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INSTABILITY THEORY OF SHOCK WAVE IN A SHOCK TUBE

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ABSTRACT: The instability theory of shock wave in a shock tube including the effects of tube wall and contact surface is studied. The experimental data of unstable shock wave coincide with one of instability criteria derived in the present paper.

KEY WORDS: instability theory, shock wave, shock tube

Previous works on instability theory of shock wave[1-3] deal with a plane shock wave with infinite extent and the results do not agree with experimental data in shock tubes[4]. In the present paper, the effects of tube wall and contact surface are taken into account. In our physical model a stationary shock front in a tube is located at $x=0$ and a contact discontinuity is located at $x=-L$ downstream. The instability theory of shock wave developed in Ref.[3] is extended to solve the present problem. It is also assumed that the equation of state for the gas is arbitrary.

Basic equations for the disturbed quantities are

$$\frac{\partial \tilde{u}_x}{\partial t}+w\frac{\partial \tilde{u}_x}{\partial x}+V\frac{\partial \tilde{p}}{\partial x}=0$$

$$\frac{\partial \tilde{u}_y}{\partial t}+w\frac{\partial \tilde{u}_y}{\partial x}+V\frac{\partial \tilde{p}}{\partial y}=0$$

$$\frac{\partial \tilde{p}}{\partial t}+w\frac{\partial \tilde{p}}{\partial x}+\frac{c^2}{V}\left(\frac{\partial \tilde{u}_x}{\partial x}+\frac{\partial \tilde{u}_y}{\partial y}\right)=0$$

$$\left(\frac{\partial}{\partial t}+w\frac{\partial}{\partial x}\right)\left(\tilde{p}+\frac{c^2}{V^2}\tilde{V}\right)=0$$

where V is the specific volume, c the sound speed and w the gas velocity downstream.

The general solution of the basic equations can be written as

$$\tilde{p}=\cos ky\cdot\exp(-i\omega t)\cdot[D\exp^{(il^{(1)}x)}+E\exp^{(il^{(2)}x)}]$$

$$\tilde{V}=\cos ky\cdot\exp(-i\omega t)\cdot\left[A\exp\left(i\frac{\omega}{w}x\right)-D\frac{V^2}{c^2}\exp(il^{(1)}x)-E\frac{V^2}{c^2}\exp(il^{(2)}x)\right]$$

$$\tilde{u}_x=\cos ky\cdot\exp(-i\omega t)\cdot\left[Bk\exp\left(i\frac{\omega}{w}x\right)+D\frac{Vl^{(1)}}{\omega-wl^{(1)}}\exp(il^{(1)}x)+E\frac{Vl^{(2)}}{\omega-wl^{(2)}}\exp(il^{(2)}x)\right]$$

$$\tilde{u}_y=i\sin ky\cdot\exp(-i\omega t)\cdot\left[-B\frac{\omega}{w}\exp\left(i\frac{\omega}{w}x\right)+D\frac{Vk}{\omega-wl^{(1)}}\exp(il^{(1)}x)+E\frac{Vk}{\omega-wl^{(2)}}\exp(il^{(2)}x)\right]$$

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where A, B, D and E are arbitrary constants and

$$l^{(1)},\ l^{(2)}=\frac{-w\omega\pm\sqrt{-c^4k^2+w^2c^2k^2+c^2\omega^2}}{c^2-w^2}$$

In order to satisfy the boundary condition

$$\tilde{u}_y=0 \qquad \text{when} \qquad y=\pm a$$

where a is the semi-width of shock tube, we take

$$k=\frac{n\pi}{a} \qquad n=1,\ 2,\ 3,\ \cdots$$

Suppose that there is no disturbance upstream, that is, all disturbed quantities tend to zero as x tends to positive infinity. Then, there are two classes of boundary conditions. One class of boundary conditions are posed on the disturbed shock front. The expression of disturbed shock is assumed to be

$$\tilde{x}=g(y,t)=x_0\cos ky\cdot\exp(-i\omega t)$$

then, the normal and tangent unit vectors are

$$\boldsymbol{n}=(1,\ -ikx_0\cdot\cos ky\cdot\exp(-i\omega t))$$

$$\boldsymbol{t}=(ikx_0\cdot\cos ky\cdot\exp(-i\omega t),\ -1)$$

The conservation laws across shock front are

$$\tilde{u}_x=(w-w_0)\cdot\left[\frac{\tilde{p}}{p-p_0}-\frac{\tilde{V}}{V_0-V}\right]\Big/2$$

$$\tilde{u}_y=(w_0-w)\cdot\frac{\partial g}{\partial y}$$

$$\tilde{V}=\left(\frac{dV}{dp}\right)_H\cdot\tilde{p}$$

$$-\frac{2}{w_0}\frac{\partial g}{\partial t}=\frac{\tilde{p}}{p-p_0}+\frac{\tilde{V}}{V_0-V}$$

where w_0 is the gas velocity upstream and subscript H denotes the Hugoniot curve.

Another class of boundary conditions is posed on the contact surface. The condition adopted is

$$\tilde{u}_x=0 \qquad \text{when} \qquad x=-L$$

Then, the dispersion relation can be derived.

Introducing Ω, let

$$\Omega=-i\frac{\omega}{ck}$$

The dispersion relation obtained is the following two equations, namely

$$\Omega^2-M^2+1=0$$

and

$$f\Omega^2+g\cdot\Omega+h=0$$

where

$$f=4M^2-\alpha$$

$$g\doteq M(1-M^2)(2-\alpha)/(kL)$$

$$h=M^2\cdot\alpha w_0/w$$

and

$$\alpha=1+j^2\left(\frac{dV}{dp}\right)_H \qquad j^2=\frac{w}{V}$$

From the definition of Ω, it is obvious that the instability of shock wave corresponds to the fact that the dispersion equation has a positive real root or a complex root with a positive real part.

The solution of the first equation is

$$\Omega=\pm i\cdot\sqrt{1-M^2}$$

which shows the stability of shock wave.

The solution of the second equation is

(1) When

$$j^2\cdot\left(\frac{dV}{dp}\right)_H>4M^2-1$$

there are two real roots. Because $f<0$, $h>0$, therefore one of two roots is positive and the other is negative. The positive real root indicates the instability of shock wave.

(2) When

$$j^2\cdot\left(\frac{dV}{dp}\right)_H<-1$$

there are two real roots. Because $f>0$, $h<0$, therefore one of two roots is positive and the other is negative. The positive real roots indicates the instability of shock wave.

(3) When

$$-1<j^2\cdot\left(\frac{dV}{dp}\right)_H<4M^2-1$$

because $f<0$, $h>0$, the condition for having a positive real root or a complex root with a positive real part is

$$g<0$$

or

$$j^2\cdot\left(\frac{dV}{dp}\right)_H>1$$

(1) When

$$M<1/2$$

The solution is neither a positive real root nor a complex root with a positive real part. This case corresponds to the stability of shock wave.

(2) When

$$M>1/2$$

the condition for the instability of shock wave is

$$1<j^2\cdot\left(\frac{dV}{dp}\right)_H<4M^2-1$$

In the shock wave experiment by Griffiths et al.[4], the experimental data of unstable shock wave correspond to the first case mentioned above, i.e.

$$j^2\cdot\left(\frac{dV}{dp}\right)_H>4M^2-1$$

In the experiment, for example, $j^2\cdot(dV/dp)_H$ and $4M^2-1$ are approximately 0.06 and -0.82 for CO_2 and nearly 0.0 and -0.75 for Ar. Meanwhile the criterion given in Refs. [1—3] is

$$j^2\cdot\left(\frac{dV}{dp}\right)_H<-1 \quad \text{or} \quad j^2\cdot\left(\frac{dV}{dp}\right)_H>1+2M$$

which does not agree with the experiment. It seems that a more accurate theory including the effects of moving contact surface and rarefaction wave is needed.

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