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THE WAVE ATTENUATION ON THE MUD BED*

Zhou Xianchu (周显初) Wang Jianfeng (王剑峰)
(*Institute of Mechanics, Chinese Academy of Sciences, Beijing* 100080, *China*)

ABSTRACT: In the paper the wave attenuation in a two layer fluid system is studied. The fluid in the top layer is ideal and that in the lower layer is the Voigt model of the viscoelastic medium. A dispersion relation is derived and the rate of the wave decay is computed. The approximate explicit expressions of the decay rate for different water depth are given, where the viscoelasticity is either very large or very small. Compared with the numerical results, our results are very accurate, which can be used by an engineer.

KEY WORDS: water wave, wave in stratified fluid, interaction between wave and mud

I. INTRODUCTION

In offshore engineering the classical water wave theory can be used under the assumption that the fluid is ideal with a rigid and nonporous bottom. Because of the energy dissipation due to percolation, viscosity and bottom friction, wave attenuates in the coast engineering, especially for a mud bottom.

The effect of mud bottom on wave attenuation can be verified from the following reports. Tubman and Suhayda[1] discovered that the energy loss of surface wave due to the mud movement is much larger than that due to the percolation and friction. Gade[2] reported that there is a "mud hole" in the Gulf of Mexico, which is used as an emergency harbor by fishing boats because of the great attenuation of waves due to the mud bottom. It is said that there is a mud hole near the Yellow River mouth too. Macpherson[3] reported that during the SW monsoon season, incident storm waves on the mud bottom of Kerala, India, are almost completely damped out within a distance of only 4—8 wavelengths. All these show that it is not sufficient to consider the bottom friction only and it is neccessary to study the mechanism of wave attenuation on a mud bed.

Early in the 1950s, Gade[4] studied this mechanism. He made experiments for the wave attenuation in a channel with a two-fluid system of kerosine and water-sugar solution, derived the dispersion relation of waves under the assumption of long waves and studied the wave attenuation. Dalrymple and Liu[5] studied the wave attenuation in two layers of fluid. They used a model with three boundary layers and two layers of ideal fluid to replace the model with two layers of viscous fluid, and obtained a explicit expression for the coefficient of wave attenuation. Hsiao et al[6]. and Macpherson[3] studied the wave attenuation in two-fluid system with ideal and viscous fluid in upper and lower layers, respectively. In Hsiao et al.'s paper, the condition of continuous pressure on the interface replaced the correct condition of continuous stress, and the error occurred. Macpherson introduced a potential function and a stream function, considered the whole flow as the superposition of the ideal flow and the viscous rotational flow, studied the case where the depth of the lower layer is infinite, derived the coefficient of the wave decay and compared it with numerical results.

The viscous model is suitable for describing the movement of the floating mud. But for the mud lying on the bottom, the viscoelastic model gives a more realistic constitutive equation for the mud in nature. It has been demonstrated by Magniot[7] with the laboratory measurements that the orbit motion induced by wave does occur in mud layer and that soft

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layer can exhibit properties similar to those found in fluids. So, the wave attenuation on the mud can be reduced to a wave problem in two layers of fluid.

Based on the wave theory with small amplitude, this paper studies the surface wave attenuation in two layers of fluid. The fluid in the top layer is ideal, and that in the lower layer is viscoelastic. The governing equations for Voigt fluid and boundary conditions are discussed and the accurate dispersion relation is given in II. When the viscosity or elasticity are very large or very small, the explicit expressions for wave damping coefficients suitable for different depth of upper and lower layers are derived in III. At last, in IV, we check the accuracy of these explicit expressions by comparing them with numerical results.

II. DERIVATION OF DISPERSION RELATION

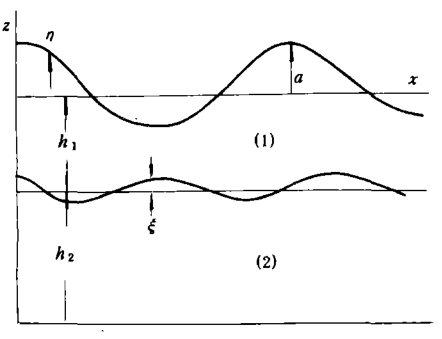


Fig. 1 Schematic figure for the two layer fluid model

The coordinate system and the rotations are shown in Fig. 1. The wave is propagating in x direction. The fluid in the top layer is ideal with depth h_1 and the lower layer is viscoelastic with depth h_2. Due to the stability condition, $\rho_2 > \rho_1$. The free surface displacement is denoted by $\eta(x,y) = a \cdot \exp(i(kx-\sigma t))$ and the interface between two fluids is $\xi(x,t) = b \cdot \exp(i(kx-\sigma t))$. The fluids both in upper and lower layers are homogeneous, the fluid in lower layer is a Voigt body and the stress-strain relationship is

$$\tau = G\varepsilon + \mu\dot{\varepsilon} \tag{1}$$

where τ is shear stress, G shear modulus, μ dynamic viscosity, ε shear strain and $\dot{\varepsilon}$ rate of shear strain. If the mud is incompressible, G and μ are constant and the convective term is of the second order and is neglected. The equation for lower-layer fluid is expressed as [8]

$$\frac{\partial^2 u}{\partial t^2} = \frac{G}{\rho}\nabla^2 u + \nu\nabla^2\frac{\partial u}{\partial t} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial t} \tag{2}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{G}{\rho}\nabla^2 w + \nu\nabla^2\frac{\partial w}{\partial t} - \frac{1}{\rho}\frac{\partial^2 p}{\partial z\partial t} \tag{3}$$

where u and w are the horizontal and vertical velocity components, respectively, p pressure, $\nu = \mu/\rho$ kinetic viscosity. ρ density of lower-layer fluid.

The stream function ψ is introduced as

$$u = \psi_z \qquad w = -\psi_x \tag{4}$$

Since we consider linear wave motions we can assume

$$\psi = S(z)\exp[i(kx-\sigma t)] \tag{5}$$

Substituting (5) into (2) and (3) and eliminating p we get the following fourth-order ordinary different equation for $S(z)$

$$\frac{d^4 s}{dz^4} + \left(-2k^2 + \frac{\sigma i}{\nu_e}\right)\frac{d^2 S}{dz^2} + \left(k^4 - \frac{i\sigma k^2}{\nu_e}\right)S = 0 \tag{6}$$

where

$$\nu_e = \nu + \frac{iG}{\rho\sigma} \tag{7}$$

is effective viscosity. It is a complex, its real part is the viscosity and the imaginary part is a measure of elasticity.

The solution of Eq. (6) is

$$S(z)=A\exp(kz)+B\exp(-kz)+Ce^{mz}+D\exp(-mz) \tag{8}$$

where A, B, C, and D are constants determined by the boundary conditions, and

$$m^2=k^2-\frac{i\sigma}{\nu_e} \tag{9}$$

Assuming that the upper layer fluid is incompressible and the motion is irrotational we introduce the potential function φ, then

$$\nabla^2\varphi=0 \tag{10}$$

$$u=\varphi_x \qquad w=\varphi_z \tag{11}$$

Now we consider the boundary conditions. There are two conditions on the free surface

$$p=0 \qquad \text{at} \quad z=\eta(x,t) \tag{12}$$

$$w_1=\eta_t \qquad \text{at} \quad z=\eta(x,t) \tag{13}$$

On the bottom, there are two conditions

$$u=w=0 \qquad \text{at} \quad z=-(h_1+h_2) \tag{14}$$

On the interface, the normal stress and the shear stress are continuous

$$\sigma_{1zz}=\sigma_{2zz} \qquad \text{at} \quad z=-h_1+\xi(x,t) \tag{15}$$

$$\tau_{2xz}=0 \qquad \text{at} \quad z=-h_1+\xi(x,t) \tag{16}$$

the normal velocity is continuous and is equal to the normal velocity of the interface

$$w_1=w_2=\xi_t \qquad \text{at} \quad z=-h_1+\xi(x,t) \tag{17}$$

where the subscripts 1 and 2 indicate the upper and lower layers, respectively. The boundary condition (15) different from Hsiao's condition of continuous pressure. The later is only an approximation when $|\nu_e|$ is small, [6] does not show how much is the error due to this approximation and there is no other paper discussing it either. So, we use the accurate condition of the continuous stress and the results are different from [6].

The solution for the top layer fluid satisfying the Eq.(10) and the boundry conditions (12) and (13) are

$$\varphi=-\frac{iag}{\sigma}\left[\operatorname{ch}kz+\frac{\sigma^2}{gk}\operatorname{sh}kz\right]\exp[i(kx-\sigma t)] \tag{18}$$

The boundary condition (15) can be written as

$$p_1(-h_1)+\frac{\partial p_1}{\partial z}\xi=p_2(-h_1)+\frac{\partial p_2}{\partial z}\xi-2\rho_2\nu_2\frac{\partial w}{\partial z}-2G\int\frac{\partial w}{\partial z}dt \tag{19}$$

Expressing the boundary conditions (14)—(17) by φ, η and A, B, C, D in (8) at the balance position, we get

$$KA\exp[-k(h_1+h_2)]-kB\exp[k(h_1+h_2)]+Cm\exp[-m(h_1+h_2)]-mD\exp[m(h_1+h_2)]=0$$
$$\text{at} \quad z=-(h_1+h_2) \tag{20}$$

$$A\exp[-k(h_1+h_2)]+B\exp[k(h_1+h_2)]+C\exp[-m(h_1+h_2)]+D\exp[m(h_1+h_2)]=0$$
$$\text{at} \quad z=-(h_1+h_2) \tag{21}$$

$$\frac{-ki}{\rho_2\nu_e}\left[(\rho_2-\rho_1)gb+\rho_1 ga\left(\operatorname{ch}kh_1-\frac{\sigma^2}{gk}\operatorname{sh}kh_1\right)\right]=A\exp(-kh_1)(m^2+k^2)k$$
$$-B\exp(kh_1)(m^2+k^2)k+2mk^2C\exp(-mh_1)-2mk^2D\exp(mh_1) \qquad \text{at} \quad z=-h_1 \tag{22}$$

$$2k^2(A\exp(-kh_1)+B\exp(kh_1))+(m^2+k^2)(C\exp(-mh_1)+D\exp(mh_1))=0$$
$$\text{at} \quad z=-h_1 \tag{23}$$

$$-\frac{ga}{\sigma}\left[\operatorname{sh} kh_1-\frac{\sigma^2}{gk}\operatorname{ch} kh_1\right]=A\exp(-kh_1)+B\exp(kh_1)+C\exp(-mh_1)+D\exp(mh_1)=\frac{\sigma}{k}b \quad \text{at} \quad z=-h_1 \tag{24}$$

Solving A, B, C and D from (20), (21), (23) and (24), substituting them into (22), we obtain the dispersion relation

$$\Omega\left[(1-\gamma)\left(\frac{kg}{\sigma^2}\operatorname{th} kh_1-1\right)-\gamma\left(1-\frac{\sigma^2}{gk}\operatorname{th} kh_1\right)\right]=\operatorname{th} kh_1-\frac{\sigma^2}{gk} \tag{25}$$

where

$$\Omega=\frac{1}{\Lambda}k(m^2-k^2)^2[4k\operatorname{ch} kh_2\operatorname{sh} mh_2-4m\operatorname{ch} mh_2\operatorname{ch} kh_2] \tag{26a}$$

$$\Lambda=4k(m^2+k^2)^2(k\operatorname{sh} mh_2\operatorname{sh} kh_2-m\operatorname{ch} kh_2\operatorname{ch} mh_2)+16k^4m(m\operatorname{sh} kh_2\operatorname{sh} mh_2-k\operatorname{ch} mh_2\operatorname{ch} kh_2)+16k^3m(m^2+k^2) \tag{26b}$$

$$\gamma=\rho_1/\rho_2 \tag{26c}$$

Now we discuss the dispersion relation.

When the depth of the lower layer fluid is very small, $h_2\to 0$. from (26a), $\Omega\to 0$, then the dispersion relation (25) becomes

$$\sigma^2=gk\operatorname{th} kh_1 \tag{27}$$

(27) is just the dispersion relation for upper layer water waves.

When the density of the lower layer fluid is very large, $\gamma\to 0$, we can imagine that the lower-layer fluid is just like a solid and the dispersion relation should agree with that for the top layer water. (25) has confirmed that fact.

If the fluid in the lower layer is very tough, very sticky and hardly different from the solid, $|\nu_e|\to\infty$, then, $m\to k$, $\Omega\to 0$, the dispersion relation (25) becomes that for the top layer fluid too.

If the viscosity and elasticity of the lower-layer fluid are all small, the fluid in the lower layer can be thought to be an ideal one, then, $m\to\infty$, $\Omega\to\operatorname{th} kh_2$, the dispersion relation (25) becomes

$$\sigma^4(\rho_2\operatorname{cth} kh_2\operatorname{cth} kh_1+\rho_1)-\sigma^2\rho_2(\operatorname{cth} kh_2+\operatorname{cth} kh_1)gk+(\rho_2-\rho_1)g^2k^2=0 \tag{28}$$

This is just the dispersion relation for a two layer fluid [9].

Using (9), it can be proved that the dispersion relation (25) is consistant with (3.20) in [3].

From (24), the amplitude ratio between the interface wave and the free surface wave is

$$\frac{b}{a}=\operatorname{ch} kh_1-\frac{gk}{\sigma^2}\operatorname{sh} kh_1 \tag{29}$$

It is valuable to mention the case of pure elasticity, $\nu=0$. From (9), m is real or pure imaginary. Ω must be real according to (26a). Therefore, k determined by (25) is real. It means that wave does not decay in the case of pure elasticity. It is the same as Mallard and Dalrymple's conclusion.

III. EXPLICIT EXPRESSIONS FOR WAVE ATTENUATION COEFFICIENTS

Usually a numerical method must be used to solve the root $k=k_r+k_i$ of the dispersion relation (25). To show the effect of each factor on wave attenuation, to supply a simple formula for engineering design, and to provide a reliable method to check the numerical results, it is necessary to simplify (25) to get a simple explicit expression for wave attenuation coefficient.

Under the assumption that the depth of the lower layer fluid is infinite and that wave length is long and ν_e is very large or very small, [3] got the explicit expressions for the wave attenuation. The explicit expressions will be derived here without first two assumptions and the application range of the explicit expressions will be extented. Usually the wave attenuation coefficients are small, so we let

$$k=k_0+k_1=k_0+k_{1r}+ik_i \tag{30}$$

where k_0 is real, k_1 is a complex correction and $|k_1/k_0| \ll 1$. Therefore, $\exp(-k_i x)$ is a factor of wave attenuation and k_i is the wave attenuation coefficient.

3.1 $v_e=v+iG/\rho\sigma \to 0$ or $k^2|v_e|/\sigma \ll 1$

From (9),

$$|k_0^2/m^2| \doteq k_0^2|v_e|/\sigma \ll 1 \tag{31}$$

$$m \doteq \sqrt{\sigma/(2v_e)}\,(-1+i)\left(1+\frac{1}{2}ik_0^2 v_e/\sigma\right) \tag{32}$$

where $|\ \ |$ is the module of a complex. Expanding (25) in powers of small k/m, we get

$$\Omega=\text{th}\,kh_2\left[1-\frac{k}{m}\text{th}\,mh_2(\text{cth}\,kh_2-\text{th}\,kh_2)-\frac{k^2}{m^2}(\text{th}^2 mh_2\,\text{sech}^2 kh_2+4)\right]+O(k^3/m^3) \tag{33}$$

(1) $h_2 \to 0$, $k_0 h_2 \ll |m|h_2 \ll 1$

$$\Omega=kh_2\left[\frac{1}{3}(mh_2)^2-4k^2/m^2\right] \tag{34}$$

Substituting (34) into (35), expanding (25) in powers of small k_1, then

$$\sigma^2=gk_0\text{th}\,k_0h_1 \tag{35}$$

$$D=\frac{\sqrt{gh_1}}{\sigma}(\text{I}_\text{m}k_1)=\frac{\rho_1}{\rho_2}k_0h_2\frac{k_0^2 v}{\sigma}\left[4+\frac{k_0^2h_2^2}{3k_0^4|v_e|^2/\sigma^2}\right]\frac{1}{\frac{1}{2}\text{sh}\,2k_0h_1+k_0h_1} \tag{36}$$

Since $|mh_2| \ll 1$, $\sigma h_2^2/|v_e| \ll 1$, so $D \ll 1$ in (36).

(2) $kh_2 \ll 1$, $|mh_2| \gg 1$, $kh_1 \ll 1$

The first two terms in (33) should be retained for Ω and the first is the main one. Expanding (25), we have

$$\sigma^2=gk_0^2(h_1+h_2) \tag{37}$$

$$k_1=\frac{1}{2(h_1+h_2)}\left[(1-\gamma)\frac{k_0h_1h_2}{h_1+h_2}+\frac{\gamma\sigma^2k_0h_1h_2}{g}+\frac{k_0h_2}{mh^2}\text{th}\,mh_2\right] \tag{38}$$

In (38) there is an imaginary part in the third term only, and

$$m=m_r+im_i=\begin{cases}\sqrt{\sigma/(G/\rho\sigma)}\left(\dfrac{-1}{2}\dfrac{v}{G/\rho\sigma}+i\right) & v \ll G/\rho\sigma\\ \sqrt{\sigma/2v}\,(-1+i) & v \gg G/\rho\sigma\end{cases} \tag{39}$$

$$\text{Im}\left(\frac{\text{th}\,mh_2}{mh_2}\right)=\frac{-m_i\text{th}\,m_rh_2+\frac{1}{2}m_r\sin 2m_ih_2\,\text{sech}^2 m_rh_2}{(m_i^2+m_r^2)h_2(\cos^2 m_ih_2+\text{th}^2 m_rh_2\sin^2 m_ih_2)} \tag{40}$$

Using $|mh_2| \gg 1$

$$D=\frac{\sqrt{gh}}{\sigma}\text{Im}\,k_1=\frac{(h_1/h_2)^{1/2}}{4(1+h_1/h_2)^{3/2}}\frac{1}{\sqrt{\sigma/2v}\;h_2} \qquad v \gg G/\rho\sigma \tag{41a}$$

$$D=\frac{\operatorname{th}\left(\frac{v}{2G/\rho\sigma}\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right)-\frac{1}{4}\frac{v}{G/\rho\sigma}\sin\left(2\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right)\operatorname{sech}^2\left(\frac{v}{2G/\rho\sigma}\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right)}{\left[\cos^2\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2+\operatorname{th}^2\left(\frac{v}{2G/\rho\sigma}\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right)\sin^2\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right]}$$

$$\cdot(h_1/h_2)^{1/2}\Big/\left[2(1+h_1/h_2)^{3/2}\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right]\qquad v\ll G/\rho\sigma \tag{41b}$$

If $|mh_2|\gg G/(\rho\sigma v)$, (41b) becomes

$$D=(h_1/h_2)^{1/2}\Big/\left[2(1+h_1/h_2)^{3/2}\sqrt{\frac{\sigma}{G/\rho\sigma}}h_2\right]\qquad v\ll G/\rho\sigma \tag{42}$$

(3) $kh_2=O(1)$, $|mh_2|\gg 1$, $kh_1\ll 1$

In this case, Ω includes the first two terms in (33), and the first is the main term. We expand (25) and obtain

$$\sigma^2=gk_0\operatorname{th}k_0h_2 \tag{43}$$

$$D=\frac{\sqrt{gh_1}}{\sigma}(\operatorname{Im}k_1)=\frac{k_0h_2}{\frac{1}{2}\operatorname{sh}2k_0h_2+k_0h_2}\sqrt{\frac{k_0h_1}{\operatorname{th}k_0h_2}}\frac{1}{2\sqrt{\frac{\sigma}{2v}}h_2}\qquad v\gg G/\rho\sigma \tag{44}$$

when $v\ll G/\rho\sigma$ the only modification is to multiply (44) by a factor similar to the first factor in (41b)

(4) $k_0h_2\ll 1$, $|mh_2|=O(1)$, $k_0h_1=O(1)$

Ω still contains the first two terms in (33), which have the same order

$$\sigma^2=gk_0\operatorname{th}k_0h_1 \tag{45}$$

$$D=\frac{\sqrt{2}\gamma}{2}\sqrt{\frac{k_0^2v}{\sigma}}\sqrt{\frac{k_0h_1}{\operatorname{th}k_0h_1}}\frac{1}{\frac{1}{2}\operatorname{sh}2k_0h_1+k_0h_1}$$

$$\frac{\operatorname{th}\sqrt{\frac{\sigma}{2v}}h_2-\frac{1}{2}\sin 2\sqrt{\frac{\sigma}{2v}}h_2\operatorname{sech}^2\sqrt{\frac{\sigma}{2v}}h_2}{\cos^2\sqrt{\frac{\sigma}{2v}}h_2+\operatorname{th}^2\sqrt{\frac{\sigma}{2v}}h_2\sin^2\sqrt{\frac{\sigma}{2v}}h_2}\qquad v\gg G/\rho\sigma \tag{46}$$

If $v\ll G/\rho\sigma$, the last factor in (46) must be replaced by a factor similar to the first factor in (41b).

(5) $h_2\to\infty$, i. e. $k_0h_2\gg 1$

$$\Omega=1-4(k^2/m^2)$$

k_0 satisfies the following equation

$$(1-\gamma)k^2g^2\operatorname{th}k_0h_1-gk_0\sigma^2+\gamma\sigma^4\operatorname{th}k_0h_1=\left(1+\frac{4k_0^2}{m^2}\right)(gk_0\sigma^2\operatorname{th}k_0h_1-\sigma^4)$$

The solution of this equation is

$$\sigma^2=gk_0\qquad\qquad \sigma^2=gk_0\operatorname{th}k_0h_1\frac{1-\gamma}{1+\gamma\operatorname{th}k_0h_1} \tag{47a}$$

For surface wave, we use $\sigma^2=gk_0$, then

$$k_1=\frac{4k_0^3v_ei}{\sigma}\frac{1-\operatorname{th}k_0h_1}{1+(2\gamma-1)\operatorname{th}k_0h_1} \tag{47b}$$

If $k_0 h_1 \ll 1$, i.e. in the long wave case,

$$D = 4 k_0^2 h_1^2 \nu / \sqrt{g h_1^3} \tag{48}$$

The attenuation coefficient in (48) is independent of G, and agrees with that in [3].

3.2 $\nu_e = \nu + iG/\rho\sigma \to \infty$, i.e. $\sigma/(k^2|\nu_e|) \ll 1$

From (9),

$$m = k[1 - i\sigma/(2k^2\nu_e)] \tag{49}$$

After some careful operations and neglecting the higher order, we have

$$\Omega = \frac{-i\sigma}{2k^2\nu_e}\left(\frac{1}{2}\,\mathrm{sh}\,2kh_2 - kh_2\right)\Big/(\mathrm{ch}^2 kh_2 + k^2h_2^2) \tag{50}$$

From the dispersion relation (25),

$$\sigma^2 = gk_0\,\mathrm{th}\,k_0 h_1 \tag{51}$$

$$D = \frac{r}{2}\sqrt{\frac{k_0 h_1}{\mathrm{th}\,k_0 h_1}}\;\frac{\sigma}{k_0^2}\;\frac{\nu}{\nu^2 + G^2/\rho^2\sigma^2}\;\frac{\frac{1}{2}\,\mathrm{sh}\,2k_0h_2 - k_0h_2}{\mathrm{ch}^2 k_0 h_2 + k_0^2 h_2^2}\cdot\frac{1}{\frac{1}{2}\,\mathrm{sh}2\,k_0h_1 + k_0h_1} \tag{52}$$

From the above formula, the attenuation coefficient is inversely proportional to ν when $\nu \gg G/\rho\sigma$. If $\nu \ll G/\rho\sigma$, D is proportional to ν.

For the case of long wave and thick mud layer, $k_0 h_2 \gg 1$, $k_0 h_1 \ll 1$, (52) is simplified as follows

$$D = \frac{\gamma}{4}\,\frac{g\sqrt{gh_1}}{\sigma^2}\,\frac{\nu}{\nu^2 + G^2/\rho^2\sigma^2} = \frac{\gamma}{4}\,\frac{1}{k_0h_1}\,\frac{\sigma}{k_0^2}\,\frac{\nu}{\nu^2 + G^2/\rho^2\sigma^2} \tag{53}$$

It is the same result as that in [3]. The formula in [3] is a special case of (52) only.

If two layers are both thin, i. e. $kh_1 \ll 1$, $kh_2 \ll 1$, (52) is simplified to

$$D = \frac{2}{3}(k_0h_2)^3\,\frac{\gamma}{4}\,\frac{1}{k_0h_1}\,\frac{\sigma}{k_0^2}\,\frac{\nu}{\nu^2 + G^2/\rho^2\sigma^2} \tag{54}$$

This means that compared with the case $h_2 \to \infty$, the attenuation coefficient in the case $h_2 \to 0$, can almost be neglected when $|\nu_e|$ is large.

From the above explicit expressions, we can see that

1) when $kh_1 \gg 1$, i.e. the water in the upper layer is very deep or the wave length is short, waves do not decay due to this mechanism, $D=0$. It is consistent with the physical intuition.

2) In the case of long wave, large viscosity and $(\sigma/k^2|\nu_e|)/k_0h_1 = O(1)$, i.e. $\nu = O((g^3h_1)^{1/2}/\sigma^2)$, we have $D = O(1)$, and $D \ll 1$ in other cases.

The classification in different cases is dependent on k when $\nu_e \to 0$, and the formulae for k_0 are different in different cases. So there are some difficulties to classify each case. We show here that in all cases k_0 can be computed by

$$\sigma^2 = gk_0\,\mathrm{th}\,k_0'(h_1 + h_2) \tag{55}$$

(35), (37), (43), (45) and (47a) are the first term approximation of (55) in each case. All approximate k_0 in the next paragraph are computed by (55) when $|\nu_e|$ is very small.

IV. COMPARISON WITH THE NUMERICAL RESULTS

In this paragraph, we compare the approximate formulae (the dashed lines in figures) with the numerical results (the solid lines in figures).

The first model is Gade's experiment. The top layer fluid is kerosine with density

0.8593 g/cm^3 and depth 3.81cm. A water-sugar solution with density 1.504g/cm^3 and viscosity 2.6 cm^2/s is in the lower-layer. The wave frequency σ is 4.4857. In Fig.2, we use two approximate formulae (44) and (46) to approach the curve. The approximation is very good except in the neighbourhood of the highest point, where the approximate formulae give an overestimated value. In the region where $H_2=(\sigma/2\nu)^{1/2}h_2$ is small, our approximation is almost the same as Gade's long wave approximation. At large H_2, our results have almost the same accuracy as those from the boundary layer theory of [5], but the application region is wider. In this example, small k_0h_1 in (44) is applicable up to 0.2, but k_0h_2 as order $O(1)$ is only 0.2— 0.6. Small k_0h_2 in (46) is applicable up to 0.18, but k_0h_1 as order $O(1)$ is only 0.2— 0.25, a little bit larger than k_0h_2. If small quantities k_0h_1 in (44) and k_0h_2 in (46) are smaller, the approximation will be better.

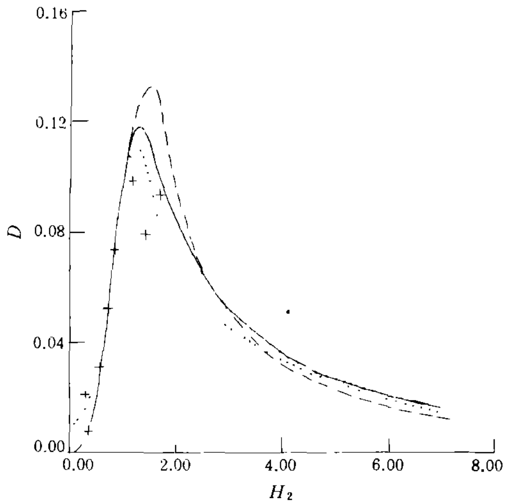
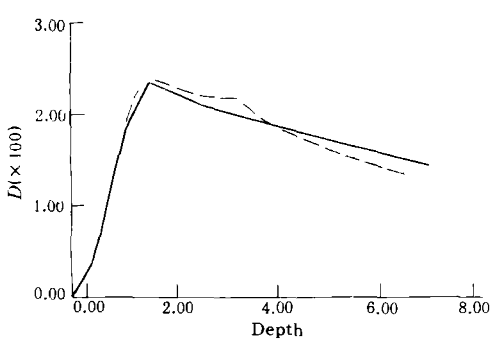


Fig.2 Comparison of D by different method for Gade's experiment model
— numerical result (25) -- approximation (44) and (46)
+ Gade's experimental point
.... Gade's long wave theory and
.... the boundary layer theory in [5]

The relationship between the attenuation and the mud depth is shown in Fig.3 and 4. The parameters in Fig.3 are $\rho_1=1.028$ g/cm^3 $h_1=400$cm, $\rho_2=1.800$ g/cm^3, $G=0$, $\nu=10^3$cm^2/s and $\sigma=0.52$. The two formulae (42) and (46) are used to approach the curve. The approximation is very good when $H_2<2.6$, and the accuracy is good for the whole curve. In this example, small quantities are all near 0.25. $|\nu_e|$ is large in Fig. 4. The parameters are $\rho_1=1$g/cm^3, $\rho_2=2$g/cm^3, $h_1=1000$cm, $G=2\times10^8$ dyn/cm^2, $\nu=10^3$cm^2/s and $\sigma=.52$. In this case only one formula (52) or (54) is needed to cover the whole curve. The attenuation is very small and the case can be considered as of no attenuation.



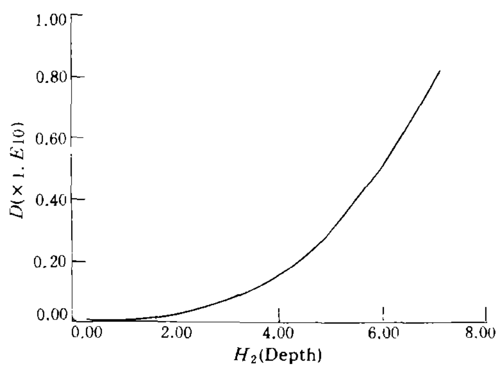


Fig.3 Attenuation coefficient versus dimensionless mud depth

Fig. 5 and 6 show the relationship between the attenuation and the dimensionless viscosity $(\nu/(gh_1^3))^{1/2}$). Because of small error and the use of logarithm coordinate system, the two curves are overlapped in the figure. The parameters in Fig.5 are $\rho_1=1$g/cm^3, $\rho_2=2$g/cm^3, $h_1=10^3$cm, $h_2=5\times10^3$cm, $G=0$ and $\sigma=0.52$. It is the case where the depth of the mud is infinite. In Fig.6, the depth of the mud has a limited thickness. The parameters are the same as in Fig. 5 except $h_2 = 5 \times 10^2$cm. It is shown in the two figures that there is an optimum viscosity where the attenuation is maxmum. Our approximation reveals the optimum value very well. The maximum attenuation in Figs.5 and 6 are the intersection points of (47b) and (52) and of (46) and (52), respectively.

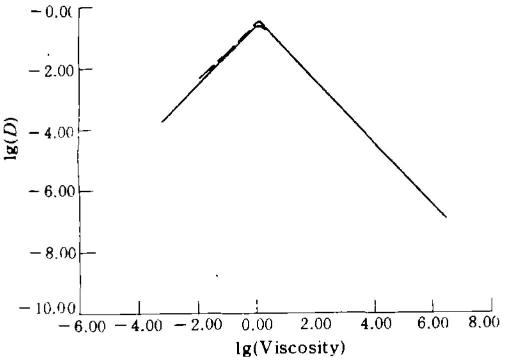


Fig 5 Attenuation coefficient versus dimensionless viscosity $\gamma/\sqrt{gh_1^3}$

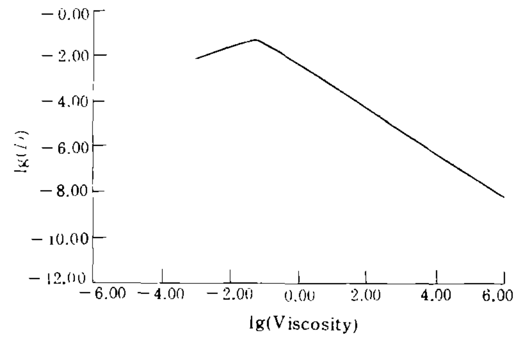


Fig.6 Attenuation coefficient versus dimensionless viscosity $\gamma/\sqrt{gh_1^3}$

Though each approximate formula is derived in each special case, from the above comparison they are shown to have a good accuracy in a large region and only one or two formulae are needed to cover a whole curve. This does offer some convenience for discussing the effects of each physical factor and can relieve us from the numerical computation.

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