3} x \sin^2 \theta \epsilon \ln \epsilon + O(\epsilon). \end{aligned} \quad (29)$$

#### 3.2. Outer temperature field within the droplet

By using the following outer expansion for the scaled temperature field within the droplet in Eq. (23)

$$T' = \frac{1}{\epsilon^2} T'_{-2} + T'_0 + O(\epsilon), \quad (30)$$

the equation in its leading order can be written as

$$U' \frac{\partial T'_{-2}}{\partial r} + \frac{V'}{r} \frac{\partial T'_{-2}}{\partial \theta} = 0. \quad (31)$$

Its solution is

$$T'_{-2} = F_0(\psi'), \quad (32)$$

where  $\psi' = \frac{3}{4} \sin^2 \theta (r^4 - r^2)$  is the streamfunction within the droplet. The unknown function  $F_0(\psi')$  can be obtained from the following equation for the temperature field  $T'_0$  in its second order

$$U' \frac{\partial T'_0}{\partial r} + \frac{V'}{r} \frac{\partial T'_0}{\partial \theta} = \lambda \Delta F_0 + \frac{\lambda}{2\beta}. \quad (33)$$

To solve Eq. (33), we use the coordinate transformation from  $(r, \theta)$  to  $(m, q)$ , with streamlines and their orthogonal lines as coordinate axes defined as

$$m = -\frac{16}{3}\psi',$$

$$q = \frac{r^4 \cos^4 \theta}{2r^2 - 1}. \tag{34}$$

Eq. (33) can thus be written in the  $(m, q)$  coordinate system as follows

$$\frac{V_q}{h_q} \frac{\partial T'_0}{\partial q} = \frac{\lambda}{h_m h_q h_\phi} \frac{\partial}{\partial m} \left( \frac{h_q h_\phi}{h_m} \frac{dF_0}{dm} \right) + \frac{\lambda}{2\beta}, \tag{35}$$

where  $h_m, h_q, h_\phi$  are the metrical coefficients for the transformation. Multiplying both sides of Eq. (35) by  $h_m h_q h_\phi$  and integrating with respect to  $q$  for one circuit along a streamline, we obtain the following equation

$$\oint V_q h_m h_\phi \frac{\partial T'_0}{\partial q} dq = \lambda \frac{\partial}{\partial m} \left[ J(m) \frac{dF_0}{dm} \right] + \frac{\lambda}{2\beta} H(m), \tag{36}$$

where  $H(m) = \oint h_m h_q h_\phi dq$  and  $J(m) = \oint \frac{h_q h_\phi}{h_m} dq$ . From the equation of continuity, it can be shown that  $V_q h_m h_\phi$  is a constant. We thus obtain a solution of Eq. (36) as follows

$$F_0 = -\frac{1}{2\beta} \left[ B + \int_0^m \frac{dx}{J(x)} \int_1^x H(s) ds \right]. \tag{37}$$

Near  $m = 0$ ,  $J(m)$  and  $H(m)$  can be expanded, respectively

$$J(m) = \frac{16}{3} - 5m + O(m^2 \ln m) \tag{38}$$

and

$$H(m) = \frac{3}{8} \ln 2 - \frac{1}{16} \ln m + O(m \ln m). \tag{39}$$

Using Eqs. (38) and (39), we thus obtain from Eq. (37)

$$T'_{-2}(r, \theta) = F_0 = -\frac{1}{2\beta} \left[ B - \frac{1}{16} m + \frac{3}{256} (3 \ln 2 - 1 \frac{3}{4}) m^2 - \frac{3}{512} m^2 \ln m \right] + O(m^3 \ln m). \tag{40}$$

By using the boundary layer approximation

$$x' = \frac{1-r}{\sqrt{\lambda \epsilon}}, \tag{41}$$

the temperature field near interface can be expressed as follows

$$t'(x', \theta) = \frac{\sqrt{\lambda}}{4\beta} x' \sin^2 \theta \frac{1}{\epsilon} + \frac{3\lambda}{16\beta} x'^2 \sin^4 \theta \ln \epsilon + o(\ln \epsilon). \tag{42}$$

### 3.3. Inner temperature fields in the leading order

From Eqs. (19) and (20), the leading orders of the inner temperature fields in the continuous phase and within the droplet can be obtained, respectively. A schematic diagram of the inner and outer regions with the leading orders near the surface of droplet is shown in Fig. 2. By using inner expansions for the continuous phase and the fluid in the droplet

$$t(x, \theta) = t_{-1} \frac{1}{\epsilon} + t_{10} \ln \epsilon + t_0 + t_{11} \epsilon \ln \epsilon + O(\epsilon), \tag{43}$$

$$t'(x', \theta) = t'_{-1} \frac{1}{\epsilon} + t'_{10} \ln \epsilon + t'_0 + t'_{11} \epsilon \ln \epsilon + O(\epsilon) \tag{44}$$

and the inner variables given in Eqs. (28) and (41), the scaled energy equations for the inner temperature fields in the leading orders can be written as follows

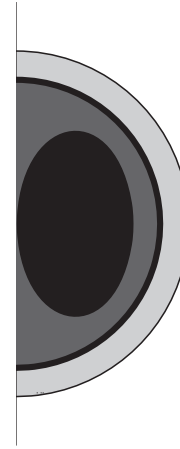


Fig. 2. A schematic diagram of the inner and outer temperature fields denoted by different colors near the surface of droplet in a coordinate system moving with the droplet velocity  $V_\infty$ . The light/dark gray and the white/black regions represent the inner  $[t/T' \sim O(\frac{1}{2})/O(\frac{1}{2})]$  and outer  $[T/T' \sim O(1)/O(\frac{1}{2})]$  temperature fields in the continuous phase/within the droplet, respectively.

$$-3x \cos \theta \frac{\partial t_{-1}}{\partial x} + \frac{3}{2} \sin \theta \frac{\partial t_{-1}}{\partial \theta} = \frac{\partial^2 t_{-1}}{\partial x^2}, \tag{45}$$

$$-3x' \cos \theta \frac{\partial t'_{-1}}{\partial x'} + \frac{3}{2} \sin \theta \frac{\partial t'_{-1}}{\partial \theta} = \frac{\partial^2 t'_{-1}}{\partial x'^2}. \tag{46}$$

The boundary conditions are

$$t_{-1}(0, \theta) = t'_{-1}(0, \theta),$$

$$\delta \frac{\partial t_{-1}}{\partial x}(0, \theta) = -\frac{\partial t'_{-1}}{\partial x'}(0, \theta), \tag{47}$$

$$t_{-1}(x \rightarrow \infty, \theta) \rightarrow 0,$$

$$t'_{-1}(x' \rightarrow \infty, \theta) \rightarrow B + \frac{\delta}{4} x' \sin^2 \theta,$$

where  $\delta = \sqrt{\lambda}/\beta$ . We transform the independent variables from  $[(x, x'), \theta]$  to  $[(\eta, \eta'), \xi]$  and the functions from  $(t_{-1}, t'_{-1})$  to  $(f_0, f'_0)$  as

$$(\eta, \eta') = \left( \frac{3}{2} x \sin^2 \theta, \frac{3}{2} x' \sin^2 \theta \right), \tag{48}$$

$$\xi = \frac{1}{2} (2 - 3 \cos \theta + \cos^3 \theta) = \frac{1}{2} (2 + \cos \theta) (1 - \cos \theta)^2$$

and

$$f_0(\eta, \xi) = t_{-1}(x, \theta),$$

$$f'_0(\eta', \xi) = t'_{-1}(x', \theta) - B - \frac{\delta}{4} x' \sin^2 \theta. \tag{49}$$

The corresponding energy equations for  $f_0, f'_0$  and boundary conditions can be written as follows

$$\frac{\partial f_0}{\partial \xi} = \frac{\partial^2 f_0}{\partial \eta^2},$$

$$\frac{\partial f'_0}{\partial \xi} = \frac{\partial^2 f'_0}{\partial \eta'^2} \tag{50}$$

and

$$f_0(0, \xi) = f'_0(0, \xi) + B,$$

$$\delta \frac{\partial f_0}{\partial \eta}(0, \xi) = -\frac{\partial f'_0}{\partial \eta'}(0, \xi) - \frac{\delta}{6}, \tag{51}$$

$$f_0(\eta \rightarrow \infty, \xi) = 0,$$

$$f'_0(\eta' \rightarrow \infty, \xi) = 0.$$

To solve Eq. (50), initial conditions are provided below

$$\begin{aligned} f_0(\eta, 0) &= 0, \\ f'_0(\eta', 0) &= f'_0(\eta', \xi(\pi)) = f'_0(\eta', 2) = g_0(\eta'), \\ g_0(\eta' \rightarrow \infty) &\rightarrow 0. \end{aligned} \tag{52}$$

Following the methods given by Harper and Moore [17], the solution of Eq. (50) can be obtained as

$$\begin{aligned} f_0(\eta, \xi) &= \frac{1}{1+\delta} \left\{ \left( B - \frac{\delta}{6}\eta \right) \operatorname{erfc} \left( \frac{\eta}{2\sqrt{\xi}} \right) + \frac{\delta\sqrt{\xi}}{3\sqrt{\pi}} \exp \left( -\frac{\eta^2}{4\xi} \right) \right. \\ &\quad \left. + \frac{1}{\sqrt{\pi\xi}} \int_0^\infty g_0(\eta^*) \exp \left[ -\frac{(\eta+\eta^*)^2}{4\xi} \right] d\eta^* \right\}, \end{aligned} \tag{53}$$

$$\begin{aligned} f'_0(\eta', \xi) &= \frac{\delta}{1+\delta} \left\{ -\left( B + \frac{1}{6}\eta' \right) \operatorname{erfc} \left( \frac{\eta'}{2\sqrt{\xi}} \right) + \frac{\sqrt{\xi}}{3\sqrt{\pi}} \exp \left( -\frac{\eta'^2}{4\xi} \right) \right\} \\ &\quad + \frac{1}{2\sqrt{\pi\xi}} \int_0^\infty g_0(\eta^*) \left\{ \exp \left[ -\frac{(\eta'-\eta^*)^2}{4\xi} \right] + \frac{1-\delta}{1+\delta} \right. \\ &\quad \left. \exp \left[ -\frac{(\eta'+\eta^*)^2}{4\xi} \right] \right\} d\eta^*. \end{aligned} \tag{54}$$

From Eqs. (49) and (53), we obtain the inner temperature field in its leading order near the surface of droplet

$$\begin{aligned} t_{-1}(0, \theta) &= f_0(0, \xi) \\ &= \frac{1}{1+\delta} \left[ B + \frac{\delta\sqrt{\xi}}{3\sqrt{\pi}} + \frac{1}{\sqrt{\pi\xi}} \int_0^\infty g_0(\eta^*) \exp \left( -\frac{\eta^{*2}}{4\xi} \right) d\eta^* \right] \\ &= \frac{1}{1+\delta} \left[ B + \frac{\delta\sqrt{\xi}}{3\sqrt{\pi}} + \frac{2}{\sqrt{\pi}} \int_0^\infty g_0(2\xi^{1/2}\zeta) \exp(-\zeta^2) d\zeta \right]. \end{aligned} \tag{55}$$

When the inner expansion in the temperature field (43) is truncated at the zero order, we rewrite Eq. (17) as

$$V_\infty = \frac{1}{2+3\alpha} \int_0^\pi \sin \theta \cos \theta \left[ t_{-1}(0, \theta) \frac{1}{\epsilon} + t_{10}(0, \theta) \ln \epsilon \right] d\theta. \tag{56}$$

Since  $\epsilon = 1/\sqrt{MaV_\infty}$ , the migration speed of the droplet is evaluated as

$$V_\infty \approx a_1^2 Ma - 2a_{10} \ln Ma + a_0, \tag{57}$$

where

$$a_1 = \frac{1}{2+3\alpha} \int_0^\pi \sin \theta \cos \theta t_{-1}(0, \theta) d\theta \tag{58}$$

and

$$a_{10} = \frac{1}{2+3\alpha} \int_0^\pi \sin \theta \cos \theta t_{10}(0, \theta) d\theta. \tag{59}$$

Substituting Eqs. (55) into Eq. (58), we obtain

$$\begin{aligned} a_1 &= \frac{\delta}{3\sqrt{\pi}(2+3\alpha)(1+\delta)} \int_0^\pi \sin \theta \cos \theta \xi^{1/2} d\theta \\ &\quad + \frac{2}{\sqrt{\pi}(2+3\alpha)(1+\delta)} \int_0^\pi \sin \theta \cos \theta \left[ \int_0^\infty g_0(2\xi^{1/2}\zeta) \right. \\ &\quad \left. \times \exp(-\zeta^2) d\zeta \right] d\theta. \end{aligned} \tag{60}$$

To determine the function  $g_0$  in Eq. (60), we use the boundary condition within the droplet at the front and rear stagnation points in Eq. (52)

$$\begin{aligned} g_0(\eta') &= \frac{\delta}{1+\delta} \left\{ -\left( B + \frac{1}{6}\eta' \right) \operatorname{erfc} \left( \frac{\eta'}{2\sqrt{2}} \right) + \frac{\sqrt{2}}{3\sqrt{\pi}} \exp \left( -\frac{\eta'^2}{8} \right) \right\} + \frac{1}{2\sqrt{2\pi}} \\ &\quad \times \int_0^\infty g_0(\eta^*) \left\{ \exp \left[ -\frac{(\eta'-\eta^*)^2}{8} \right] + \frac{1-\delta}{1+\delta} \exp \left[ -\frac{(\eta'+\eta^*)^2}{8} \right] \right\} d\eta^*. \end{aligned} \tag{61}$$

The integral of the third term on the right-hand side of Eq. (61) is approximated as

$$\begin{aligned} \int_0^\infty g_0(\eta^*) h(\eta', \eta^*) d\eta^* &= \int_0^{\eta'_i} g_0(\eta^*) h(\eta', \eta^*) d\eta^* + g_0(\eta'_i) \\ &\quad \times \int_{\eta'_i}^\infty h(\eta', \eta^*) d\eta^*. \end{aligned} \tag{62}$$

Then, Eq. (61) is evaluated in a linear system of equations

$$\begin{aligned} g_0(\eta') &= -\frac{1}{4\sqrt{2\pi}} g_0(\eta'_i) \left\{ \exp \left[ -\frac{(\eta'-\eta'_i)^2}{8} \right] + \frac{1-\delta}{1+\delta} \exp \left[ -\frac{(\eta'+\eta'_i)^2}{8} \right] \right\} \Delta\eta^* \\ &\quad - \frac{1}{4\sqrt{2\pi}} g_0(\eta'_{N+1}) \left\{ \exp \left[ -\frac{(\eta'-\eta'_{N+1})^2}{8} \right] + \frac{1-\delta}{1+\delta} \exp \left[ -\frac{(\eta'+\eta'_{N+1})^2}{8} \right] \right\} \Delta\eta^* \\ &\quad - \frac{1}{2\sqrt{2\pi}} \sum_{j=2}^N g_0(\eta'_j) \left\{ \exp \left[ -\frac{(\eta'-\eta'_j)^2}{8} \right] + \frac{1-\delta}{1+\delta} \exp \left[ -\frac{(\eta'+\eta'_j)^2}{8} \right] \right\} \Delta\eta^* \\ &\quad - \frac{1}{2} g_0(\eta'_{N+1}) \left[ \operatorname{erfc} \left( \frac{\eta'_{N+1}+\eta'}{2\sqrt{2}} \right) + \frac{1-\delta}{1+\delta} \operatorname{erfc} \left( \frac{\eta'_{N+1}+\eta'}{2\sqrt{2}} \right) \right] \\ &= \frac{\delta}{1+\delta} \left[ -\left( B + \frac{1}{6}\eta' \right) \operatorname{erfc} \left( \frac{\eta'}{2\sqrt{2}} \right) + \frac{\sqrt{2}}{3\sqrt{\pi}} \exp \left( -\frac{\eta'^2}{8} \right) \right], \end{aligned} \tag{63}$$

where  $\eta'_{N+1} = \eta'_i$  and  $\Delta\eta^* = \eta'_i/N$ . The physical coefficients used in the space experiment [12] with the uniform temperature gradient  $G = 12$  K/cm for the continuous phase of Fluorinert FC-75 and the droplet of 5cst silicon oil at  $T = 333$  K are adopted here to yield  $\alpha = 0.342$ ,  $\beta = 0.571$  and  $\lambda = 0.299$ . A typical value for  $\eta'_i$  is chosen as 3. Using the trial and error method to satisfy the above approximation, we determine the unknown constant  $B = 0.1935$  and obtain the dependence of  $g_0$  on  $\eta'$  as shown in Fig. 3. From Eq. (60), we can determine the root-mean-square of the leading order term of the migration speed as

$$a_1 = 1.15 \times 10^{-2}. \tag{64}$$

Although equations and boundary conditions describing the second order term of the migration speed can be obtained, we are unable to find an analytical result for  $t_{10}$  in Eq. (56). Under the truncation after the leading order term in Eq. (57), we obtain the migration speed of the droplet

$$V_\infty \approx 1.3 \times 10^{-4} \text{ Ma}. \tag{65}$$

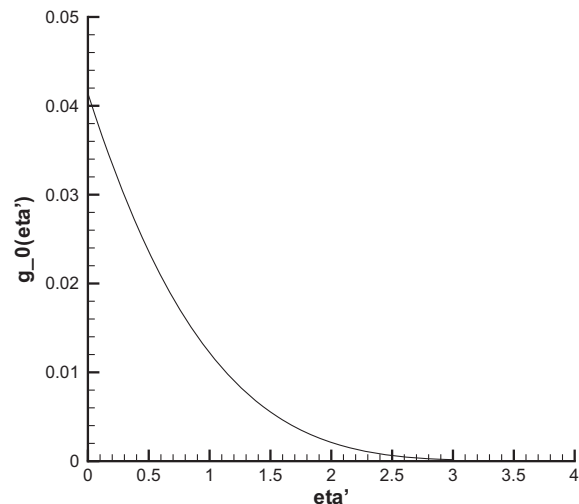


Fig. 3. Function  $g_0$  versus  $\eta'$  determined from Eq. (63).

By using the migration speed  $V_\infty$ , the adding thermal source  $\Omega$  in Eq. (22) can be determined as

$$\Omega = G\nu_0 V_\infty \left(1 + \frac{\lambda}{2\beta}\right) \approx 0.1 \text{ Ma K/s.} \quad (66)$$

#### 4. Conclusions and discussion

In this paper, the conservative integral thermal flux across the surface for a steady thermocapillary droplet migration in a uniform temperature gradient at large  $Re$  and  $Ma$  numbers has been guaranteed by adding a thermal source in the droplet. Under the assumptions of quasi-steady state and non-deformation of the droplet, we have made an analytical result for the steady thermocapillary migration of droplet with the thermal source at large  $Re$  and  $Ma$  numbers. The result shows that the thermocapillary droplet migration speed slowly increases with the increase of  $Ma$  number.

We emphasize that the modeling of the adding thermal source in the droplet is used to investigate the effect of integral thermal flux across the surface on the thermocapillary droplet migration process. It is one of the physical means to satisfy the conservative integral thermal flux boundary condition in the steady migration process at large  $Ma$  numbers [15]. On the one hand, in general, when the droplet moves forward under the uniform temperature gradient, the thermal energy in the droplet is imported from the upper surface of the droplet and exported from the lower surface. The adding thermal source in the droplet can decrease/increase the inner thermal flux from the interface/inside to the inside/interface at the upper/lower surface of the droplet and satisfy the thermal flux boundary condition at the surface of droplet for the steady migration process. On the other hand, at large  $Ma$  numbers, the heat advection around the droplet is a more significant mechanism for heat transfer across/around the droplet. It is always keeping the weak transport of thermal energy from outside of the droplet to inside. Thus, the adding thermal source is conducive to providing the thermal energy in the droplet to meet the requirement put forward by the quasi-steady migration process. Meanwhile, the heat transport across the droplet surface becomes weaker than the heat convection as  $Ma$  number increases. To reach the steady migration process, more thermal energy in the droplet will be provided. The proportional relationship of the thermal source to  $Ma$  number in Eq. (66) qualitatively confirms the above requirement of more thermal energy in the droplet.

In final, some suggestions are provided in order that the thermal source in the droplet can be implemented in a real experiment. As given in Eqs. (1) and (66) and shown in Fig. 1, the thermal source depending on  $Ma$  number will be added in the droplet when it starts to move under the uniform temperature gradient. The heat radiant technology may be one of the possible physical means to add the thermal source in the droplet.

#### Conflict of Interest

None declared.

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