Journal of Thermal Stresses
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/uths20

General Solutions for Three-Dimensional Thermoelasticity of Two-Dimensional Hexagonal Quasicrystals and an Application

Lianzhi Yang [a b], Liangliang Zhang [a b], Fan Song [c] & Yang Gao [a]

[a] College of Science, China Agricultural University, Beijing, China

[b] College of Engineering, China Agricultural University, Beijing, China

[c] State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing, China

Published online: 03 Mar 2014.

To cite this article: Lianzhi Yang, Liangliang Zhang, Fan Song & Yang Gao (2014) General Solutions for Three-Dimensional Thermoelasticity of Two-Dimensional Hexagonal Quasicrystals and an Application, Journal of Thermal Stresses, 37:3, 363-379, DOI: 10.1080/01495739.2013.869149

To link to this article: http://dx.doi.org/10.1080/01495739.2013.869149

Journal of Thermal Stresses, 37: 363–379, 2014

ISSN: 0149-5739 print/1521-074X online
DOI: 10.1080/01495739.2013.869149

GENERAL SOLUTIONS FOR THREE-DIMENSIONAL THERMOELASTICITY OF TWO-DIMENSIONAL HEXAGONAL QUASICRYSTALS AND AN APPLICATION

Lianzhi Yang[1,2], Liangliang Zhang[1,2], Fan Song[3], and Yang Gao[1]
[1]*College of Science, China Agricultural University, Beijing, China*
[2]*College of Engineering, China Agricultural University, Beijing, China*
[3]*State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing, China*

By introducing four displacement functions, the governing equations of the three-dimensional thermoelasticity of two-dimensional hexagonal quasicrystals are decoupled into two uncorrelated problems. Two higher-order displacement functions are introduced to represent the general solutions, which are eighth-order and fourth-order, respectively, for the two problems. By taking a decomposition and superposition procedure, the general solutions are further simplified in seven cases in terms of six quasi-harmonic displacement functions. To show the application of the general solutions obtained, a closed form solution is obtained for an infinite space containing a penny-shaped crack, subjected to a uniformly distributed temperature at the crack surface.

Keywords: General solutions; Thermoelasticity; Two-dimensional quasicrystals

INTRODUCTION

Since the first observation of icosahedral quasicrystals (QCs) in Al-Mn alloys around 1984 [1], the mechanical and thermal properties of QCs have been intensively investigated in experimental and theoretical analyses [2, 3], which show their complex structures and unusual properties. Unlike crystals and glassy solids, QCs have a long-range quasi-periodic translational order and a long-range orientational order, and process noncrystallo-graphic rational symmetry, e.g., fivefold, eightfold, tenfold, or twelvefold symmetry axes [4]. All QCs are spatial and of three-dimensional (3D) bodies [5].

According to the quasi-periodic directions of QCs, there are three kinds of QCs, which respectively are one-, two-, and three-dimensional QCs. The one-dimensional (1D) QCs are the ones in which the atom arrangement is quasi-periodic in one direction and periodic in the other two directions. The two-dimensional (2D) QCs belong to ones in which the atom arrangement is quasi-periodic in two directions and periodic in the other one. The 3D QCs behave in such a manner that the arrangement presents quasi-periodicity in all three directions.

Received 15 March 2013; accepted 24 July 2013.
Address correspondence to Yang Gao, College of Science, China Agricultural University, P.O. Box 74, Beijing 100083, China. E-mail: gaoyangg@gmail.com

Elasticity is one of the most important properties of QCs. Within the framework of the Landau–Lifshitz phenomenological theory of elementary excitation of condensed matter, Bak [6, 7] and Levine et al. [8] formulated the elastic energy theory of the QCs, in which the elastic fields of QCs include the phonon field and phason field. Phonons are related to translations of atoms (standard elasticity), while phasons are related to rearrangements of atomic configurations. Both phonons and phasons are considered as continuous medium filed variables within the elastic theory of QCs. Ding et al. [9] established the generalized linear elastic theory of QCs, which provides us with a fundamental theory based on the notion of a continuum model to describe the elastic behavior of QCs. For comprehensive and detailed presentation for the linear elasticity of QCs, the review by Hu et al. [10] and a monograph by Fan [11] are recommended.

General solutions play an important role in the analysis of the initial-boundary value problems of elasticity, because not only do they have theoretical merits themselves, they can also be benchmarks to clarify various numerical methods such as the finite element method and the boundary element method. A review about the progress in the study of general solutions of elasticity and their application is presented by Wang et al. [12]. Due to the introduction of phason field, the equations of QC elasticity are much more complicated than those of classical elasticity for crystals, and the analytical solutions are difficult to obtain. In recent years, some mathematical physics methods and function theories, such as the cylindrical coordinate system, Fourier representations, Hankel transform and operator matrix, have been used to seek the general solutions of some QCs [13, 14].

An increasing number of QCs with good thermal stability make thermoelasticity analyses for QCs become increasingly more important. Furthermore, in view of the fact that the QCs have a potential to be used as the components in drilling and nuclear storage facilities, it is very necessary to study the influence of temperature on QCs. For the general solutions of QCs, thermal effort is always beyond the scope of the studies. Recently, Wang and Zhang [15] have derived the general solutions for plane-strain thermoelasticity of 2D decagonal QCs; Li and Li [16] have deduced the general solutions for 3D thermoelasticity of 1D hexagonal QCs.

The general solutions for 3D thermoelasticity of more complicated configurations of QCs have not been attempted. The purpose of this article is to develop our previous work for 3D elasticity of 2D hexagonal QCs [14], and to obtain the general solutions of 3D thermoelasticity for the 2D QCs. To illustrate the application of the general solutions obtained, a closed form solution is derived for an infinite thermoelastic space containing a penny-shaped crack with radius a_0, subjected to a uniformly distributed temperature T_0 at the crack surface.

BASIC EQUATIONS

A 2D QC is defined as a 3D body of whose atom arrangement is periodic direction in the x_3-axis and quasi-periodic in the $x_1 - x_2$ plane referred to a coordinate system (x_1, x_2, x_3). There are 10 systems, i.e., triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal, pentagonal, decagonal, octagonal systems and 57 point groups [17]. Among them, six systems and 31 point groups are of crystal rotational symmetry; four systems and 26 point groups are of non-crystal rotational symmetry. The hexagonal system is of crystal rotational symmetry and consists of two Laue classes, specifically, Laue class 9 and Laue class 10.

The deformed state of the QCs requires a combined consideration of interrelated phonon and phason fields. In 2D QCs, a phason displacement field w_n $(n = 1, 2)$ exists in addition to a phonon displacement u_i $(i = 1, 2, 3)$. According to 2D QC linear elastic theory established by Ding et al. [9] , the strain-displacement relations for 2D QCs are given by

$$\varepsilon_{ij} = \left(\partial_j u_i + \partial_i u_j\right)/2, \quad w_{nj} = \partial_j w_n \tag{1}$$

where $j = 1, 2, 3$, repeated indices imply summation and $\partial_j = \partial/\partial x_j$, ε_{ij} and w_{nj} denote the phonon strain and phason strain, respectively.

In the absence of body force, the balance laws require

$$\partial_j \sigma_{ij} = 0, \quad \partial_j H_{nj} = 0 \tag{2}$$

where σ_{ij} and H_{nj}, respectively, denote the phonon and phason stresses.

For 2D hexagonal QCs, the point groups $6mm$, 622, $\bar{6}m2$, $6/mmm$ belong to Laue class 10. The linear constitutive equations of the QCs take the following form [10, 11]:

$$\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_1 w_{11} + R_2 w_{22} - \beta_1 T \\
\sigma_{22} &= C_{12}\varepsilon_{11} + C_{11}\varepsilon_{22} + C_{13}\varepsilon_{33} + R_2 w_{11} + R_1 w_{22} - \beta_1 T \\
\sigma_{33} &= C_{13}\varepsilon_{11} + C_{13}\varepsilon_{22} + C_{33}\varepsilon_{33} + R_3 w_{11} + R_3 w_{22} - \beta_3 T \\
\sigma_{23} &= \sigma_{32} = 2C_{44}\varepsilon_{23} + R_4 w_{23} \\
\sigma_{31} &= \sigma_{13} = 2C_{44}\varepsilon_{13} + R_4 w_{13} \\
\sigma_{12} &= \sigma_{21} = 2C_{66}\varepsilon_{12} + R_6 w_{12} + R_6 w_{21} \\
H_{11} &= R_1\varepsilon_{11} + R_2\varepsilon_{22} + R_3\varepsilon_{33} + K_1 w_{11} + K_2 w_{22} \\
H_{22} &= R_2\varepsilon_{11} + R_1\varepsilon_{22} + R_3\varepsilon_{33} + K_2 w_{11} + K_1 w_{22} \\
H_{23} &= 2R_4\varepsilon_{23} + K_4 w_{23} \\
H_{12} &= 2R_6\varepsilon_{12} + K_3 w_{12} + K_6 w_{21} \\
H_{13} &= 2R_4\varepsilon_{13} + K_4 w_{13} \\
H_{21} &= 2R_6\varepsilon_{12} + K_6 w_{12} + K_3 w_{21}
\end{aligned} \tag{3}$$

where $C_{11}, C_{12}, C_{13}, C_{33}, C_{44}$ represent the elastic constants in phonon field, K_1, K_2, K_3, K_4 are the elastic constants in phason field, R_1, R_2, R_3, R_4 are the phonon-phason coupling elastic constants, β_1, β_3 are the thermal constants, T is the variation of the temperature, and $2C_{66} = C_{11} - C_{12}$, $K_6 = K_1 - K_2 - K_3$, $2R_6 = R_1 - R_2$. It is noted that the phonon elastic constants in QCs can be measured by some experimental methods such as X-ray diffractions, neutron scattering and so on [18–21]. However, the experiments testing the elastic constants of QCs are mainly concentrated on the decagonal and icosahedral QCs. Up to now, the relevant data such as phason constants and phonon-phason coupling constants for 2D hexagonal QCs are still lacking.

According to the Fourier law, the relation between the thermal flux vector $\boldsymbol{q}$ and the temperature T is

$$q_1 = -k_{11}\partial_1 T, \quad q_2 = -k_{11}\partial_2 T, \quad q_3 = -k_{33}\partial_3 T \tag{4}$$

where k_{11} and k_{33} are coefficients of thermal conductivity.

Assuming that the thermoelastic loading changes slowly with time and without consideration of the rate of entropy, the uncoupled thermoelastic theory of QCs is adopted in the following analysis. Consequently, in a steady-state, the heat conductivity equation is

$$\partial_1 q_1 + \partial_2 q_2 + \partial_3 q_3 = 0 \tag{5}$$

Substituting Eq. (4) into Eq. (5), it can be seen that

$$\left(k_{11}\Lambda + k_{33}\partial_3^2\right) T = 0 \tag{6}$$

where $\Lambda = \partial_1^2 + \partial_2^2$ is the planar Laplacian.

Substituting Eqs. (1) and (3) into Eq. (2), the equilibrium equations expressed by phonon displacement u_i and phason displacement w_n can be written as follows:

$$\begin{aligned}
&\left(C_{11}\partial_1^2 + C_{66}\partial_2^2 + C_{44}\partial_3^2\right) u_1 + (C_{12} + C_{66})\,\partial_1\partial_2 u_2 + (C_{13} + C_{44})\,\partial_1\partial_3 u_3 \\
&\quad + \left(R_1\partial_1^2 + R_6\partial_2^2 + R_4\partial_3^2\right) w_1 + (R_2 + R_6)\,\partial_1\partial_2 w_2 - \beta_1\partial_1 T = 0 \\
&\left(C_{11}\partial_2^2 + C_{66}\partial_1^2 + C_{44}\partial_3^2\right) u_2 + (C_{12} + C_{66})\,\partial_1\partial_2 u_1 + (C_{13} + C_{44})\,\partial_2\partial_3 u_3 \\
&\quad + \left(R_1\partial_2^2 + R_6\partial_1^2 + R_4\partial_3^2\right) w_2 + (R_2 + R_6)\,\partial_1\partial_2 w_1 - \beta_1\partial_2 T = 0 \\
&(C_{13} + C_{44})\,\partial_1\partial_3 u_1 + (C_{13} + C_{44})\,\partial_2\partial_3 u_2 + \left(C_{44}\Lambda + C_{33}\partial_3^2\right) u_3 + (R_3 + R_4)\,\partial_1\partial_3 w_1 \\
&\quad + (R_3 + R_4)\,\partial_2\partial_3 w_2 - \beta_3\partial_3 T = 0 \\
&\left(R_1\partial_1^2 + R_6\partial_2^2 + R_4\partial_3^2\right) u_1 + (R_2 + R_6)\,\partial_1\partial_3 u_2 + (R_3 + R_4)\,\partial_1\partial_3 u_3 \\
&\quad + \left(K_1\partial_1^2 + K_3\partial_2^2 + K_4\partial_3^2\right) w_1 + (K_2 + K_6)\,\partial_1\partial_2 w_2 = 0 \\
&\left(R_1\partial_2^2 + R_6\partial_1^2 + R_4\partial_3^2\right) u_2 + (R_2 + R_6)\,\partial_1\partial_3 u_1 + (R_3 + R_4)\,\partial_2\partial_3 u_3 \\
&\quad + \left(K_1\partial_2^2 + K_3\partial_1^2 + K_4\partial_3^2\right) w_2 + (K_2 + K_6)\,\partial_1\partial_2 w_1 = 0
\end{aligned} \tag{7}$$

Introduce displacement functions F, f, G and g to represent the displacement components in Eq. (7) as

$$\begin{aligned}
u_1 &= \partial_1 F + \partial_2 f, \quad u_2 = \partial_2 F - \partial_1 f \\
w_1 &= \partial_1 G + \partial_2 g, \quad w_2 = \partial_2 G - \partial_1 g
\end{aligned} \tag{8}$$

The use of the displacement functions in Eq. (8) and temperature T allows us to reduce the balance Eq. (7) and heat conductivity Eq. (6) to two uncoupled problems, namely, problem I:

$$\boldsymbol{AU} = \boldsymbol{0} \tag{9}$$

and problem II:

$$\overline{\boldsymbol{A}}\overline{\boldsymbol{U}} = \boldsymbol{0} \tag{10}$$

where the vectors $U = [F, G, u_3, T]^T$, $\overline{U} = [f, g]^T$ (the superscript "T" denotes the transpose), A and $\overline{A}$ are 4×4 and 2×2 differential operator matrices, respectively, such as

$$A = \begin{bmatrix} C_{11}\Lambda + C_{44}\partial_3^2 & R_1\Lambda + R_4\partial_3^2 & (C_{13} + C_{44})\,\partial_3 & -\beta_1 \\ R_1\Lambda + R_4\partial_3^2 & K_1\Lambda + K_4\partial_3^2 & (R_3 + R_4)\,\partial_3 & 0 \\ (C_{13} + C_{44})\,\Lambda\partial_3 & (R_3 + R_4)\,\Lambda\partial_3 & C_{44}\Lambda + C_{33}\partial_3^2 & -\beta_3\partial_3 \\ 0 & 0 & 0 & k_{11}\Lambda + k_{33}\partial_3^2 \end{bmatrix} \tag{11}$$

$$\overline{A} = \begin{bmatrix} C_{66}\Lambda + C_{44}\partial_3^2 & R_6\Lambda + R_4\partial_3^2 \\ R_6\Lambda + R_4\partial_3^2 & K_3\Lambda + K_4\partial_3^2 \end{bmatrix} \tag{12}$$

In terms of the matrix Eqs. (9) and (10), the equilibrium Eq. (7) and heat conductivity Eq. (6) are decoupled into two uncorrelated problems: problem I, associated with displacement functions F, G, u_3 and temperature T, and problem II, associated with displacement functions f and g. It seems to be extremely difficult to find the solution by means of direct integration due to the complexity of the equations. Next, we will introduce two displacement functions to simplify the above complicated equations of the two problems. Furthermore, a decomposition and superposition procedure is manipulated on the complicated governing equation expressed by the two displacement functions.

General Solutions of Problem I

By means of the operator method [12, 22], the general solutions of problem I will be developed. To find out the general solutions of problem I, we introduce a 4×4 differential operator matrix $\boldsymbol{B}$, components B_{ij} of which are "algebraic complement minors" of $\boldsymbol{A}$ in Eq. (9), i.e.,

$$\boldsymbol{AB} = \boldsymbol{BA} = A_0\boldsymbol{I} \tag{13}$$

where A_0 is the "determinant" of the differential operator $\boldsymbol{A}$ and $\boldsymbol{I}$ is the unit matrix. Then the general solution of Eq. (9) can be expressed as

$$\boldsymbol{U} = \boldsymbol{B\varphi}$$

where the displacement function vector φ satisfies the following equation:

$$A_0\varphi = 0 \tag{14}$$

The "determinant" A_0 of $\boldsymbol{A}$ yields

$$A_0 = \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\left(a\partial_3^6 + b\Lambda\partial_3^4 + c\Lambda^2\partial_3^2 + d\Lambda^3\right)$$

with the coefficient a, b, c, and d given by

$$
\begin{aligned}
a &= C_{33}\left(C_{44}K_4 - R_4^2\right)\\
b &= C_{44}\left(K_4C_{44} - R_4^2\right) + C_{33}\left(C_{44}K_1 + C_{11}K_4 - 2R_1R_4\right) + 2R_4\left(C_{13} + C_{44}\right)\left(R_3 + R_4\right)\\
&\quad - C_{44}\left(R_3 + R_4\right)^2 - K_4\left(C_{13} + C_{44}\right)^2\\
c &= C_{33}\left(C_{11}K_1 - R_1^2\right) + C_{44}\left(C_{44}K_1 + C_{11}K_4 - 2R_1R_4\right) + 2R_1\left(C_{13} + C_{44}\right)\left(R_3 + R_4\right)\\
&\quad - C_{11}\left(R_3 + R_4\right)^2 - K_1\left(C_{13} + C_{44}\right)^2\\
d &= C_{44}\left(C_{11}K_1 - R_1^2\right)
\end{aligned}
$$

Now introduce a displacement function H, which satisfies the following equation:

$$\nabla_1^2\nabla_2^2\nabla_3^2\nabla_4^2 H = 0 \tag{15}$$

in which the quasi-harmonic differential operators ∇_q^2 are expressed as

$$\nabla_q^2 = \Lambda + \frac{1}{s_q^2}\partial_3^2 \tag{16}$$

$q = 1, 2, 3, 4$, $s_4^2 = k_{11}/k_{33}$, s_1^2, s_2^2 and s_3^2 are the three characteristic roots (or eigenvalues) of the following cubic algebra equation:

$$as^6 - bs^4 + cs^2 - d = 0 \tag{17}$$

By virtue of the operator analysis technique [12, 22], the four general solutions of problem I can be obtained as

$$F = B_{q1}H,\ G = B_{q2}H,\ u_3 = B_{q3}H,\ T = B_{q4}H \tag{18}$$

where components of the matrix $\boldsymbol{B}$ are in detail

$$
\begin{aligned}
B_{11} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\left[\left(K_1\Lambda + K_4\partial_3^2\right)\left(C_{44}\Lambda + C_{33}\partial_3^2\right) - \left(R_3 + R_4\right)^2\Lambda\partial_3^2\right]\\
B_{12} &= B_{21} = \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\left[\left(C_{13} + C_{44}\right)\left(R_3 + R_4\right)\Lambda\partial_3^2\right.\\
&\quad \left. - \left(C_{44}\Lambda + C_{33}\partial_3^2\right)\left(R_1\Lambda + R_4\partial_3^2\right)\right]\\
B_{13} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\Lambda\partial_3\left[\left(R_3 + R_4\right)\left(R_1\Lambda + R_4\partial_3^2\right) - \left(C_{13} + C_{44}\right)\left(K_1\Lambda + K_4\partial_3^2\right)\right]\\
B_{14} &= B_{24} = B_{34} = 0\\
B_{22} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\left[\left(C_{11}\Lambda + C_{44}\partial_3^2\right)\left(C_{44}\Lambda + C_{33}\partial_3^2\right) - \left(C_{13} + C_{44}\right)^2\Lambda\partial_3^2\right]\\
B_{23} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\Lambda\partial_3\left[\left(C_{13} + C_{44}\right)\left(R_1\Lambda + R_4\partial_3^2\right) - \left(R_3 + R_4\right)\left(C_{11}\Lambda + C_{44}\partial_3^2\right)\right]\\
B_{31} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\partial_3\left[\left(R_3 + R_4\right)\left(R_1\Lambda + R_4\partial_3^2\right) - \left(C_{13} + C_{44}\right)\left(K_1\Lambda + K_4\partial_3^2\right)\right]\\
B_{32} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\partial_3\left[\left(C_{13} + C_{44}\right)\left(R_1\Lambda + R_4\partial_3^2\right) - \left(R_3 + R_4\right)\left(C_{11}\Lambda + C_{44}\partial_3^2\right)\right]\\
B_{33} &= \left(k_{11}\Lambda + k_{33}\partial_3^2\right)\left[\left(C_{11}\Lambda + C_{44}\partial_3^2\right)\left(K_1\Lambda + K_4\partial_3^2\right) - \left(R_1\Lambda + R_4\partial_3^2\right)^2\right]
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
B_{41} &= \beta_3\left(R_1\Lambda + R_4\partial_3^2\right)\left(R_3 + R_4\right)\partial_3^2 - \beta_3\left(C_{13} + C_{44}\right)\left(K_1\Lambda + K_4\partial_3^2\right)\partial_3^2 \\
&\quad + \beta_1\left[\left(K_1\Lambda + K_4\partial_3^2\right)\left(C_{44}\Lambda + C_{33}\partial_3^2\right) - \left(R_3 + R_4\right)^2\Lambda\partial_3^2\right] \\
B_{42} &= -\beta_3\left(C_{11}\Lambda + C_{44}\partial_3^2\right)\left(R_3 + R_4\right)\partial_3^2 + \beta_3\left(C_{13} + C_{44}\right)\left(R_1\Lambda + R_4\partial_3^2\right)\partial_3^2 \\
&\quad - \beta_1\left[\left(R_1\Lambda + R_4\partial_3^2\right)\left(C_{44}\Lambda + C_{33}\partial_3^2\right) - \left(C_{13} + C_{44}\right)\left(R_3 + R_4\right)\Lambda\partial_3^2\right] \\
B_{43} &= \beta_3\partial_3\left(C_{11}\Lambda + C_{44}\partial_3^2\right)\left(K_1\Lambda + K_4\partial_3^2\right) - \beta_3\partial_3\left(R_1\Lambda + R_4\partial_3^2\right)^2 \\
&\quad + \beta_1\Lambda\partial_3\left[\left(R_3 + R_4\right)\left(R_1\Lambda + R_4\partial_3^2\right) - \left(C_{13} + C_{44}\right)\left(K_1\Lambda + K_4\partial_3^2\right)\right] \\
B_{44} &= \left(C_{11}\Lambda + C_{44}\partial_3^2\right)\left[\left(K_1\Lambda + K_4\partial_3^2\right)\left(C_{44}\Lambda + C_{33}\partial_3^2\right) - \left(R_3 + R_4\right)^2\Lambda\partial_3^2\right] \\
&\quad - \left(R_1\Lambda + R_4\partial_3^2\right)\left[\left(R_1\Lambda + R_4\partial_3^2\right)\left(C_{44}\Lambda + C_{33}\partial_3^2\right) - \left(R_3 + R_4\right)\left(C_{13} + C_{44}\right)\Lambda\partial_3^2\right] \\
&\quad + \left(C_{13} + C_{44}\right)\Lambda\partial_3^2\left[\left(R_1\Lambda + R_4\partial_3^2\right)\left(R_3 + R_4\right) - \left(K_1\Lambda + K_4\partial_3^2\right)\left(C_{13} + C_{44}\right)\right]
\end{aligned}
$$

If the subscript q is taken to be 1, 2 or 3, it can be seen that in Eq. (19) $B_{q4} = 0$. Accordingly, three sets of general solutions with $T=0$ will be obtained, which are actually the elastic general solutions of 2D hexagonal QCs without thermal effect. Taking $q = 4$ and writing out the algebraic cofactors, the general solution of problem I can be expressed as

$$
F = B_{41}H, \quad G = B_{42}H, \quad u_3 = B_{43}H, \quad T = B_{44}H \tag{20}
$$

or

$$
\begin{aligned}
F &= \left(a_1\partial_3^4 + b_1\Lambda\partial_3^2 + c_1\Lambda^2\right)H, \quad G = \left(a_2\partial_3^4 + b_2\Lambda\partial_3^2 + c_2\Lambda^2\right)H \\
u_3 &= \left(a_3\partial_3^4 + b_3\Lambda\partial_3^2 + c_3\Lambda^2\right)\partial_3 H, \quad T = \left(a\partial_3^6 + b\Lambda\partial_3^4 + c\Lambda^2\partial_3^2 + d\Lambda^3\right)H
\end{aligned} \tag{21}
$$

where

$$
\begin{aligned}
a_1 &= \beta_3\left[R_4\left(R_3 + R_4\right) - \left(C_{13} + C_{44}\right)K_4\right] + \beta_1 K_4 C_{33} \\
b_1 &= \beta_3\left[R_1\left(R_3 + R_4\right) - \left(C_{13} + C_{44}\right)K_1\right] + \beta_1\left[K_1 C_{33} + C_{44}K_4 - \left(R_3 + R_4\right)^2\right] \\
c_1 &= \beta_1 K_1 C_{44} \\
a_2 &= \beta_3\left[R_4\left(C_{13} + C_{44}\right) - C_{44}\left(R_3 + R_4\right)\right] - \beta_1 R_4 C_{33} \\
b_2 &= \beta_3\left[\left(C_{13} + C_{44}\right)R_1 - \left(R_3 + R_4\right)C_{11}\right] + \beta_1\left[\left(C_{13} + C_{44}\right)\left(R_3 + R_4\right) - R_1 C_{33} - R_4 C_{44}\right] \\
c_2 &= -\beta_1 R_1 C_{44} \\
a_3 &= \beta_3\left(C_{44}K_4 - R_4^2\right) \\
b_3 &= \beta_3\left(C_{11}K_4 + K_1 C_{44} - 2R_1 R_4\right) + \beta_1\left[\left(R_3 + R_4\right)R_4 - \left(C_{13} + C_{44}\right)K_4\right] \\
c_3 &= \beta_3\left(C_{11}K_1 - R_1^2\right) + \beta_1\left[R_1\left(R_3 + R_4\right) - \left(C_{13} + C_{44}\right)K_1\right]
\end{aligned}
$$

Since the displacement function H satisfies the eighth-order differential Eq. (15), it is not easy to obtain rigorous analytic solutions directly. By utilizing the generalized

Almansi's theorem [23], the displacement function H can be expressed by four quasi-harmonic equations H_q in five distinct forms as

Case 1:

$$H = H_1 + H_2 + H_3 + H_4, \quad \text{when } s_1^2 \neq s_2^2 \neq s_3^2 \neq s_4^2 \neq s_1^2 \tag{22}$$

Case 2:

$$H = H_1 + H_2 + H_3 + x_3 H_4, \quad \text{when } s_1^2 \neq s_2^2 \neq s_3^2 = s_4^2 \neq s_1^2 \tag{23}$$

Case 3:

$$H = H_1 + x_3 H_2 + H_3 + x_3 H_4, \quad \text{when } s_1^2 = s_2^2 \neq s_3^2 = s_4^2 \tag{24}$$

Case 4:

$$H = H_1 + H_2 + x_3 H_3 + x_3^2 H_4, \quad \text{when } s_1^2 \neq s_2^2 = s_3^2 = s_4^2 \tag{25}$$

Case 5:

$$H = H_1 + x_3 H_2 + x_3^2 H_3 + x_3^3 H_4, \quad \text{when } s_1^2 = s_2^2 = s_3^2 = s_4^2 \tag{26}$$

where H_q satisfy the following second-order equations:

$$\nabla_Q^2 H_q = 0 \ (q = 1, 2, 3, 4) \tag{27}$$

in which the upper case subscript Q takes the same number as the corresponding lower case q, but with *no summation convention*.

Therefore, the eighth-order Eq. (15) has been replaced with four quasi-harmonic equations. Considering different cases of four characteristic roots, the general solution of Eq. (20) shall take five forms. Next, the five cases of the general solutions for problem I are deduced individually.

Case 1:

$$s_1^2 \neq s_2^2 \neq s_3^2 \neq s_4^2 \neq s_1^2$$

In the case of four distinct characteristic roots, the solution of the eighth-order partial differential Eq. (15) can be represented according to Eq. (22). Substituting Eq. (22) into Eq. (21), and by using Eq. (27), the general solution of Eq. (20) can be written as

$$F = \lambda_q \partial_3^4 H_q, \quad G = \theta_q \partial_3^4 H_q, \quad u_3 = \eta_q \partial_3^5 H_q, \quad T = \xi_q \partial_3^6 H_q \tag{28}$$

where

$$\lambda_q = a_1 - b_1 \frac{1}{s_q^2} + c_1 \frac{1}{s_q^4}, \quad \theta_q = a_2 - b_2 \frac{1}{s_q^2} + c_2 \frac{1}{s_q^4},$$

$$\eta_q = a_3 - b_3 \frac{1}{s_q^2} + c_3 \frac{1}{s_q^4}, \quad \xi_q = a - b \frac{1}{s_q^2} + c \frac{1}{s_q^4} - d \frac{1}{s_q^6}$$

For further simplification, by assuming

$$\psi_q = \lambda_Q \partial_3^4 H_q \tag{29}$$

Eq. (28) becomes

$$F = \delta_{Qq}\psi_q, \quad G = m_{1q}\psi_q, \quad u_3 = m_{2q}\partial_3\psi_q, \quad T = m_{3q}\partial_3^2\psi_q \tag{30}$$

where δ_{Qq} is the Kronecker delta symbol, $m_{1q} = \theta_q/\lambda_Q$, $m_{2q} = \eta_q/\lambda_Q$, $m_{3q} = \xi_q/\lambda_Q$. When $q = 1, 2, 3$, $m_{3q} = 0$. From Eqs. (27) and (29), it can be seen that ψ_q satisfy the following equations:

$$\nabla_Q^2\psi_q = 0(q = 1, 2, 3, 4) \tag{31}$$

Case 2:

$$s_1^2 \neq s_2^2 \neq s_3^2 = s_4^2 \neq s_1^2$$

After the same manipulation as case 1, by using the expression of H in Eq. (23), and by introducing

$$\begin{aligned} &\psi_1 = \lambda_1\partial_3^4 H_1 + \left(4a_1 - 2b_1\tfrac{1}{s_3^2}\right)\partial_3^3 H_4, \quad \psi_2 = \lambda_2\partial_3^4 H_2 \\ &\psi_3 = \lambda_3\partial_3^4 H_3, \quad \psi_4 = \lambda_3\partial_3^3 H_4 \end{aligned} \tag{32}$$

the general solution in case 2 can be researched as follows:

$$\begin{aligned} F &= \psi_1 + \psi_2 + \psi_3 + x_3\partial_3\psi_4, \quad G = m_{11}\psi_1 + m_{12}\psi_2 + m_{13}\left(\psi_3 + x_3\partial_3\psi_4\right) + m_{15}\psi_4 \\ u_3 &= m_{21}\partial_3\psi_1 + m_{22}\partial_3\psi_2 + m_{23}\left(\partial_3\psi_3 + x_3\partial_3^2\psi_4\right) + m_{25}\partial_3\psi_4 \\ T &= m_{31}\partial_3^2\psi_1 + m_{32}\partial_3^2\psi_2 + m_{33}\left(\partial_3^2\psi_3 + x_3\partial_3^3\psi_4\right) + m_{35}\partial_3^2\psi_4 \end{aligned} \tag{33}$$

where

$$\begin{aligned} m_{15} &= \frac{1}{\lambda_3}\left[\left(4a_2 - 2b_2\frac{1}{s_3^2}\right) - m_{11}\left(4a_1 - 2b_1\frac{1}{s_3^2}\right)\right] \\ m_{25} &= \frac{1}{\lambda_3}\left[\left(5a_3 - 3b_3\frac{1}{s_3^2} + c_3\frac{1}{s_3^4}\right) - m_{21}\left(4a_1 - 2b_1\frac{1}{s_3^2}\right)\right] \\ m_{35} &= \frac{1}{\lambda_3}\left[\left(6a - 4b\frac{1}{s_3^2} + 2c\frac{1}{s_3^4}\right) - m_{31}\left(4a_1 - 2b_1\frac{1}{s_3^2}\right)\right] \end{aligned}$$

In view of Eqs. (27) and (32), it is seen that ψ_q satisfy Eq. (31).

Case 3:

$$s_1^2 = s_2^2 \neq s_3^2 = s_4^2 \neq s_1^2$$

Based on the expression of H of case 3 in Eq. (24), and by introducing

$$\begin{aligned} &\psi_1 = \lambda_1\partial_3^4 H_1 + \left(4a_1 - 2b_1\frac{1}{s_1^2}\right)\partial_3^3 H_2 + \left(4a_1 - 2b_1\frac{1}{s_3^2}\right)\partial_3^3 H_4, \ \psi_2 = \lambda_2\partial_3^3 H_2 \\ &\psi_3 = \lambda_3\partial_3^4 H_3, \ \psi_4 = \lambda_3\partial_3^3 H_4 \end{aligned} \tag{34}$$

the general solution in case 3 can be derived as follows:

$$\begin{aligned}F &= \psi_1 + x_3\partial_3\psi_2 + \psi_3 + x_3\partial_3\psi_4\\ G &= m_{11}\left(\psi_1 + x_3\partial_3\psi_2\right) + m_{16}\psi_2 + m_{13}\left(\psi_3 + x_3\partial_3\psi_4\right) + m_{15}\psi_4\\ u_3 &= m_{21}\left(\partial_3\psi_1 + x_3\partial_3^2\psi_2\right) + m_{26}\partial_3\psi_2 + m_{23}\left(\partial_3\psi_3 + x_3\partial_3^2\psi_4\right) + m_{25}\partial_3\psi_4\\ T &= m_{31}\left(\partial_3^2\psi_1 + x_3\partial_3^3\psi_2\right) + m_{36}\partial_3^2\psi_2 + m_{33}\left(\partial_3^2\psi_3 + x_3\partial_3^3\psi_4\right) + m_{35}\partial_3^2\psi_4\end{aligned} \tag{35}$$

where

$$\begin{aligned}m_{16} &= \tfrac{1}{\lambda_1}\left[\left(4a_2 - 2b_2\tfrac{1}{s_1^2}\right) - m_{11}\left(4a_1 - 2b_1\tfrac{1}{s_1^2}\right)\right]\\ m_{26} &= \tfrac{1}{\lambda_1}\left[\left(5a_3 - 3b_3\tfrac{1}{s_1^2} + c_3\tfrac{1}{s_1^4}\right) - m_{21}\left(4a_1 - 2b_1\tfrac{1}{s_1^2}\right)\right]\\ m_{36} &= \tfrac{1}{\lambda_1}\left[\left(6a - 4b\tfrac{1}{s_1^2} + 2c\tfrac{1}{s_1^4}\right) - m_{31}\left(4a_1 - 2b_1\tfrac{1}{s_1^2}\right)\right]\end{aligned}$$

From Eqs. (27) and (34), it is seen that ψ_q also satisfy Eq. (31).

Case 4:

$$s_1^2 \neq s_2^2 = s_3^2 = s_4^2$$

Based on the expression of H of case 4 in Eq. (25), and by introducing

$$\begin{aligned}&\psi_1 = \lambda_1\partial_3^4 H_1 + \left(4a_1 - 2b_1\tfrac{1}{s_2^2}\right)\partial_3^3 H_3 + \left(12a_1 - 2b_1\tfrac{1}{s_2^2}\right)\partial_3^2 H_4, \quad \psi_2 = \lambda_2\partial_3^4 H_2\\ &\psi_3 = \lambda_2\partial_3^3 H_3 + \left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\partial_3^2 H_4, \quad \psi_4 = \lambda_2\partial_3^2 H_4\end{aligned} \tag{36}$$

the general solution in case 4 can be obtained as follows:

$$\begin{aligned}F &= \psi_1 + \psi_2 + x_3\partial_3\psi_3 + x_3^2\partial_3^2\psi_4\\ G &= m_{11}\psi_1 + m_{12}\left(\psi_2 + x_3\partial_3\psi_3 + x_3^2\partial_3^2\psi_4\right) + m_{15}\psi_3 + x_3 m_{17}\partial_3\psi_4 + m_{18}\psi_4\\ u_3 &= m_{21}\partial_3\psi_1 + m_{22}\left(\partial_3\psi_2 + x_3\partial_3^2\psi_3 + x_3^2\partial_3^3\psi_4\right) + m_{25}\partial_3\psi_3 + x_3 m_{27}\partial_3^2\psi_4 + m_{28}\partial_3\psi_4\\ T &= m_{31}\partial_3^2\psi_1 + m_{32}\left(\partial_3^2\psi_2 + x_3\partial_3^3\psi_3 + x_3^2\partial_3^4\psi_4\right) + m_{35}\partial_3^2\psi_3 + x_3 m_{37}\partial_3^3\psi_4 + m_{38}\partial_3^2\psi_4\end{aligned} \tag{37}$$

where

$$\begin{aligned}m_{17} &= \tfrac{1}{\lambda_2}\left[\left(8a_2 - 4b_2\tfrac{1}{s_2^2}\right) - m_{12}\left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\right]\\ m_{18} &= \tfrac{1}{\lambda_2}\left[\left(12a_2 - 2b_2\tfrac{1}{s_2^2}\right) - m_{11}\left(12a_1 - 2b_1\tfrac{1}{s_2^2}\right) - m_{15}\left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\right]\\ m_{27} &= \tfrac{1}{\lambda_2}\left[\left(10a_3 - 6b_3\tfrac{1}{s_2^2} + 2c_3\tfrac{1}{s_2^4}\right) - m_{22}\left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\right]\\ m_{28} &= \tfrac{1}{\lambda_2}\left[\left(20a_3 - 6b_3\tfrac{1}{s_2^2}\right) - m_{21}\left(12a_1 - 2b_1\tfrac{1}{s_2^2}\right) - m_{25}\left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\right]\\ m_{37} &= \tfrac{1}{\lambda_2}\left[\left(12a - 8b\tfrac{1}{s_2^2} + 4c_3\tfrac{1}{s_2^4}\right) - m_{32}\left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\right]\\ m_{38} &= \tfrac{1}{\lambda_2}\left[\left(30a - 12b\tfrac{1}{s_2^2} + 2c_3\tfrac{1}{s_2^4}\right) - m_{31}\left(12a_1 - 2b_1\tfrac{1}{s_2^2}\right) - m_{35}\left(8a_1 - 4b_1\tfrac{1}{s_2^2}\right)\right]\end{aligned}$$

ψ_q also satisfy Eq. (31).

Case 5:

$$s_1^2 = s_2^2 = s_3^2 = s_4^2$$

Based on the expression of H of case 5 in Eq. (26), and by introducing

$$\begin{aligned}
\psi_1 &= \lambda_1 \partial_3^4 H_1 + \left(4a_1 - 2b_1 \tfrac{1}{s_1^2}\right) \partial_3^3 H_2 + \left(12a_1 - 2b_1 \tfrac{1}{s_1^2}\right) \partial_3^2 H_3 + 24a_1 \partial_3 H_4 \\
\psi_2 &= \lambda_1 \partial_3^3 H_2 + \left(8a_1 - 4b_1 \tfrac{1}{s_1^2}\right) \partial_3^2 H_3 + \left(36a_1 - 6b_1 \tfrac{1}{s_1^2}\right) \partial_3 H_4 \\
\psi_3 &= \lambda_1 \partial_3^2 H_3 + \left(12a_1 - 6b_1 \tfrac{1}{s_1^2}\right) \partial_3 H_4, \;\; \psi_4 = \lambda_1 \partial_3 H_4
\end{aligned} \tag{38}$$

the general solution in case 5 can be developed as follows:

$$\begin{aligned}
F &= \psi_1 + x_3 \partial_3 \psi_2 + x_3^2 \partial_3^2 \psi_3 + x_3^3 \partial_3^3 \psi_4 \\
G &= m_{11}\left(\psi_1 + x_3 \partial_3 \psi_2 + x_3^2 \partial_3^2 \psi_3 + x_3^3 \partial_3^3 \psi_4\right) + m_{15}\left(\psi_2 + 2x_3 \partial_3 \psi_3 + 3x_3^2 \partial_3^2 \psi_4\right) \\
&\quad + m_{18}\left(\psi_3 + 3x_3 \partial_3 \psi_4\right) + m_{19} \psi_4 \\
u_3 &= m_{21}\left(\partial_3 \psi_1 + x_3 \partial_3^2 \psi_2 + x_3^2 \partial_3^3 \psi_3 + x_3^3 \partial_3^4 \psi_4\right) \\
&\quad + m_{25}\left(\partial_3 \psi_2 + 2x_3 \partial_3^2 \psi_3 + 3x_3^2 \partial_3^3 \psi_4\right) + m_{28}\left(\partial_3 \psi_3 + 3x_3 \partial_3^2 \psi_4\right) + m_{29} \partial_3 \psi_4 \\
T &= m_{31}\left(\partial_3^2 \psi_1 + x_3 \partial_3^3 \psi_2 + x_3^2 \partial_3^4 \psi_3 + x_3^3 \partial_3^5 \psi_4\right) + m_{35}\left(\partial_3^2 \psi_2 + 2x_3 \partial_3^3 \psi_3 + 3x_3^2 \partial_3^4 \psi_4\right) \\
&\quad + m_{38}\left(\partial_3^2 \psi_3 + 3x_3 \partial_3^3 \psi_4\right) + m_{39} \partial_3^2 \psi
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
m_{19} &= \tfrac{1}{\lambda_1}\left[(24a_2 - 24m_{11}a_1) - m_{18}\left(12a_1 - 6b_1 \tfrac{1}{s_1^2}\right) - m_{15}\left(36a_1 - 6b_1 \tfrac{1}{s_1^2}\right)\right] \\
m_{29} &= \tfrac{1}{\lambda_1}\left[\left(60a_3 - 6b_3 \tfrac{1}{s_1^2} - 24m_{21}a_1\right) - m_{28}\left(12a_1 - 6b_1 \tfrac{1}{s_1^2}\right) - m_{25}\left(36a_1 - 6b_1 \tfrac{1}{s_1^2}\right)\right] \\
m_{39} &= \tfrac{1}{\lambda_1}\left[\left(120a_3 - 24b_3 \tfrac{1}{s_1^2} - 24m_{21}a_1\right) - m_{38}\left(12a_1 - 6b_1 \tfrac{1}{s_1^2}\right) - m_{35}\left(36a_1 - 6b_1 \tfrac{1}{s_1^2}\right)\right]
\end{aligned}$$

ψ_q again satisfy Eq. (31).

Therefore, problem I of 2D hexagonal QCs is governed by the four quasi-harmonic governing equations in Eq. (31).

General Solutions of Problem II

Problem II for the thermoelasticity in 2D hexagonal QCs is the same as problem II of the pure elasticity in our previous work [14]. For the sake of descriptive integrality, the general solutions for problem II are presented briefly here.

Introduce a displacement function $\overline{H}$, which satisfies the following fourth-order partial differential equation:

$$\overline{\nabla}_1^2 \overline{\nabla}_2^2 \overline{H} = 0 \tag{40}$$

where the quasi-harmonic differential operators $\overline{\nabla}_n^2$ are expressed as $\overline{\nabla}_n^2 = \Lambda + \frac{1}{\bar{s}_n^2} \partial_3^2$, $\bar{s}_1^2$ and $\bar{s}_2^2$ are the two characteristic roots of the following equation:

$$\left(C_{44}K_4 - R_4^2\right)\bar{s}^4 - \left(C_{66}K_4 + C_{44}K_3 - 2R_4R_6\right)\bar{s}^2 + \left(C_{66}K_3 - R_6^2\right) = 0$$

One group of general solutions of problem II [14] is

$$f = \left(K_3\Lambda + K_4\partial_3^2\right)\overline{H},\ g = -\left(R_6\Lambda + R_4\partial_3^2\right)\overline{H}$$

According to Almansi's theorem, there exists displacement function $\overline{H}$ in the following two forms:

Case 6:

$$\overline{H} = \overline{H}_1 + \overline{H}_2,\ \text{when } \bar{s}_1^2 \neq \bar{s}_2^2$$

Case 7:

$$\overline{H} = \overline{H}_1 + x_3\overline{H},\ \text{when } \bar{s}_1^2 = \bar{s}_2^2$$

$\overline{H}_n$ satisfy the following quasi-harmonic governing equations

$$\overline{\nabla}_N^2\overline{H}_n = 0 \tag{41}$$

in which the upper case subscript N takes the same number as the corresponding lower case n, but with *no summation convention*.

Considering different cases of the two characteristic roots, the general solution of problem II shall take two forms individually.

Case 6:

$$\bar{s}_1^2 \neq \bar{s}_2^2$$

Introducing

$$\bar{\psi}_n = \left(-K_3\frac{1}{\bar{s}_N^2} + K_4\right)\partial_3^2\overline{H}_n$$

the general solution of case 6 is

$$f = \delta_{Nn}\bar{\psi}_n, \quad g = m_{4n}\bar{\psi}_n \tag{42}$$

where $m_{4n} = \left(R_6\frac{1}{\bar{s}_n^2} - R_4\right)\Big/\left(-K_3\frac{1}{\bar{s}_N^2} + K_4\right)$. $\bar{\psi}_n$ satisfy the following equations

$$\overline{\nabla}_N^2\bar{\psi}_n = 0 \tag{43}$$

Case 7:

$$\bar{s}_1^2 = \bar{s}_2^2$$

Introducing

$$\bar{\psi}_1 = \left(-K_3\frac{1}{\bar{s}_1^2} + K_4\right)\partial_3^2\overline{H}_1, \quad \bar{\psi}_2 = \left(-K_3\frac{1}{\bar{s}_1^2} + K_4\right)\partial_3\overline{H}_2$$

the general solution of case 7 is

$$f = \bar{\psi}_1 + x_3\partial_3\bar{\psi}_2 + m_{43}\bar{\psi}_2, \quad g = m_{41}\bar{\psi}_1 + m_{42}x_3\partial_3\bar{\psi}_2 + m_{44}\bar{\psi}_2 \tag{44}$$

where $m_{43} = 2K_4 \Big/ \left(-K_3 \frac{1}{\bar{s}_1^2} + K_4\right)$, $m_{44} = -2R_4 \Big/ \left(-K_3 \frac{1}{\bar{s}_1^2} + K_4\right)$. $\bar{\psi}_n$ also satisfy the Eq. (43).

As a result, problem II of 2D hexagonal QCs is governed by the two quasi-harmonic governing equations in Eq. (43).

The general solutions of the thermoelasticity of 2D hexagonal QCs can be expressed in terms of the six quasi-harmonic functions ψ_q and $\bar{\psi}_n$, which are very simple and useful. In consideration of different cases in which the six characteristic roots s_q^2 and $\bar{s}_n^2$ are distinct or possibly equal to each other, the general solutions possess different forms. If the six characteristic roots are distinct, the general solutions are

$$\begin{aligned} &u_1 = \delta_{Qq}\partial_1\psi_q + \delta_{Nn}\partial_2\bar{\psi}_n, \quad u_2 = \delta_{Qq}\partial_2\psi_q - \delta_{Nn}\partial_1\bar{\psi}_n, \quad u_3 = m_{2q}\partial_3\psi_q \\ &w_1 = m_{1q}\partial_1\psi_q + m_{4n}\partial_2\bar{\psi}_n, \quad w_2 = m_{1q}\partial_2\psi_q - m_{4n}\partial_1\bar{\psi}_n, \quad T = m_{3q}\partial_3^2\psi_q \end{aligned} \tag{45}$$

In cylindrical coordinates, the general solutions are

$$\begin{aligned} &u_r = \delta_{Qq}\partial_r\psi_q + \tfrac{1}{r}\delta_{Nn}\partial_\varphi\bar{\psi}_n, \quad u_\varphi = \tfrac{1}{r}\delta_{Qq}\partial_\varphi\psi_q - \delta_{Nn}\partial_r\bar{\psi}_n, \quad u_3 = m_{2q}\partial_3\psi_q \\ &w_r = m_{1q}\partial_r\psi_q + \tfrac{1}{r}m_{4n}\partial_\varphi\bar{\psi}_n, \quad w_\varphi = \tfrac{1}{r}m_{1q}\partial_\varphi\psi_q - m_{4n}\partial_r\bar{\psi}_n, \quad T = m_{3q}\partial_3^2\psi_q \end{aligned} \tag{46}$$

Eq. (45) and Eq. (46) are the general solutions for 3D thermoelasticity of 2D hexagonal QCs in terms of displacement function ψ_q and $\bar{\psi}_n$. Under given boundary conditions, the analytic solutions can be obtained by solving Eqs. (45) or (46), (31) and (43).

An Infinite Thermoelastic Space Containing a Penny-Shaped Crack

Consider a 2D hexagonal QC space weakened by a flat crack S located in the plane $x_3 = 0$, which is parallel to the quasi-periodic plane. The center of the crack is located at the origin of the coordinate system. Assume that the upper and lower surfaces of the crack have the same temperature distribution $T_0(x_1, x_2)$. Thus the problem can be turned into a mixed boundary value problem of the half-space $x_3 \geq 0$, with the following mixed boundary conditions on the plane $x_3 = 0$:

$$\sigma_{33} = 0, \ T = T_0(x_1, x_2), \ for \quad x_3 = 0, \ (x_1, x_2) \in S \tag{47}$$

$$u_3 = \frac{\partial T}{\partial x_3} = 0, \ for \quad x_3 = 0, \ (x_1, x_2) \notin S \tag{48}$$

$$\sigma_{23} = \sigma_{31} = 0, \ H_{23} = H_{13} = 0, \ for \quad x_3 = 0, \ -\infty < (x_1, x_2) < \infty \tag{49}$$

where S denotes the crack surface. It is noted that in Eqs. (47)–(49), the boundary conditions are not relative with the phason displacements in the fact that the x_3-axis is the periodic direction.

The same boundary problem in a thermoelastic transversely isotropic solid [24] was solved by means of potential functions and Fourier–Hankel transformation. The potential functions of 2D hexagonal QCs are assumed as the same formulation with those of the transversely isotropic solid, i.e.,

$$\psi_q = h_{q1}\varpi_1 + h_{q2}\varpi_2, \bar{\psi}_n = 0, \quad (q = 1, 2, 3, 4, n = 1, 2) \tag{50}$$

where h_{q1} and h_{q2} are constants to be determined, and

$$\begin{aligned}\varpi_1(x_1,x_2,x_3) &= \iint_S \frac{\omega(N_0)}{L(M,N_0)}dS \\ \varpi_2(x_1,x_2,x_3) &= \iint_S \vartheta(N_0)\{x_3\ln[L(M,N_0)+x_3]-L(M,N_0)\}\,dS\end{aligned} \tag{51}$$

where ω and ϑ are respectively the crack surface displacement $u_3(x_1,x_2,0)$ and temperature gradient $\partial_3 T(x_1,x_2,x_3)|_{x_3=0}$, and $L(M,N_0)$ is the distance between point $M(x_1,x_2,x_3)$ and $N_0\left(x_1^0,x_2^0,0\right)\in S$. By virtue of the potential of a simple layer [24], we obtain

$$\begin{aligned}&\partial_3\varpi_1|_{x_3=0}=0,\ \partial_3^3\varpi_2\big|_{x_3=0}=0, \quad for\ \ (x_1,x_2)\notin S \\ &\partial_3\varpi_1|_{x_3=0}=-2\pi\omega=-2\pi u_3(x_1,x_2,0), \quad for\ \ (x_1,x_2)\in S \\ &\partial_3^3\varpi_2\big|_{x_3=0}=-2\pi\vartheta=-2\pi\ \partial_3 T|_{x_3=0}, \quad for\ \ (x_1,x_2)\in S\end{aligned} \tag{52}$$

Substituting Eq. (50) into Eq. (45), and into the linear constitutive Eq. (3), there are

$$\begin{aligned}&\sigma_{33}=\gamma_{1q}\partial_3^2\psi_q, \quad \sigma_{23}=\sigma_{32}=\gamma_{2q}\partial_2\partial_3\psi_q, \quad \sigma_{31}=\sigma_{13}=\gamma_{2q}\partial_1\partial_3\psi_q \\ &H_{23}=\gamma_{3q}\partial_2\partial_3\psi_q,\ H_{13}=\gamma_{3q}\partial_1\partial_3\psi_q\end{aligned} \tag{53}$$

where

$$\begin{aligned}&\gamma_{1q}=C_{33}m_{2q}-\left(C_{13}+R_3m_{1q}\right)\frac{1}{s_Q^2}-\beta_3m_{3q} \quad \gamma_{2q}=C_{44}\left(\delta_{Qq}+m_{2q}\right)+R_4m_{1q} \\ &\gamma_{3q}=R_4\left(\delta_{Qq}+m_{2q}\right)+K_4m_{1q}\end{aligned} \tag{54}$$

Making use of Eq. (52) and boundary condition Eq. (48), the following equations can be obtained

$$\begin{cases} m_{2q}h_{q1}=-\frac{1}{2\pi}, & m_{2q}h_{q2}=0 \\ m_{3q}h_{q2}=-\frac{1}{2\pi}, & m_{3q}h_{q1}=0 \end{cases} \tag{55}$$

To satisfy the boundary condition Eq. (49), it is assumed that

$$\gamma_{2q}h_{qn}=0, \quad \gamma_{3q}h_{qn}=0,\ (n=1,2) \tag{56}$$

Eqs. (55) and (56) give

$$\begin{Bmatrix} h_{1n} \\ h_{2n} \\ h_{3n} \\ h_{4n} \end{Bmatrix} = -\frac{1}{2\pi}\begin{bmatrix} \gamma_{21}\gamma_{22}\gamma_{23}\gamma_{24} \\ \gamma_{31}\gamma_{32}\gamma_{33}\gamma_{34} \\ m_{21}m_{22}m_{23}m_{24} \\ m_{31}m_{32}m_{33}m_{34} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ \delta_{1n} \\ \delta_{2n} \end{Bmatrix}, \quad (n=1,2) \tag{57}$$

h_{q1} and h_{q2} can be obtained from Eq. (57).

Substituting Eq. (50) into Eq. (47), we get

$$\begin{aligned} &\gamma_{1q} h_{q1} \partial_3^2 \varpi_1 + \gamma_{1q} h_{q2} \partial_3^2 \varpi_2 = 0 \\ &m_{3q} h_{q1} \partial_3^2 \varpi_1 + m_{3q} h_{q2} \partial_3^2 \varpi_2 = T_0 \end{aligned} \tag{58}$$

By virtue of Eqs. (55) and (16), Eq. (58) can be transformed to

$$\begin{aligned} &\gamma_{1q} h_{q1} s_Q^2 \Lambda \iint_S \frac{\omega(N_0)}{L(N, N_0)} dS - \gamma_{1q} h_{q2} \iint_S \frac{\vartheta(N_0)}{L(N, N_0)} dS = 0 \\ &m_{3q} h_{q2} \iint_S \frac{\vartheta(N_0)}{L(N, N_0)} dS = T_0(N) \end{aligned} \tag{59}$$

where $L(N, N_0)$ is the distance between two points $N(x_1, x_2, 0)$ and $N_0\left(x_1^0, x_2^0, 0\right)$, both on the crack surface. By virtue of Eq. (55), we can derive

$$\iint_S \frac{\vartheta(N_0)}{L(N, N_0)} dS = -2\pi T_0(N), \quad \Lambda \iint_S \frac{\omega(N_0)}{L(N, N_0)} dS = -\frac{2\pi \gamma_{1q} h_{q2} T_0(N)}{\gamma_{1q} h_{q1} s_Q^2} \tag{60}$$

For a penny-shaped crack, with the radius a_0, exact solutions can be obtained by using Fabrikant's results [25]. In the case when the temperature is uniformly distributed, i.e. $T = T_0(N) = const.$, in cylinder coordinates, the solutions of Eq. (60) are derived as

$$\vartheta(r, \varphi) = -\frac{2T_0}{\pi\sqrt{a_0^2 - r^2}}, \quad \omega(r, \varphi) = \frac{2\gamma_{1q} h_{q2} T_0}{\pi \gamma_{1q} h_{q1} s_Q^2} \sqrt{a_0^2 - r^2} \tag{61}$$

Equations (60) and (61) are very similar to those presented in transversely isotropic thermoelasticity [24] and piezoelasticity [26], so the corresponding solution can be derived immediately. For brevity, the exact expressions of ϖ_1 and ϖ_2 for this problem are omitted. Only the stress σ_{33} around the penny crack is presented as

$$\sigma_{33}\big|_{x_3=0,\ r>a_0} = -\frac{4\gamma_{1q} h_{q2}}{\sqrt{r^2 - a_0^2}} a_0 T_0 \tag{62}$$

CONCLUSIONS

On the basis of the operator method and the introduction of the two displacement functions H and $\overline{H}$, the general solutions of 3D thermoelasticity of 2D hexagonal QCs are first presented. The introduced displacement functions H and $\overline{H}$ have to satisfy eighth-order and fourth-order partial differential equations, respectively. Owing to complexity of the two higher-order equations, it is difficult to obtain rigorous analytic solutions directly. Based on Almansi's theorem, and by virtue of a decomposition and superposition procedure, the general solution are further simplified in terms of six quasi-harmonic functions ψ_q and $\bar{\psi}_n$. Considering that the characteristic roots s_q^2 and $\bar{s}_n^2$ may be distinct or equal to each other, the obtained general solutions of 2D QCs involve different forms, but all are in simple forms which are conveniently applied.

To illustrate the application of the general solutions obtained, the closed form solution for infinite thermoelastic 2D hexagonal QCs containing a penny-shaped crack is derived, with the assumption that all characteristic roots are distinct. The general solutions are very convenient to study the inhomogeneity and defect problems of 2D hexagonal QCs. These also provide basis to judge the rationality of the solutions by the finite element method or the boundary element method. The analysis method in this article can also be used to solve the more complicated thermoelastic problems of 3D QCs.

ACKNOWLEDGMENT

The authors are very grateful to the anonymous reviewers for their helpful suggestions.

FUNDING

This work was supported by the National Natural Science Foundation of China (No. 11172319), Chinese Universities Scientific Fund (No. 2011JS046 and No. 2013BH008), Opening Fund of State Key Laboratory of Nonlinear Mechanics, Program for New Century Excellent Talents in University (No. NCET-13-0552), and National Science Foundation for Post-Doctoral Scientists of China (No. 2013M541086).

REFERENCES

1. D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Metallic Phase with Long-range Orientational Order and No Translational Symmetry, *Phys. Rev. Lett.*, vol. 53, pp. 1951–1953, 1984.
2. I. A. Ovid'ko, Plastic Deformation and Decay of Dislocations in Quasi-crystals, *Mater. Sci. Eng. A*, vol. 154, pp. 29–33, 1992.
3. M. Wollgarten, M. Beyss, K. Urban, H. Liebertz, and U. Köster, Direct Evidence for Plastic Deformation of Quasicrystals by Means of a Dislocation Mechanism, *Phys. Rev. Lett.,* vol. 71, pp. 549–552, 1993.
4. D. Levine and P. J. Stcinhardt, Quasicrystals: A New Class of Ordered Structures, *Phys. Rev. Lett.*, vol. 53, pp. 2477–2480, 1984.
5. R. J. Scheater and L. A. Bendersky, *Introduction to Quasicrystals*, Academic Press, San Diego, CA, 1988.
6. P. Bak, Phenomenological Theory of Icosahedral Incommensurate ("Quasiperiodic") Order in Mn-Al Alloys, *Phys. Rev. Lett.,* vol. 54, pp. 1517–1519, 1985.
7. P. Bak, Symmetry, Stability, and Elastic Properties of Icosahedral Incommensurate Crystals, *Phys. Rev. B,* vol. 32, pp. 5764–5772, 1985.
8. D. Levine, T. C. Lubensky, S. Qstlund, S. Ramaswamy, P. J. Steinhardt, and J. Toner, Elasticity and Dislocations in Pentagonal and Icosahedral Quasicrystals, *Phys. Rev. Lett.,* vol. 54, pp. 1520–1523, 1985.
9. D. H. Ding, W. G. Yang, C. Z. Hu, and R. H. Wang, Generalized Elasticity Theory of Quasicrystals, *Phys. Rev. B*, vol. 48, pp. 7003–7010, 1993.
10. C. Z. Hu, R. H. Ding, and D. H. Ding, Symmetry Groups, Physical Property Tensors, Elasticity and Dislocations in Quasicrystals, *Rep. Prog. Phys*, vol. 63, pp. 1–39, 2000.
11. T. Y. Fan, *The Mathematical Elasticity of Quasicrystals and its Applications*, Springer, Heidelberg, 2012.

12. M. Z. Wang, B. X. Xu, and C. F. Gao, Recent General Solutions in Linear Elasticity and Their Applications, *Appl. Mech. Rev.,* vol. 61, pp. 030803, 2008.
13. G. T. Liu, T. Y. Fan, and R. P. Guo, Governing Equations and General Solutions of Plane Elasticity of One-dimensional Quasicrystals, *Int. J. Solids Struct.*, vol. 41, pp. 3949–3959, 2004.
14. Y. Gao and B. S. Zhao, General Solutions of Three-dimensional Problems for Two-dimensional Quasicrystals, *Appl. Math. Model*., vol. 33, pp. 3382–3390, 2009.
15. X. Wang and J. Q. Zhang, A Steady Line Heat Source in a Decagonal Quasicrystalline Half-Space, *Mech. Res. Commun.,* vol. 32, pp. 420–428, 2005.
16. X. Y. Li and P. D. Li, Three-dimensional Thermoelastic General Solutions of One-dimensional Hexagonal Quasi-crystal and Fundamental Solutions, *Phys. Lett. A,* vol. 376, pp. 2004–2009, 2012.
17. C. Z. Hu, R. H. Wang, W. Yang, and D. H. Ding, Point Groups and Elastic Properties of Two-dimensional Quasicrystals, *Acta Cryst*. vol. 52, pp. 251–256, 1996.
18. M. A. Chernikvo, H. R. Ott, A. Bianchi, A. Migliori, and T. W. Darling, Elastic Moduli of a Single Quasicrystal of Decagonal Al-Ni-Co: Evidence for Transverse Elastic Isotropy, *Phys. Rev. Lett*., vol. 80, pp. 321–324, 1998.
19. H. Abe, N. Taruma, D. L. Bolloc'h, S. C. Moss, Y. Matsuo, Y. Ishii, and J. M. Bai, Anomalous-X-ray Scattering Associated with Short-Range Order in an $Al_{70}Ni_{50}Co_{15}$ Decagonal Quasicrystal, *Mater Sci. Eng. A,* vol. 294–296, pp. 299–302, 2000.
20. A. Letoublon, M. D. Boissieu, M. Boudard, L. Mancini, J. Gastaldi, B. Hennion, R. Caudron, and R. Bellissent, Phason Elastic Constants of the Icosahedral Al-Pd-Mn Phason Derived from Diffuse Scattering Measurements, *Phil. Mag. Lett.,* vol. 81, pp. 273–283, 2001.
21. K. Edagowa, Phonon-Phason Coupling in Decagonal Quasicrystals. *Phil. Mag.,* vol. 87, pp. 2789–2798, 2007.
22. Y. Gao and B. S. Zhao, A General Treatment of Three-Dimensional Elasticity of Quasicrystals by an Operator Method, *Phys. Status Solidi*(b), vol. 243, pp. 4007–4019, 2006.
23. R. A. Eubanks and F. Sternberg, On the Axisymmetric Problem of Elastic Theory for a Medium with Transverse Isotropy, *J. Rat. Mech. Anal*., vol. 3, pp. 89–101, 1954.
24. H. J. Ding, W. Q. Chen, and L. Zhang, *Elasticity of Transversely Isotropic Materials*, Springer, The Netherlands, 2006.
25. V. I. Fabrikant, *Applications of Potential Theory in Mechanics: A Selection of New Results,* Kluwer Academic, The Netherlands, 1989.
26. E. Karapetian, I. Sevostianov, and M. Kachanow, Penny-shaped and Half-plane Cracks in a Transversely Isotropic Piezoelectric Solid under Arbitrary Loading, *Arch. Appl. Mech*., vol. 70, pp. 201–229, 2000.