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STOCHASTICITY NEAR RESONANCES IN A KIND OF NEAR-INTEGRABLE HAMILTONIAN SYSTEMS BASED ON SMALE HORSESHOES*

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Abstract

In this paper, a method is developed to detect the appearance of stochasticity in a kind of near-integrable Hamiltonian system with two time-scales. One is fast and the other slow. The stochasticity is showed to be chaos in the sense of Smale horseshoes actually. A stochastic web is discovered in our example, by use of the results obtained in this paper.

Key words stochasticity, Smale horseshoe, Ω-stability, time-scale, stochastic web

I. Introduction

The numerical results show an extraordinary picture of phase space: islands within islands and densely interwoven stochastic layers among them in many near-integrable Hamiltonian systems. We are interested mainly in the appearance of stochasticity near resonances. It is believed that the transversal intersections of the separatrices near resonances might be responsible for it, which is not, however, proved theoretically[1-2, 5-6, 8].

As pointed out in [1–2, 4], Melnikov method is invalid when applied to verify the intersections in the system with one time-scale explicitly, because the corresponding Melnikov function has no simple zeros, containing exponential small terms.

On the basis of the above analysis, we try to study a kind of near-integrable Hamiltonian systems with two time-scales. One is fast and the other slow. And thus we obtain some results, which can be used to show that stochasticity in the systems is chaos in the sense of Smale horseshoe. And an example is given to verify the validity of our results and a stochastic web is discovered in the exampie.

II. Preliminary

The Hamiltonian system we shall study can be described in the form:

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$$H(J,\theta,t)=H_0(J)+\varepsilon H_1(J,\theta,t)+\varepsilon^2 H_2(J,\theta,t) \tag{2.1}$$

where (J,θ) is an action-angle variable, $\frac{\Omega}{\omega}=\frac{r}{s}$, $\frac{\Omega'}{\omega'}=\frac{r'}{s'}$, $\omega=\frac{\partial H_0(J)}{\partial J}$ and r, s, r', s' integers, Ω and $\Omega'+o(\varepsilon)$ are the frequencies of H_1 and H_2 in t, respectively. Obviously, the system has two time-scales, one t and the other εt

By some transformations, which are actually diffeomorphic, we obtain a new system in the form, in the degenerate case:

$$\dot{x}=\varepsilon f(x,t)+\varepsilon^2 g(x,\varepsilon t,t) \tag{2.2}$$

where f and g are $C^r(r\geqslant 2)$ functions, periodic of T_1 and T_2 in t respectively and g is of T in τ , $\tau=\varepsilon t$.

For details, see [1] or section JV in this paper.

Suppose that $g\not\equiv 0$ and $T_1/T_2=m/n$, m, n, integers, the latter leading to the same period of both f and g, nT_1 or mT_2 in t, denoted still by T_1.

By the following diffeomorphism[2, 7]:

$$x=y+\varepsilon u(y,\tau)+\varepsilon^2 v(y,\tau,t) \tag{2.3}$$

system (2.2) can be transformed into the topologically equivalent one[2, 7]:

$$\dot{y}=\varepsilon\bar{f}(y)+\varepsilon^2[\bar{f}_1(y)+\bar{g}(y,\tau)]+\varepsilon^3 h(y,\tau,t,\varepsilon) \tag{2.4}$$

where h is continuous, bounded and

$$\bar{f}(y)=\frac{1}{T_1}\int_0^{T_1} f(y,t)dt,\quad \bar{g}(y,\tau)=\frac{1}{T_1}\int_0^{T_1} g(y,\tau,t)dt,$$

and $\bar{f}_1(y)=\frac{1}{T_1}\int_0^{T_1} Df(y,t)u(y,t)dt$, the averaging part of $Df(y,t)u(y,t)$ and u can be solved from the equation:

$$\partial u/\partial t=\tilde{f}(y,t),\quad \tilde{f}(y,t)=f(y,t)-\bar{f}(y),$$

In short, to study the dynamical behaviours of systems (2.1) and (2.2), it is enough to study those of system (2.4).

III. The Detection of Chaos in the System with Two Time-Scales

For convenience, system (2.4) in the previous section is described as follows:

$$x'=f(x)+\varepsilon g(x,\tau)+\varepsilon^2 h(x,\tau,\tau/\varepsilon,\varepsilon) \tag{3.1}$$

where f and g are $C^r(r\geqslant 2)$ functions, the latter being periodic of T in τ , $x\in D\subset \mathbf{R}^2$, D , a bounded closed set and $x'=dx/d\tau$.

We are interested in the relationship of dynamical behaviours between (3. 1) and the following system:

$$\dot{y}=f(y)+\varepsilon g(y,\varepsilon) \tag{3.2}$$

Suppose that $x(\tau,t)$ be a solution of (3.1) with initial condition $x(0,0)=x_0$ and $y(\tau)$ of (3.2) with initial condition $y(0)=y_0$.

By Gronwall inequality, we have

$$|x(\tau,t)-y(\tau)|\leqslant|x_0-y_0|+(L_1+\varepsilon L_2)\int_0^\tau|x(s,s/\varepsilon)-y(s)|ds+\varepsilon^2M\tau$$

$$\leqslant[|x_0-y_0|+\frac{\varepsilon^2M}{L_1+\varepsilon L_2}\exp[(L_1+\varepsilon L_2)\tau],$$

where L_1 and L_2 are Lipschitz constants of f and g in y, M is a supper bound of h. This implies that:

Lemma 3.1 $|x(\tau,t)-y(\tau)|=o(\varepsilon^2)$, $\tau=O(1)$ if $|x_0-y_0|=o(\varepsilon^2)$, for the above functions $x(\tau,t)$ and $y(\tau)$.

We construct the Poincaré mappings corresponding to systems (3.1) and (3.2):

$$P'_\varepsilon:\ \Sigma_{\tau_0}\longrightarrow\Sigma_{\tau_0},$$
$$P_\varepsilon:\ \Sigma_{\tau_0}\longrightarrow\Sigma_{\tau_0},$$

where $\Sigma_{\tau_0}=\{(x,\tau)\in D\times S^1_T|\tau=\tau_0\in[0,T]\}$, $S^1_T=\mathrm{R}/T$.

Notice that the error will come out when P'_ε, as the Poincaré mapping of (3.1), is used to discuss the dynamical behaviours, because h is of the different periods in τ and t, or τ/ε. When we restrict ourselves to the torus $D\times S^1_T$, τ in τ/ε is distorted.

It is fortunate that the error has no effect on our discussion up to $o(\varepsilon^2)$, known by the anologous deduction as that in Lemma 3.1.

By Lemma 3.1, we have

$$P'_\varepsilon x-P_\varepsilon x=o(\varepsilon^2)\quad\text{for every}\quad x\in D.$$

Theorem 3.1 The mappings P'_ε and P_ε satisfy the following relation:

$$P'_\varepsilon=P_\varepsilon+o(\varepsilon^2).$$

If chaos in the sense of Smale horseshoes occurs to (3.2), P_ε has an invariant set Λ([2−3]).

In 1970, S. Smale proved that P_ε is Ω-stable([2−3]), which implies that P'_ε also has an invariant set when ε is sufficiently small. The Ω-stability also makes us know that P'_ε can be used to study the dynamical properties of (3.1) when ε is sufficiently small.

Finally, we have:

Theorem 3.2 If the corresponding Melnikov function of (3.2) has simple zeros independent of ε, chaos in the sense of Smale horseshoes occurs to system (3.1).

Up to now, we can discuss chaos, or stochasticity in the systems such as (2.1) and (3.1).

IV. Application

In this section, our results in Section III will be used to study a near-integrable system and a stochastic web is discoverd, which is, in fact, a chaotic structure based on Smale horseshoes.

The system we study is as follows:

$$H(x,y,t)=\frac{1}{2}y^2+\frac{1}{2}\omega^2x^2-\varepsilon\frac{\alpha}{k}\cos(kx-\Omega t)$$
$$+2\varepsilon^2\beta x^2\sin(2\omega-q\varepsilon)t \tag{4.1}$$

where $N\omega=\Omega$, and N is an integer. The system describes the motions near resonances in a kind of accelerator[9].

Using the transformation:

$$x=(2J/\omega)^{1/2}\sin\theta,\qquad y=(2J\omega)^{1/2}\cos\theta,$$

we change system (4.1) into a new Hamiltonian system:

$$\bar{H}(J,\theta,t)=\omega J-\varepsilon\frac{a}{k}\sum_{n=-\infty}^{n=\infty}J_n\left(k\left(\frac{2J}{\omega}\right)^{1/2}\right)\cos(n\theta-\Omega t)$$

$$+\varepsilon^2\frac{2\beta J}{\omega}(1-\cos 2\theta)\sin(2\omega-q\varepsilon)t \tag{4.2}$$

where (J,θ) is an action-angle variable and $J_n(\cdot)$ are Bessel functions for all n.

Notice that $N\omega=\Omega$ and take the gernerating function:

$$F=(N\theta-\Omega t)I,$$

we have

$$\bar{H}'(I,\varphi,t)=\bar{H}(J,\theta,t)+\frac{\partial F}{\partial t} \tag{4.3}$$

a new Hamiltonian system in the action-angle (I,φ), cf, [1, 5−6].

From the previous two sections, we only need to pay attention to the system, which corresponds to (3.2):

$$\left.\begin{aligned} I'&=\frac{a}{k}J_N(k\rho)\sin\varphi+\varepsilon\frac{2\beta I}{\omega}\cos\left(\frac{2\varphi}{N}-q\tau\right)-\varepsilon a(\rho)\sin\varphi \\ \varphi'&=\frac{Na}{\omega\rho}J'_N(k\rho)\cos\varphi+\varepsilon\frac{N\beta}{\omega}\sin\left(\frac{2\varphi}{N}-q\tau\right)-\varepsilon b(\rho)\cos\varphi \end{aligned}\right\} \tag{4.4}$$

where $\rho=(2NI/\omega)^{1/2}$, $\tau=-\varepsilon t$

System (4.4) can be obtained by the procedures anologous to those in section II or in [1, 5−6]. The concrete forms of $a(\rho)$ and $b(\rho)$ are omitted, which, as we can observe, have no effect on our calculations.

The unperturbed system of (4.4) has Hamiltonian:

$$H_0(I,\varphi)=\frac{a}{k}J_N(k\rho)\cos\varphi, \tag{4.5}$$

and the sets of the corresponding hyperbolic and elliptic fixed points are

$$P_h=\{(\rho^*,\varphi^*)\,|\,J_N(k\rho^*)=0,\ \varphi^*=\pm\pi/2\},$$
$$P_e=\{(\rho^*,\varphi^*)\,|\,J'_N(k\rho^*)=0,\ \varphi^*=0,\ \pi\}.$$

The equations of the separatrices joining $(\rho^*,\pm\pi/2)$ for every ρ^* satisfying $J_N(k\rho^*)=0$ are $\rho=\rho^*$ and $\varphi=\pm\pi/2$, that is, in the form of τ:

$$(\rho(\tau),\varphi(\tau))=\left(\rho^*,2\operatorname{arctg}\left(\operatorname{th}\left(\frac{1}{2}a\tau\right)\right)\right)$$

where $a=\frac{Na}{\omega\rho^*}J'_N(k\rho^*)$ and $\varphi(\tau)$ are odd functions.

It is easy to calculate the corresponding Melnikov function:

$$M_1(\tau_0)=\frac{4a\beta I^*}{\omega}\int_0^{\infty}\cos\varphi\cos\left(\frac{2\varphi}{N}-q\tau\right)d\tau\cdot\cos q\tau_0,$$

having simple zeros independent of ε. This implies that all arc separatrices break and the transversal intersections happen under small perturbations.

The separatrices joining $(\rho_1^*, \pm\pi/2)$ and $(\rho_2^*, \pm\pi/2)$ for any ρ_1^* and ρ_2^* satisfying $J_N(k\rho)=0$, for example, AB is Fig.1, can be solved:

$$(\rho(\tau),\varphi(\tau))=\left(\rho^*(\tau), \frac{\pi}{2}\right),$$

Obviously, $J_N(k\rho^*(\tau))\to 0$, , as $\tau\to\pm\infty$.

The calculated Melnikov function is

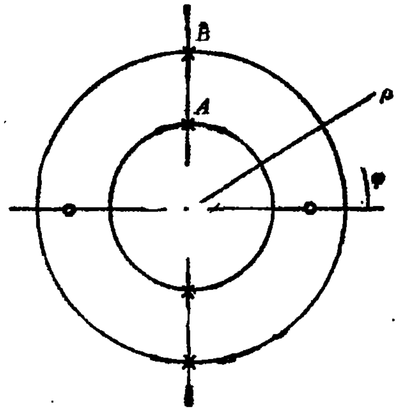
$$M_2(\tau_0)=-\int_{-\infty}^{\infty}\frac{a}{k}J_N(k\rho^*(\tau))\frac{N\beta}{\omega}\sin\left(\frac{\pi}{N}-q(\tau+\tau_0)\right)d\tau$$

$$=l_1\cos q\tau_0+l_2\sin q\tau_0.$$

where

$$l_1=-\frac{\alpha\beta N}{k\omega}\int_{-\infty}^{\infty}J_N(\rho^*(\tau))\sin\left(\frac{\pi}{N}-q\tau\right)d\tau,$$

$$l_2=-\frac{\alpha\beta N}{k\omega}\int_{-\infty}^{\infty}J_N(\rho^*(\tau))\cos\left(\frac{\pi}{N}-q\tau\right)d\tau.$$



• elliptic fixed point
× hyperbolic fixed point

Fig. 1

The Melnikov function has simple zeros independent of ε since at least one of l_1 and l_2 is not equal to zero, except for at most countable q's[10].

We conclude that all straight separatrices also break and the transversal intersections happen as ε is sufficiently small.

By Bessel function having infinite zeros and its properties, we have a web consisting of infinite separatrices for the unperturbed system of (4.4) as in Fig.1 and therefore a chaotic web occurs to (4.4), based on Smale horseshoes, as ε is sufficiently small[2-3, 11].

By Theroem 3.2, a stochastic web, a chaotic structure in the sense of Smale horseshoes actually, appears in system (4.1).

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