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HIGH-ORDER ASYMPTOTIC ANALYSIS FOR THE CRACK IN NONLINEAR MATERIAL*

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ABSTRACT: Accurate high-order asymptotic analyses were carried out for Mode II plane strain crack in power hardening materials. The second-order crack tip fields have been obtained. It is found that the amplitude coefficient k_2 of the second term of the asymptotic field is correlated to the first order field as the hardening exponent $n<n^*(n^*\approx 5)$, but as $n\geqslant n^*$, k_2 turns to become an independent parameter. Our results also indicated that, the second term of the asymptotic field has little influence on the near-crack-tip field and can be neglected when $n<n^*$. In fact, k_2 directly reflects the effects of triaxiality near the crack tip, the crack geometry and the loading mode, so that besides J-integral it can be used as another characteristic parameter in the two-parameter criterion.

KEY WORDS: asymptotic analysis, near-crack-tip field, triaxiality

I. INTRODUCTION

Since the well-known J-integral[1] and the HRR singular fields[2-4] were proposed, much theoretical and experimental work has been done in order to employ the J-integral as a ductile fracture criterion. In recent years, the critical value of the J-integral denoted by J_{IC} has been accepted as a measure of the toughness of ductile materials. But from a strict point of view, the onset of crack growth can be phrased in terms of the attainment of J_{IC} only when the region dominated by the J-characterized HRR singular fields encloses completely the fracture process zone. This naturally brings up some requirements for specimen, including configuration geometry, material properties and loading mode. For the Mode I plane strain crack problem, these requirements have been disscussed in detail by Shih and German[5], McMeeking and Parks[6], Needleman and Tvergaad[7] and Wang[8]. All of these analyses have indicated that the size requirements about the ligament for the bend crack geometries are quite different from those for the tension crack geometries. For the bend crack geometries, the ligament c roughly obeys $c\geqslant 25\,J_{IC}/\sigma_{flow}$ while for the tension crack geometries, $c\geqslant 200\,J_{IC}/\sigma_{flow}$.

As a matter of fact, the above two different conditions certainly have something to do with the triaxial stress state near the crack tip. Thus, some researchers proposed to use two parameters for characterizing the stress state near crack tip. The first parameter is J-integral, and the second one is a parameter which can reflect the triaxiality at the crack tip. Li and Wang[9] made a high-order asymptotic analysis on the Mode I plane strain crack problem, and obtained the second-order asymptotic fields. Using two amplitude coefficients k_1 and k_2, the effect of the configuration geometry can be characterized. In this way, a theoretical foundation is provided for two-parameter fracture criterion.

Recently, O'Dowd and Shih[10,11] introduced the concept of J-Q annulus. Within the annulus, the full range of both high and low triaxiality fields can be parameterized by Q and J/σ_o, where σ_o is the yield stress. Q can be interpreted as a (non-dimensional) stress triaxiality parameter, which in fact is the same as k_2 of Li and Wang[9]. Betegón and Hancock[12] have also discussed the two-parameter charaterization of elastic-plastic crack tip fields. The different near-tip fields at large-scale yielding in bend and tension geometries have been matched with the small-scale yielding fields, which are obtained by a modified boundary layer formulation based

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on the K_{I}-field and the T-stress of the asymptotic series of the elastic field.

Bradford[13] analysed a Mode II crack using the finite element method. Crack tip stress and strain fields are investigated and shown to be in agreement with the HRR fields at all loading levels from small-scale yielding to general yielding. This demonstrates that an initiation value of J may be used as a criterion for the onset of crack growth. The study mentioned above is only for the edge-cracked square plate of power hardening material with hardening exponent $n=3$. Therefore, it is still difficult to reach a general conclusion for the Mode II crack problems.

This paper is an extension of the work by Li and Wang[9]. A similar investigation is made for the Mode II plane strain crack. It is proved that the contribution of the second-order solution to the crack tip stress field is much smallar as compared with that of the first-order (i.e. the HRR solution), and that the amplitude coefficient k_2 is related to the first-order field if the hardening exponent n is smaller than n^* where $n^* \approx 5$. If $n \geqslant n^*$, k_2 becomes an independent parameter which relates not only to the material properties, but also to the crack geometry and yielding scale.

II. GOVERNING EQUATION

In this paper, the Ramberg-Osgood hardening material is investigated, whose constitutive relation in uniaxial tension can be expressed as

$$\varepsilon=\sigma+\alpha\sigma^n \tag{2.1}$$

where σ and ε are the non-dimensional stress and strain respectively, which are in turn defined as

$$\sigma=\bar{\sigma}/\bar{\sigma}_o \qquad \varepsilon=\bar{\varepsilon}/\bar{\varepsilon}_o$$

where, $\bar{\sigma}_o$ is the yield stress; and $\bar{\varepsilon}_o$ is the corresponding yield strain equal to $\bar{\sigma}_o/E$; E is the initial slope of the stress-strain curve (i.e. Young's modulus); α and n are hardening coefficient and hardening exponent, respectively. Throughout this paper, all unbarred stress and strain will be non-dimensional. Italic letters i, j, k are used for subscript indices running over 1, 2, 3, and Greek letters β, γ, ρ for subscript indices running over 1, 2.

Under multi-axial stress states, the strain is given as

$$\varepsilon_{ij}=(1+v)S_{ij}+\frac{(1-2v)}{3}\sigma_{kk}\delta_{ij}+\frac{3}{2}\alpha\sigma_e^{n-1}S_{ij} \tag{2.2}$$

where S_{ij} is the stress deviator, σ_e the effective stress and v Poisson's ratio.

$$S_{ij}=\sigma_{ij}-\delta_{ij}\sigma_{kk}/3 \qquad \sigma_e^2=\frac{3}{2}S_{ij}S_{ij}$$

In asymptotic analysis for near-crack-tip field in plane strain, (2.2) can be further written in the brief form:

$$\varepsilon_{\beta\gamma}=\varepsilon_{\beta\gamma}^e+\varepsilon_{\beta\gamma}^p=(1+v)\sigma_{\beta\gamma}+\delta_{\beta\gamma}\Gamma\sigma_{\rho\rho}+\Lambda S_{\beta\gamma} \tag{2.3}$$

where $\Lambda=\frac{3}{2}\alpha\sigma_e^{n-1}$; $\Gamma=-(1+v)v+\left(\frac{1}{2}-v\right)^2$.

If a nondimensional stress function φ is introduced, stress components can be given as

$$\left.\begin{aligned}\sigma_r&=\frac{1}{r}\left(\frac{\partial\varphi}{\partial r}+\frac{1}{r}\frac{\partial^2\varphi}{\partial\theta^2}\right)\\ \sigma_\theta&=\frac{\partial^2\varphi}{\partial r^2}\\ \tau_{r\theta}&=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right)\end{aligned}\right\} \tag{2.4}$$

where $\varphi=\bar{\varphi}/\bar{\sigma}_o L^2$, $r=\bar{r}/L$, L is the characteristic length of the specimen.

Let

$$\varphi=K_1 r^{s_1+2}\tilde{\varphi}_1(\theta)+K_2 r^{s_2+2}\tilde{\varphi}_2(\theta) \tag{2.5}$$

then, by Eq.(2.4), the stress takes the form:

$$\sigma_{\beta\gamma}=K_1 r^{s_1}[\tilde{\sigma}_{\beta\gamma 1}(\theta)+\eta r^{\Delta s_2}\tilde{\sigma}_{\beta\gamma 2}(\theta)] \tag{2.6}$$

where

$$\left.\begin{aligned}&\tilde{\sigma}_{rl}=\tilde{\varphi}_l''+(s_l+2)\tilde{\varphi}_l\\&\tilde{\sigma}_{\theta l}=(s_l+2)(s_l+1)\tilde{\varphi}_l \qquad (l=1,\ 2)\\&\tilde{\tau}_{r\theta l}=-(s_l+1)\tilde{\varphi}_l'\\&\eta=K_2/K_1 \qquad \Delta s_2=s_2-s_1\end{aligned}\right\} \tag{2.7}$$

and

$$\sigma_e^2=K_1^2 r^{2s_1}\tilde{\sigma}_{e_1}^2\left[1+\left(2\eta r^{\Delta s_2}\frac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}+\eta^2 r^{2\Delta s_2}\frac{\tilde{\sigma}_{e_2}^2}{\tilde{\sigma}_{e_1}^2}\right)\right] \tag{2.8}$$

As $r\to 0$, the quantity in parentheses of (2.8) is an infinitesimal one. Therefore

$$\sigma_e^{n-1}\approx K_1^{n-1}r^{(n-1)s_1}\tilde{\sigma}_{e1}^{n-1}\left[1+\eta(n-1)r^{\Delta s_2}\frac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e1}^2}\right] \tag{2.9}$$

where

$$\left.\begin{aligned}&\tilde{\sigma}_{e1}^2=\frac{3}{4}(\tilde{\sigma}_{r1}-\tilde{\sigma}_{\theta 1})^2+3\tilde{\tau}_{r\theta 1}^2\\&\tilde{\sigma}_{e12}=\frac{3}{4}(\tilde{\sigma}_{r1}-\tilde{\sigma}_{\theta 1})(\tilde{\sigma}_{r2}-\tilde{\sigma}_{\theta 2})+3\tilde{\tau}_{r\theta 1}\tilde{\tau}_{r\theta 2}\\&\tilde{\sigma}_{e2}^2=\frac{3}{4}(\tilde{\sigma}_{r2}-\tilde{\sigma}_{\theta 2})^2+3\tilde{\tau}_{r\theta 2}^2\end{aligned}\right\} \tag{2.10}$$

Substituting (2.6) into (2.3), we obtain

$$\left.\begin{aligned}&\varepsilon_{\beta\gamma}^e=K_1 r^{s_1}(\tilde{\varepsilon}_{\beta\gamma 1}^e+\eta r^{\Delta s_2}\tilde{\varepsilon}_{\beta\gamma 2}^e)\\&\varepsilon_{\beta\gamma}^p=\alpha K_1^n r^{ns_1}(\tilde{\varepsilon}_{\beta\gamma 1}^p+\eta r^{\Delta s_2}\tilde{\varepsilon}_{\beta\gamma 2}^p)\end{aligned}\right\} \tag{2.11}$$

where

$$\tilde{\varepsilon}_{\beta\gamma 1}^e=(1+\nu)\tilde{\sigma}_{\beta\gamma l}+\delta_{\beta\gamma}\Gamma\tilde{\sigma}_{\rho\rho l}, \qquad (l=1,\ 2) \tag{2.12}$$

$$\left.\begin{aligned}&\tilde{\varepsilon}_{r1}^p=\frac{3}{4}\tilde{\sigma}_{e1}^{n-1}(\tilde{\sigma}_{r1}-\tilde{\sigma}_{\theta 1})\\&\tilde{\varepsilon}_{\theta 1}^p=-\tilde{\varepsilon}_{r1}^p\\&\tilde{\varepsilon}_{r\theta 1}^p=\frac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\tilde{\tau}_{r\theta 1}\end{aligned}\right\} \tag{2.13}$$

$$\left.\begin{aligned}&\tilde{\varepsilon}_{r2}^p=\frac{3}{4}\tilde{\sigma}_{e1}^{n-1}\left[(n-1)\frac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}(\tilde{\sigma}_{r1}-\tilde{\sigma}_{\theta 1})+(\tilde{\sigma}_{r2}-\tilde{\sigma}_{\theta 2})\right]\\&\tilde{\varepsilon}_{\theta 2}^p=-\tilde{\varepsilon}_{r2}^p\\&\tilde{\varepsilon}_{r\theta 2}^p=\frac{3}{2}\tilde{\sigma}_{e1}^{n-1}\left[(n-1)\frac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}\tilde{\tau}_{r\theta 1}+\tilde{\tau}_{r\theta 2}\right]\end{aligned}\right\} \tag{2.14}$$

The displacement near the crack tip can be obtained by integration of Eq.(2.14). Neglecting the rigid displacement, we get

$$u_\beta=K_1 r^{s_1+1}(\tilde{u}_{\beta 1}^e+\eta r^{\Delta s_2}\tilde{u}_{\beta 2}^e)+\alpha K_1^n r^{ns_1+1}(\tilde{u}_{\beta 1}^p+\eta r^{\Delta s_2}\tilde{u}_{\beta 2}^p)+\cdots \tag{2.15}$$

where

$$\left.\begin{aligned}\tilde{u}^e_{r1}&=\tilde{\varepsilon}^e_{r1}/(1+s_1)\\ \tilde{u}^e_{\theta 1}&=(2\tilde{\varepsilon}^e_{r\theta 1}-\tilde{u}^{e\,\prime}_{r1})/s_1\end{aligned}\right\}\tag{2.16}$$

$$\left.\begin{aligned}\tilde{u}^p_{r1}&=\tilde{\varepsilon}^p_{r1}/(1+ns_1)\\ \tilde{u}^p_{\theta 1}&=(2\tilde{\varepsilon}^p_{r\theta 1}-\tilde{u}^{p\,\prime}_{r1})/ns_1\end{aligned}\right\}\tag{2.17}$$

$$\left.\begin{aligned}\tilde{u}^e_{r2}&=\tilde{\varepsilon}^e_{r2}/(1+s_1+\Delta s_2)\\ \tilde{u}^e_{\theta 2}&=(2\tilde{\varepsilon}^e_{r\theta 2}-\tilde{u}^{e\,\prime}_{r2})/(s_1+\Delta s_2)\end{aligned}\right\}\tag{2.18}$$

$$\left.\begin{aligned}\tilde{u}^p_{r2}&=\tilde{\varepsilon}^p_{r2}/(1+ns_1+\Delta s_2)\\ \tilde{u}^p_{\theta 2}&=(2\tilde{\varepsilon}^p_{r\theta 2}-\tilde{u}^{p\,\prime}_{r2})/(ns_1+\Delta s_2)\end{aligned}\right\}\tag{2.19}$$

The strain compatibility equation is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varepsilon_\theta)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\varepsilon_r-\frac{1}{r}\frac{\partial}{\partial r}\varepsilon_r-\frac{2}{r^2}\frac{\partial^2}{\partial r\partial\theta}(r\varepsilon_{r\theta})=0\tag{2.20}$$

Substituting (2.3) and (2.11) into (2.20) gives

$$\alpha K_1^n r^{ns_1-2}\Pi_1^p+\alpha K_1^n\eta r^{ns_1+\Delta s_2-2}\Pi_2^p+K_1r^{s_1-2}\Pi_1^e+K_1\eta r^{s_1+\Delta s_2-2}\Pi_2^e=0\tag{2.21}$$

where

$$\left.\begin{aligned}\Pi_1^p&=\tilde{\varepsilon}^{p\,\prime\prime}_{r1}-ns_1(ns_1+2)\tilde{\varepsilon}^p_{r1}-2(ns_1+1)\tilde{\varepsilon}^{p\,\prime}_{r\theta 1}\\ \Pi_2^p&=\tilde{\varepsilon}^{p\,\prime\prime}_{r2}-(ns_1+\Delta s_2)(ns_1+\Delta s_2+2)\tilde{\varepsilon}^p_{r2}-2(ns_1+\Delta s_2+1)\tilde{\varepsilon}^{p\,\prime}_{r\theta 2}\\ \Pi_1^e&=\tilde{\varepsilon}^{e\,\prime\prime}_{r1}-s_1\tilde{\varepsilon}^e_{r1}+s_1(s_1+1)\tilde{\varepsilon}^e_{\theta 1}-2(s_1+1)\tilde{\varepsilon}^{e\,\prime}_{r\theta 1}\\ \Pi_2^e&=\tilde{\varepsilon}^{e\,\prime\prime}_{r2}-(s_1+\Delta s_2)\tilde{\varepsilon}^e_{r2}+(s_1+\Delta s_2)(s_1+\Delta s_2+1)\tilde{\varepsilon}^e_{\theta 2}-2(s_1+\Delta s_2+1)\tilde{\varepsilon}^{e\,\prime}_{r\theta 2}\end{aligned}\right\}\tag{2.22}$$

The stress free condition on the crack face requires

$$\sigma_\theta(r,\pi)=\tau_{r\theta}(r,\pi)=0\tag{2.23}$$

which leads to

$$\left.\begin{aligned}\tilde{\varphi}_1(\pi)&=\tilde{\varphi}_1'(\pi)=0\\ \tilde{\varphi}_2(\pi)&=\tilde{\varphi}_2'(\pi)=0\end{aligned}\right\}\tag{2.24}$$

At $\theta=0$, we have

$$\left.\begin{aligned}\tilde{\varphi}_1(0)&=\tilde{\varphi}_1''(0)=0\\ \tilde{\varphi}_2(0)&=\tilde{\varphi}_2''(0)=0\end{aligned}\right\}\tag{2.25}$$

The normalized conditions are:

$$\tilde{\varphi}_1'(0)=1\qquad\tilde{\varphi}_2'(0)=1\tag{2.26}$$

Equations from (2.21) through (2.26) are the governing equations for the asymptotic fields.

III. SOLUTION OF GOVERNING EQUATIONS

It is noted that $\Delta s_2>0$, so that (2.21) first leads to

$$\Pi_1^p=0\tag{3.1}$$

therefore, the first-order field can be obtained by the solution of (3.1) with (2.24) and (2.25). This problem has been solved by Rice and Rosengren [2] and Hutchinson [3,4]. The eigenvalue is known to be

$$s_1=-\frac{1}{n+1} \tag{3.2}$$

In the following, the solution of the second-order field will be discussed.

First, assume

$$ns_1+\Delta s_2-2<s_1-2 \tag{3.3}$$

or

$$0<\Delta s_2<\frac{n-1}{n+1} \tag{3.4}$$

then the term $\alpha K_1^n \eta r^{ns_1+\Delta s_2-2}\Pi_2^p$ corresponds to the contribution of the second term of the asymptotic solution, which leads to the following equation:

$$\Pi_2^p=0 \tag{3.5}$$

Using (2.14), Eq.(3.5) can be transformed into a linear equation:

$$\Pi_2^p=D_1\tilde{\varphi}_2''''+D_2\tilde{\varphi}_2'''+D_3\tilde{\varphi}_2''+D_4\tilde{\varphi}_2'+D_5\tilde{\varphi}_2=0 \tag{3.6}$$

where $D_1 \sim D_5$ are functions of $\tilde{\varphi}_1 \sim \tilde{\varphi}_1''''$, s_1 and Δs_2, which are given in the Appendix.

Since Eq.(3.6) is a linear ordinary equation of $\tilde{\varphi}_2(\theta)$, we can first obtain the two particular solutions which satisfy the conditions

$$\text{(i)} \quad \tilde{\varphi}_2'(0)=1 \qquad \tilde{\varphi}_2'''(0)=0 \tag{3.7}$$

$$\text{(ii)} \quad \tilde{\varphi}_2'(0)=0 \qquad \tilde{\varphi}_2'''(0)=1 \tag{3.8}$$

respectively. Using the four-order Runge-Kutta method in which each integrating step length can be adjusted automatically as expected, one can integrate Eq.(3.6) from the initial conditions of (2.25) and (3.7) (or (3.8)). During the solving process, the numerical accuracy of each integration step can be controlled. In principle, it is possible to get any high accuracy solutions. But in practical calculating, the accuracy of each integration step is controlled within 10^{-8}. After obtaining the two particular solutions $\tilde{\varphi}_2^{(1)}(\theta)$ and $\tilde{\varphi}_2^{(2)}(\theta)$, we can form the general solution of Eq.(3.6) as

$$\tilde{\varphi}_2(\theta)=C_1\tilde{\varphi}_2^{(1)}(\theta)+C_2\tilde{\varphi}_2^{(2)}(\theta) \tag{3.9}$$

The satisfaction of boundary condition (2.24) yields

$$\left.\begin{aligned} C_1\tilde{\varphi}_2^{(1)}(\pi)+C_2\tilde{\varphi}_2^{(2)}(\pi)=0 \\ C_1\tilde{\varphi}_2^{(1)\prime}(\pi)+C_2\tilde{\varphi}_2^{(2)\prime}(\pi)=0 \end{aligned}\right\} \tag{3.10}$$

Eq.(3.10) has non-zero solution (C_1, C_2) if and only if its determinant is equal to zero, i.e.

$$\Delta=\tilde{\varphi}_2^{(1)}(\pi)\tilde{\varphi}_2^{(2)\prime}(\pi)-\tilde{\varphi}_{(2)}^{(1)\prime}(\pi)\tilde{\varphi}_2^{(2)}(\pi)=0 \tag{3.11}$$

Δ is obviously the function of Δs_2. We can adjust the value of Δs_2 to make Δ equal to zero, so as to get the solution of eigenfunction $\tilde{\varphi}_2(\theta)$.

Our calculation shows that no value of Δs_2 exists within the range of (3.4) which makes Δ of (3.11) vanish for $1<n<n^*$ ($n^*\approx 5$). Therefore, let $\Delta s_2=\frac{n-1}{n+1}$ (i.e. $\Delta s_2=0.5$ and $\Delta s_2=0.25$), we turn to solve the following equation derived from Eq.(2.27).

$$\Pi_2^p+\Pi_1^e=0 \tag{3.12}$$

where the assumption $\eta=1/\alpha K_1^{n-1}$ has been introduced.

Eq.(3.12) can be further written as:

$$D_1\tilde{\varphi}_2''''+D_2\tilde{\varphi}_2'''+D_3\tilde{\varphi}_2''+D_4\tilde{\varphi}_2'+D_5\tilde{\varphi}_2=-[\tilde{\varepsilon}_{r1}^{e\,\prime\prime}-s_1\tilde{\varepsilon}_{r1}^e+s_1(s_1+1)\tilde{\varepsilon}_{\theta1}^e-2(s_1+1)\tilde{\varepsilon}_{r\theta1}^{e\,\prime}] \tag{3.13}$$

Table 1
Mode II

n	5	10
Δs_2	0.66158988	0.49810168
s_2	0.49492321	0.40719259

As the first step, we can solve a particular solution $\tilde{\varphi}_2^{(0)}(\theta)$ of Eq. (3.13), which satisfies the homogeneous initial conditions.

$$\tilde{\varphi}_2(0)=\tilde{\varphi}_2'(0)=\tilde{\varphi}_2''(0)=\tilde{\varphi}_2'''(0)=0 \tag{3.14}$$

Combined with (3.9), the general solution of (3.13) becomes

$$\tilde{\varphi}_2=C_1\tilde{\varphi}_2^{(1)}+C_2\tilde{\varphi}_2^{(2)}+\tilde{\varphi}_2^{(0)} \tag{3.15}$$

where the unknown coefficients C_1 and C_2 can be determined by the boundary conditions:

$$\left.\begin{aligned} C_1\tilde{\varphi}_2^{(1)}(\pi)+C_2\tilde{\varphi}_2^{(2)}(\pi)=-\tilde{\varphi}_2^{(0)}(\pi) \\ C_1\tilde{\varphi}_2^{(1)\prime}(\pi)+C_2\tilde{\varphi}_2^{(2)\prime}(\pi)=-\tilde{\varphi}_2^{(0)\prime}(\pi) \end{aligned}\right\} \tag{3.16}$$

Since the determinant Δ of (3.16) is not equal to zero, there exists the unique solution (C_1, C_2).

The corresponding angular distributions of stress components $\tilde{\sigma}_{r2}$, $\tilde{\sigma}_{\theta 2}$, $\tilde{\tau}_{r\theta 2}$ and $\tilde{\sigma}_{e_2}$ (equal to $\tilde{\sigma}_{e_{12}}/\tilde{\sigma}_{e_1}$) are plotted in Fig.1. (The Poisson's ratio $\nu=0.3$).

However, when n increases and is equel to or larger than n^* (where $n^*\approx 5$), there indeed exist those Δs_2 which make Δ of (3.9) vanish within the range of (3.4). The calculated Δs_2 are given in Table 1. The stresses derived from Eq. (3.9) are depicted in Fig.2.

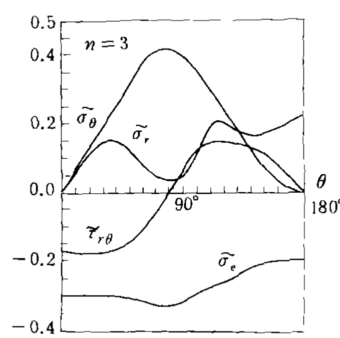

Fig.1 Angular distribution of $\tilde{\sigma}_{r2}$, $\tilde{\sigma}_{\theta 2}$, $\tilde{\tau}_{r\theta 2}$ and $\tilde{\sigma}_{e2}$ for $n=3$ (the subscript 2 omitted)

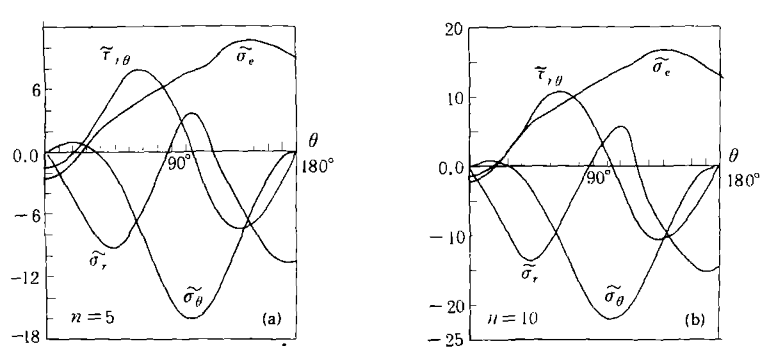

Fig.2 Angular distribution of $\tilde{\sigma}_{r2}$, $\tilde{\sigma}_{\theta 2}$, $\tilde{\tau}_{r\theta 2}$ and $\tilde{\sigma}_{e2}$ for (a) $n=5$, (b) $n=10$ (the subscript 2 omitted)

IV. DETERMINATION OF k_2

From Eqs. (2.6), we can easily find that the stresses take the following asymptotic expansion:

$$\sigma_{ij}=k_1\left(\frac{\bar{r}}{\bar{J}/\sigma_0}\right)^{s_1}\tilde{\sigma}_{ij1}(\theta)+k_2\left(\frac{\bar{r}}{\bar{J}/\sigma_0}\right)^{s_2}\tilde{\sigma}_{ij2}(\theta) \tag{4.1}$$

As mentioned in section 2, $\bar{r}=rL$, and $\bar{J}$ is the dimensional value of the J-integral, $J_1=\dfrac{\bar{J}E}{L\bar{\sigma}_0}$. The relation between k_1, k_2 and K_1, K_2 are

$$\left.\begin{aligned} k_1&=\left(\frac{\bar{J}}{\sigma_0 L}\right)^{s_1}K_1 \\ k_2&=\left(\frac{\bar{J}}{\sigma_0 L}\right)^{s_2}K_1 \end{aligned}\right\} \tag{4.2}$$

If the characteristic length is taken as $L=\bar{J}/\bar{\sigma}_0$, we have $k_1=K_1$, $k_2=K_2$. Eq.(4.1) can be also written as

$$\sigma_{ij}=k_1 r^{s_1}\tilde{\sigma}_{ij1}(\theta)+k_2 r^{s_2}\tilde{\sigma}_{ij2}(\theta) \tag{4.3}$$

here k_1 is given by

$$k_1=\left(\frac{1}{\alpha\bar{\varepsilon}_o I_1}\right)^{-s_1} \tag{4.4}$$

The coefficient k_1 depends only on material property while k_2 depends not only on the material property, but also on the specimen geometry and the yielding level. Here k_2 is the same as Q of O'Dowd and Shih [10, 11], which is interpreted as a stress triaxiality parameter.

According to Li and Wang [9], the values of k_2 can be determined by matching the two-term expansion with the full-field solution of Shih and German [5] and Needleman and Tvergaad [7] for Mode I plane strain crack. For Mode II plane strain crack problem, if $n>n^*$ (where $n^*\approx 5$), the values of k_2 can be determined in the same way as that for Mode I. Since the numerical results concerned with Mode II have not been found, the detailed values of k_2 remains to be determined.

On the other hand, if $n<n^*$, for example, $n=3$, k_2 is related to k_1 as afore-mentioned. As

$$K_2=\eta K_1 \qquad \eta=\frac{1}{\alpha K_1^{n-1}} \tag{4.5}$$

therefore

$$K_2=\frac{1}{\alpha K_1^{n-2}} \tag{4.6}$$

or

$$k_2=\frac{1}{\alpha k_1^{n-2}} \tag{4.7}$$

Table 2 gives some important values, by which the contribution of the second-order stress field for $n=3$ can be calculated. Here, E and $\bar{\varepsilon}_0$ are set to 210×10^3 MPa and 10^{-3}, respectively and $\nu=0.3$, $\alpha=1$. The value of I_1 can be obtained from Symington Shih and Ortiz [14].

The ratio of the second-order stress to the first-order stress takes the form:

$$\xi_{ij}(\theta)=|\eta r^{\left(\frac{n-1}{n+1}\right)}\tilde{\sigma}_{ij2}(\theta)/\tilde{\sigma}_{ij1}(\theta)| \tag{4.8}$$

Over the range: $0<r<50$, the maximum values of $\xi_{ij}(\theta)$ are given in Table 3.

Table 2

Mode II plane strain crack, $n=3$

s_1	s_2	$\tilde{\tau}_{r\theta 1}(0)$	$\tilde{\tau}_{r\theta 2}(0)$	$\tilde{\sigma}_{\theta 1}(45°)$	$\tilde{\sigma}_{\theta 2}(45°)$	I_1	k_1	k_2	η
−0.25	0.25	0.750	−0.171	−0.893	0.301	0.95	5.696	0.176	0.0308

Table 3

Mode II

$\xi_{r\theta\max}(0°)$	$\xi_{\theta\max}(45°)$
0.0187	0.0276

From Table 3, it can be seen that the contribution of the second-order stress is very small and can be ignored. In other words, the first-order stress field (i.e. the HRR solution) is a good approach to the near-crack-tip field. This agrees completely with that in Bradford[13].

V. CONCLUSIONS

For Mode II plane strain crack, if $n<n^*$ (where $n^*\approx 5$), the second term of the asymptotic expansion series is coupled with the first elastic strain term of the same series. It is also indicated that the second-order stresses are too small as compared with the frist-order ones and can be neglected. On the other hand, the amplitude k_2 for $n\geqslant n^*$ has no relation with k_1, but depends on material properties, crack geometry and yieling level. As we are short of the further numerical results, the value of k_2 is not determined yet. However, comparing the results with that of Li and Wang[9], we can find that the value of s_2 in Mode II is larger than the corresponding one in Mode I. That is to say, as the crack tip is approached, the second term of the series becomes less important to the near-tip stress field for Mode II than that in Mode I.

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APPENDIX

Using Eqs. (2.10), (2.13) and (2.14), the plastic strain can be written as

$$\left.\begin{aligned}\tilde{\varepsilon}^{p}_{r2}&=A(\tilde{\sigma}_{r2}-\tilde{\sigma}_{\theta 2})+B\tilde{\tau}_{r\theta 2}\\ \tilde{\varepsilon}^{p}_{r\theta 2}&=B(\tilde{\sigma}_{r2}-\tilde{\sigma}_{\theta 2})/2+C\tilde{\tau}_{r\theta 2}\end{aligned}\right\}\tag{A.1}$$

where

$$\left.\begin{aligned}A&=\Omega\{[(n-1)\tilde{S}^2_{r1}+\tilde{g}]\,\tilde{g}^2\}\\ B&=\Omega\{2(n-1)\tilde{S}_{r1}\tilde{\tau}_{r\theta 1}\tilde{g}^2\}\\ C&=\Omega\{2[(n-1)\tilde{\tau}^2_{r\theta 1}+\tilde{g}]\,\tilde{g}^2\}\end{aligned}\right\}\tag{A.2}$$

$$\tilde{S}_{r1}=\frac{1}{2}(\tilde{\sigma}_{r1}-\tilde{\sigma}_{\theta 1})\qquad \tilde{g}=\tilde{S}^2_{r1}+\tilde{\tau}^2_{r\theta 1}\tag{A.3}$$

$$\Omega=(3^{\frac{n+1}{2}}/4)\,\tilde{g}^{\frac{n-7}{2}}$$

Substituting (A.1) into (2.22) and (3.12), we can obtain (3.12). In (3.12), the coefficients $D_i\,(i=1\sim 5)$ are given as follows.

$$\left.\begin{aligned}D_1&=A\\ D_2&=2A'-(ns_1+\Delta s_2+s_2+2)B\\ D_3&=A''-(ns_1+\Delta s_2+2s_2+3)B'-[(ns_1+\Delta s_2)(ns_1+\Delta s_2+2)\\ &\quad+(s_2+2)s_2]A+2(ns_1+\Delta s_2+1)(s_2+1)C\\ D_4&=-(s_2+1)B''-2(s_2+2)s_2A'+[(ns_1+\Delta s_2)(ns_1+\Delta s_2+2)(s_2+1)\\ &\quad+(ns_1+\Delta s_2+1)(s_2+2)s_2]B+2(ns_1+\Delta s_2+1)(s_2+1)C'\\ D_5&=-(s_2+2)s_2A''+(ns_1+\Delta s_2)(ns_1+\Delta s_2+2)(s_2+2)s_2A\\ &\quad+(ns_1+\Delta s_2+1)(s_2+2)s_2B'\end{aligned}\right\}\tag{A.4}$$