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INSTABILITY THEORY OF SHOCK WAVE IN A CHANNEL

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Abstract

The instability theory of shock wave was extended from the case with an infinite front[1] *to the case of a channel with a rectangular cross section. First, the mathematical formulation of the problem was given which included a system of disturbed equations and three kinds of boundary conditions. Then, the general solutions of the equations upstream and downstream were given and each contained five constants to be determined. Thirdly, under one boundary condition and one assumption, it was proved that all of the disturbances in front of the shock front and one of the two acoustic disturbances behind the shock front should be zero. The boundary condition was that all of the disturbed physical quantities should approach to zero at infinity. The assumption was that only the unstable shock wave was concerned here. So it was reasonable to assume* $\omega=i\gamma$, γ *was the instability growth rate and was a positive real number. Another kind of boundary conditions was that the normal disturbed velocities should be zero at the solid wall of the channel, and it led to the result that the wave number of disturbances could only be a set of discrete values. Finally, a total of five conservation equations across the disturbed shock front was the third kind of boundary conditions which was used to determine the remained four undetermined constants downstream and an undetermined constant representing the amplitude of disturbed shock front. Then a dispersion relation was derived. The results show that a positive real* γ *does exist, so the assumption made above is self-consistent, and there are two modes, instead of one, for unstable shock. One mode corresponds to* $\gamma=-W\cdot k$ $(W<0)$*. It is a newly discovered mode and represents an absolute instability of shock. The instability criterion derived from another mode is nearly the same as the one obtained in* [2, 3]*, in addition, its growth rate is newly derived in this paper, and on this basis, it is further pointed out that at* $j^2(\partial V/\partial P)_H=1+2M$*, the shock wave is most unstable, i.e. its nondimensional growth rate* $\Gamma=\infty$.

If ω *is assumed to be a complex number with* $\mathrm{Im}(\omega)\geqslant 0$ *instead of being assumed a pure imaginary number at the beginning, it can be proved in Section V that there are still two modes for the instability criteria, besides, the roots* ω *of the dispersion equation are still imaginary.*

Key words shock wave, shock wave in a channel, instability theory of shock wave, shock wave stability

I. Introduction

There are two approaches in the study on shock wave stability. One of them was initiated by Landau and Lifshitz[4] (what is called the evolutionarity of shock wave), and for the case of one dimensional small disturbances, the stability criterion they obtained is $M_1>1$ and $M_2<1$. In 1982, Xu[5] proved that for two dimensional small disturbances, $M_1>1$ and $M_2<1$ is no longer the necessary and sufficient condition for the stability of shock wave front, no matter the front is infinite or bounded in a channel. In fact, the stability (or evolutionary) criterion is influenced yet by the frequency of the two dimensional small disturbance. Generally speaking, shock wave front is nonevolutionary (unstable) if the small disturbances it suffers are not one dimensional.

The other approach was initiated by Dýakov[2] who used a method called normal mode analysis, then Swan and Fowles[3] followed. They supposed that the Hugoniot curve of shock wave is arbitrary, then the instability region obtained is

$$j^2(\partial V/\partial P)_H<-1 \quad \text{or} \quad j^2(\partial V/\partial P)_H>1+2M$$

Fowles and Houwing[6] showed that when the above condition is satisfied, a shock wave front will split spontaneously into two shock waves and a contact discontinuity. Book[7] used the above criterion to discuss the stability problem of Sedov's point explosion wave. Xu[1] gave further the expression of the growth rate, when the above criterion for instability is satisfied.

The small disturbances behind an infinite plane shock wave consist of four independent disturbances, which are one entropy, one vortex as well as two sound disturbances. In discussing the stability problem of shock wave, everyone of the four disturbances should share the same ω and k, but l might be different in general. There is a flaw in either Dyakov's or Swan and Fowles' paper. They discard, in fact, one sound disturbance and remain another one but without any proof or announcement. Such a proof is supplemented in Xu's work[1], and as a consequence, two (instead of one) instability modes are obtained. One mode is a newly discovered one, which represents an absolute instability of shock wave; the other is the same as the one obtained by Dýakov[2], Swan and Fowles [3].

In this paper, shock wave instability in channel with a rectangular cross section is discussed. Since it is three dimensional in essence, the small disturbances downstream consist of five (instead of four) independent disturbances: one entropy, two vortex, and two sound disturbances. As long as our interest is focused on the problem of shock instability, the condition that the amplitudes of the disturbed physical quantities at infinity (both downstream and upstream) should be zero leads to the effect that no disturbance is allowed to occur upstream, and only one of the two sound disturbances is allowed to occur downstream, so we need not to assume that there is no disturbances upstream beforehand. After substituting the remained four independent small disturbances into the conservation equations at the shock front, we obtained a dispersion relation. It shows that, similar to the result obtained by Xu[1], there are still two instability modes for shock wave. What is new is that the Dýakov's criterion for instability is now modified as

$$j^2\left(\frac{\partial V}{\partial P}\right)<-1 \quad \text{or} \quad j^2\left(\frac{\partial V}{\partial P}\right)_H\geqslant 1+2M$$

II. The Equations for Small Disturbance and Its General Solution

The Catesian coordinates system $Oxyz$ is fixed on the shock, its x-axis conincides with the axis of channel, and the undisturbed shock front is located at $x=0$. Gas flows from the region $x>0$ (with velocity W_0) to $x<0$ (with velocity W). We use as more as possible the same symbols that Swan and Fowles[3] used. Suppose that

$$M_0=-W_0/C_0>1,\quad M=-W/C<1$$

and the Hugoniot curve of shock might be arbitrary.

The continuity, momentum and isentropic equations for small disturbances downstream are

$$\frac{\partial \bar{u}_x}{\partial t}+W\frac{\partial \bar{u}_x}{\partial x}+V\frac{\partial \bar{p}}{\partial x}=0$$

$$\frac{\partial \bar{u}_y}{\partial t}+W\frac{\partial \bar{u}_y}{\partial x}+V\frac{\partial \bar{p}}{\partial y}=0$$

$$\frac{\partial \bar{u}_z}{\partial t}+W\frac{\partial \bar{u}_z}{\partial x}+V\frac{\partial \bar{p}}{\partial z}=0$$

$$\frac{\partial \bar{p}}{\partial t}+W\frac{\partial \bar{p}}{\partial x}+\frac{C^2}{V}\left(\frac{\partial \bar{u}_x}{\partial x}+\frac{\partial \bar{u}_y}{\partial y}+\frac{\partial \bar{u}_z}{\partial z}\right)=0$$

$$\frac{\partial \bar{s}}{\partial t}+W\frac{\partial \bar{s}}{\partial x}=0 \quad \text{or} \quad \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial x}\right)\left(\bar{p}+\frac{C^2}{V^2}\bar{v}\right)=0$$

If we substitute W, V and C with W_0, V_0 and C_0, we also get the corresponding governing equations for upstream region.

The general solution of the disturbance equations downstream is:

$$\bar{u}_x=(\cos k_y\cdot y)\cdot(\cos k_z\cdot z)\exp[-i\omega t]\Big[Bk_y\exp[il_1x]+E\cdot k_z\exp[il_1x]$$
$$+F\frac{Vl^{(1)}}{\omega-Wl^{(1)}}\exp[il^{(1)}x]+G\cdot\frac{Vl^{(2)}}{\omega-Wl^{(2)}}\exp[il^{(2)}x]\Big]$$

$$\bar{u}_y=i(\sin k_y y)\cdot(\cos k_z z)\exp[-i\omega t]\Big[-Bl_1\exp[il_1x]$$
$$+F\frac{Vk_y}{\omega-Wl^{(1)}}\exp[il^{(1)}x]+G\cdot\frac{Vk_y}{\omega-Wl^{(2)}}\exp[il^{(2)}x]\Big]$$

$$\bar{u}_z=i(\cos k_y y)\cdot(\sin k_z z)\exp[-i\omega t]\Big[-El_1\exp[il_1x]$$
$$+F\frac{Vk_z}{\omega-Wl^{(1)}}\exp[il^{(1)}x]+G\cdot\frac{Vk_z}{\omega-Wl^{(2)}}\exp[il^{(2)}x]\Big]$$

$$\bar{v}=(\cos k_y y)\cdot(\cos k_z z)\exp[-i\omega t]\Big[A\exp[il_1x]$$
$$-F\frac{V^2}{C^2}\exp[il^{(1)}x]-G\frac{V^2}{C^2}\exp[il^{(2)}x]\Big]$$

$$\bar{p}=(\cos k_y y)\cdot(\cos k_z z)\exp[-i\omega t]\,[F\exp[il^{(1)}x]+G\exp[il^{(2)}x]]$$

where

$$x<0,\quad |y|\leqslant a,\quad |z|\leqslant b$$
$$k_y=n\pi/a\qquad (n=1,2,3,\cdots)$$

$$k_z = m\pi/b \qquad (m=1,2,3,\cdots)$$

$$l_1 = \omega/W, \quad k^2 = k_y^2 + k_z^2$$

$$l^{(1)} = \frac{-W\omega + \sqrt{C^2\omega^2 + C^2k^2W^2 - C^4k^2}}{C^2 - W^2}$$

$$l^{(2)} = \frac{-W\omega - \sqrt{C^2\omega^2 + C^2k^2W^2 - C^4k^2}}{C^2 - W^2}$$

A, B, E, F, G are arbitrary constants, represent the amplitudes of one entropy, two vortex and two sound disturbances respectively. It is easy to see that the boundary conditions at the solid wall of channel have been satisfied automatically, i.e. $\bar{u}_y|_{y=\pm a}=0$ and $\bar{u}_z|_{z=\pm b}=0$.

Similarly, the general solution of disturbance equations upstream is:

$$\bar{u}_x = \cos k_y y \cdot \cos k_z z \exp[-i\omega t]\Big[B_0 k_y \exp[il_{10}x] + E_0 k_z \exp[il_{10}x] + F_0 \frac{V_0 l_0^{(1)}}{\omega - W_0 l_0^{(1)}} \exp[il_0^{(1)}x] + G_0 \frac{V_0 l_0^{(2)}}{\omega - W_0 l_0^{(2)}} \exp[il_0^{(2)}x]\Big]$$

$$\bar{u}_y = i\sin k_y y \cdot \cos k_z z \exp[-i\omega t]\Big[-B_0 l_{10} \exp[il_{10}x] + F_0 \frac{V_0 k_y}{\omega - W_0 l_0^{(1)}} \exp[il_0^{(1)}x] + G_0 \frac{V_0 k_y}{\omega - W_0 l_0^{(2)}} \exp[il_0^{(2)}x]\Big]$$

$$\bar{u}_z = i\cos k_y y \cdot \sin k_z z \exp[-i\omega t]\Big[-E_0 l_{10} \exp[il_{10}x] + F_0 \frac{V_0 k_z}{\omega - W_0 l_0^{(1)}} \exp[il_0^{(1)}x] + G_0 \frac{V_0 k_z}{\omega - W_0 l_0^{(2)}} \exp[il_0^{(2)}x]\Big]$$

$$\bar{v} = \cos k_y y \cdot \cos k_z z \exp[-i\omega t]\Big[A_0 \exp[il_{10}x] - F_0 \frac{V_0^2}{C_0^2} \exp[il_0^{(1)}x] - G_0 \frac{V_0^2}{C_0^2} \exp[il_0^{(2)}x]\Big]$$

$$\bar{p} = \cos k_y y \cdot \cos k_z z \exp[-i\omega t]\,[F_0 \exp[il_0^{(1)}x] + G_0 \exp[il_0^{(2)}x]]$$

where

$$x > 0, \quad |y| \leqslant a, \quad |z| \leqslant b$$

$$k_y = n\pi/a \qquad (n=1,2,3,\cdots)$$

$$k_z = m\pi/b \qquad (m=1,2,3,\cdots)$$

$$l_{10} = \omega/W_0, \quad k = k_y^2 + k_z^2$$

$$l_0^{(1)} = \frac{W_0\omega + \sqrt{C_0^2\omega^2 + C_0^2k^2W_0^2 - C_0^4k^2}}{W_0^2 - C_0^2}$$

$$l_0^{(2)} = \frac{W_0\omega - \sqrt{C_0^2\omega^2 + C_0^2k^2W_0^2 - C_0^4k^2}}{W_0^2 - C_0^2}$$

A_0, B_0, E_0, F_0, G_0 are arbitrary constants, represent the amplitudes of an entropy, two vortex and two sound disturbances respectively, and W_0, C_0, V_0 are flow velocity, sound speed and specific volume upstream respectively. Also, the boundary conditions at the solid walls are automatically satisfied by the general solution upstream.

The remained boundary conditions to be satisfied could be devided into two classes. One of them is the conservation relations for various disturbance quantities at the shock front, and will be discussed in Section III. The other which will be discussed now is that the amplitudes of disturbed physical quantities should be zero at infinity, i.e. as

$$x \to \pm\infty, \ \bar{u}_x, \ \bar{u}_y, \ \bar{u}_z, \ \bar{v}, \ \bar{p} \to 0$$

From the above demands, the following inequalities should be satisfied:

$\mathrm{Im}(l)<0$ for $x<0$, where l represents l_1 or $l^{(1)}$ or $l^{(2)}$; and

$\mathrm{Im}(l_0)>0$ for $x>0$, where l_0 represents l_{10} or $l_0^{(1)}$ or $l_0^{(2)}$.

Since the gas concerned is nondissipative, and we are only interested in the instability problem, we suppose that $\omega=i\gamma$, where γ is a positive real number. Substituting ω into the expression of l, we have:

$\mathrm{Im}(l_1)<0$, $\mathrm{Im}(l^{(1)})>0$ and $\mathrm{Im}(l^{(2)})<0$ for $x<0$, as well as

$\mathrm{Im}(l_{10})<0$, $\mathrm{Im}(l_0^{(1)})<0$ and $\mathrm{Im}(l_0^{(2)})<0$ for $x>0$.

Hence in the upstream region ($x>0$), all of the five disturbances should be discarded (correspondingly, each of the A_0, B_0, E_0, F_0, G_0 should be zero); and in the downstream region ($x<0$), only the sound disturbance related to $l^{(1)}$ should be discarded ($F=0$).

It is unnecessary and somewhat unnatural to suppose beforehand that ω is a pure imaginary number, and we will discuss the general case that ω is complex in Section V. It will be showed that even when ω is complex, no new instability region appears.

III. The System of Conservation Equations Across the Shock and the Dispersion Relation

Suppose that the geometry of the disturbed shock is:

$$x=g(y,z,t)=g_0\cdot\cos k_y y\cdot\cos k_z z\cdot\exp[-i\omega t]$$

then

the unit normal vector is: $\boldsymbol{n}=(1,-\partial g/\partial y,-\partial g/\partial z)$

the unit tangent vectors are: $\boldsymbol{t}_y=(-\partial g/\partial y,-1,0)$

$\boldsymbol{t}_z=(-\partial g/\partial z,0,-1)$

From the momentum equation, the tangent velocities (upstream and downstream) must be equal at the disturbed shock, so we have

$$\text{at } x=0 \quad \begin{cases} (W+\bar{u}_x,\bar{u}_y,\bar{u}_z)\cdot\boldsymbol{t}_y=(W_0,0,0)\boldsymbol{t}_y \\ (W+\bar{u}_x,\bar{u}_y,\bar{u}_z)\cdot\boldsymbol{t}_z=(W_0,0,0)\boldsymbol{t}_z \end{cases}$$

Similarly, the jump in the normal components of velocity across the shock ($x=0$) is given by

$$(W+\bar{u}_x, \ \bar{u}_y, \ \bar{u}_z)\boldsymbol{n}-(W_0,0,0)\cdot\boldsymbol{n}-[(V_0-V-\bar{v})\cdot \ (P-P_0+\bar{p})]^{\frac{1}{2}}$$

The Hugoniot curve of shock and the mass conservation equation demand that at $x=0$

$$\bar{p}=(\partial P/\partial V)_H\cdot\bar{v}$$

$$(\bar{D}-W_0)^2=V_0^2(P+\bar{p}-P_0)\cdot(V_0-\bar{v}-V)^{-1}$$

where $\bar{D}$ is the velocity of disturbed shock, and

$$\bar{D}=\partial g/\partial t=-i\omega g_0\cos k_y y\cdot\cos k_z z\exp[-i\omega t]$$

The solution we are interested in is nontrivial, i.e. not all of the five constants A, B, E, G and g_0 are zero, and there are five conservation equations also at the shock, we then obtain the dispersion relation

$$2\frac{W}{W_0}\cdot\omega\left(k^2+\frac{\omega^2}{W^2}\right)=\left(\frac{\omega^2}{WW_0}+k^2\right)(\omega-Wl^{(2)})\left[1+j^2\left(\frac{\partial V}{\partial P}\right)_H\right]$$

where

$$j^2=(\rho_0 W_0)^2=\left(\frac{W_0}{V_0}\right)^2=(\rho W)^2=\left(\frac{W}{V}\right)^2$$

$$k^2=k_y^2+k_z^2$$

and $l^{(2)}$ is a given function of ω. Substituting this function for $l^{(2)}$ into the above dispersion relation, we get an algebraic equation of ω, it is the roots of the algebraic equation that determine the modes of instability. Let $\Omega=\omega/Ck$, the algebraic equation of ω is transformed into the one of Ω:

$$2\cdot\frac{W}{W_0}(1-M^2)\Omega(\Omega^2+M^2)$$

$$=\left(\frac{W}{W_0}\Omega^2+M^2\right)(\Omega-M\sqrt{\Omega^2+M^2-1})\left[1+j^2\left(\frac{\partial V}{\partial P}\right)_H\right]$$

IV. Two Modes of Instability

It can easily be seen from the dispersion relation that there is a solution

$$\Omega=iM$$

which makes $\Omega+M^2=0$. It can also be rewritten as

$$\Gamma=\gamma/Ck=M$$

or

$$\gamma=-W\cdot k \qquad (W<0)$$

In the following, a transformation is made for solving the dispersion relation.

Let

$$\Omega=i(1-M^2)^{1/2}/\mathrm{sh}\theta \quad \text{or} \quad \Gamma=\gamma/Ck=(1-M^2)^{1/2}/\mathrm{sh}\theta$$

then

$$\frac{l^{(2)}}{k}=i\frac{M-\mathrm{ch}\theta}{\sqrt{1-M^2}\,\mathrm{sh}\theta}$$

Substituting them into the algebraic equation of Ω, we obtain the dispersion equation in the final form that is easy to solve:

$$(M\mathrm{ch}\theta-1)(f\mathrm{ch}^2\theta+g\mathrm{ch}\theta+h)=0$$

in which

$$f=M^2\cdot[1+j^2(\partial V/\partial P)_H]$$

$$g=2M(1-M^2)W/W_0$$

$$h=-\left[M^2+\frac{W}{W_0}(1-M^2)\right]\left[1+j^2\left(\frac{\partial V}{\partial P}\right)_H\right]+2\frac{W}{W_0}(1-M^2)$$

The first instability mode correspondes to the root $\mathrm{ch}\theta=1/M$, then, $\mathrm{sh}\theta=(1-M^2)^{1/2}/M$, $\Omega=iM, \Gamma=M, \gamma=CkM=-Wk>0 (W<0)$, this root corresponds a new instability mode which is called the "absolute instability". It did not appear in either Dýakov's paper[2] or Swan-Fowles' paper[3], and has already mentioned above.

We now discuss the second instability mode. It can be proved directly that if

$$j^2(\partial V/\partial P)_H<-1 \quad \text{or} \quad j^2(\partial V/\partial P)_H>1+2M$$

then we have the following two tenable inequalities:

(1) $g^2-4fh>0$

i.e., the two roots of the quadratic equation of $\mathrm{ch}\theta$ are real number, and

(2) $|f+h|>|g|$ and $|h|>|f|$

i.e., both of the absolute values of the two real roots are greater than one, though one of the roots is positive and the other is negative.

The positive root with magnitude greater than one corresponds to a positive $\mathrm{sh}\theta$, from the expression of Ω, it is easy to see that it is this root that corresponds to the instability, and the value of $\mathrm{ch}\theta$ is

$$\mathrm{ch}\theta=[-g\pm\sqrt{g^2-4fh}]/2f$$

Case 1 If $j^2(\partial V/\partial P)_H<-1$, then $f<0$ and $fh<0$, the positive real root is

$$\mathrm{ch}\theta=[-g-\sqrt{g^2-4fh}]/2f$$

and the growth rate of instability Γ is

$$\Gamma=\frac{-2\sqrt{1-M^2}\cdot f}{\sqrt{2g^2-4fh-4f^2+2g\sqrt{g^2-4fh}}}$$

Case 2 If $j^2(\partial V/\partial P)_H>1+2M$, then $f>0$ and $fh<0$, the positive real root is

$$\mathrm{ch}\theta=[-g+\sqrt{g^2-4fh}]/2f$$

and correspondingly, the growth rate of instability Γ is

$$\Gamma=\frac{2\sqrt{1-M^2}\cdot f}{\sqrt{2g^2-4fh-4f^2-2g\sqrt{g^2-4fh}}}$$

It is casy to check that γ(or Γ) we obtain in this section is positive indeed, so it is consistent with the assumption made in Section II.

V. In Case of Complex ω

In this section, we relax our restriction (the assumption of a pure imaginary ω) and consider a general case in which ω might be complex. We stilll focus our attention on the problem of instability. What we will prove is that:

(1) the form of the dispersion relation is the same as the one obtained in the previous section;

(2) the parameter region that instability occurs is also the same as the one obtained in the previous section.

So far, we avoid to discuss the problem of shock wave stability in general. The reason is that if the problem discussed is shock wave stability, then the disturbances might occur upstream, and the whole formulation of the problem should be reconsidered. In this case, one of the appropriate research approaches might be the theory on shock wave evolutionarity (see Landau and Lifshitz[4] and Xu[5]).

A complex ω corresponds a complex $\mathrm{sh}\theta$, $\mathrm{ch}\theta$ and θ, so we suppose this time

$$\theta=\theta_r+i\theta_i$$

then

$$\mathrm{ch}\theta=C_r+iC_i=\mathrm{ch}\theta_r\cos\theta_i+i\,\mathrm{sh}\theta_r\sin\theta_i$$
$$\mathrm{sh}\theta=S_r+iS_i=\mathrm{sh}\theta_r\cos\theta_i+i\,\mathrm{ch}\theta_r\sin\theta_i$$

and

$$\Omega=\frac{\sqrt{1-M^2}}{S_r^2+S_i^2}\cdot S_i+i\frac{\sqrt{1-M^2}}{S_r^2+S_i^2}\cdot S_r$$

$$\mathrm{Im}\left(\frac{l^{(1)}}{k},\ \frac{l^{(2)}}{k}\right)=\frac{1}{S_r^2+S_i^2}\cdot\frac{1}{\sqrt{1-M^2}}\cdot\mathrm{sh}\theta_r[M\cos\theta_i\pm\mathrm{ch}\theta_r]$$

$$\mathrm{Im}\left(\frac{l_1}{k}\right)=-\frac{1}{M}\frac{\sqrt{1-M^2}}{S_r^2+S_i^2}\cdot S_r$$

From the expresson of Ω above, it is easy to see that the sufficient and necessary condition for shock instability is $S_r>0$, and while $S_r>0$, we always have $\mathrm{Im}(l_1/k)<0$, so one entropy and two vortex disturbances downstream should not be discarded. From the expression of $\mathrm{Im}(l^{(1)}/k,\ l^{(2)}/k)$, we see that one value of $(M\cos\theta_i\pm\mathrm{ch}\theta_r)$ is always positive and the other is negative. The prerequisite of retaining the sound disturbance G (retaining the $l^{(2)}$ branch) and discarding the sound disturbance F (discarding the $l^{(1)}$ branch) is only $\theta_r>0$ and $-\pi/2<\theta_i<\pi/2$ owing to $S_r>0$.

In fact, the assumption that $\theta_r>0$ and $-\pi/2<\theta_i<\pi/2$ does not lose generality. Because if $\theta_r<0$, then $\pi/2<\theta_i<3\pi/2$ owing to $S_r>0$, and a new dispersion relation can be obtained with sound disturbance F branch retained and sound disturbance G branch discarded. It is easy to show that the signs of the roots $\mathrm{ch}\theta$ are changed while the values of growth rates Ω are unchanged.

On the other hand, since we already have four independent disturbances (unknowns) behind the shock, from a mathematical view, the dispersion relation could not be determined if the disturbances upstream are not zero. Therefore, we are going to prove that if the shock is unstable, then the disturbed physical quantities upstream have to be zero. Let

$$\frac{\omega}{C_0k}=\Omega_0=i\frac{\sqrt{M_0^2-1}}{\mathrm{ch}\varphi}$$

where φ is complex,

$$\varphi=\varphi_r+i\varphi_i$$
$$\mathrm{ch}\varphi=\bar{C}_r+i\bar{C}_i=\mathrm{ch}\varphi_r\cdot\cos\varphi_i+i\,\mathrm{sh}\varphi_r\cdot\sin\varphi_i$$
$$\mathrm{sh}\varphi=\bar{S}_r+i\bar{S}_i=\mathrm{sh}\varphi_r\cos\varphi_i+i\,\mathrm{ch}\varphi_r\sin\varphi_i$$

and

$$\mathrm{Im}\left(\frac{l_0^{(1)}}{k},\ \frac{l_0^{(2)}}{k}\right)=\frac{1}{\bar{C}_r^2+\bar{C}_i^2}\cdot\frac{1}{\sqrt{M_0^2-1}}\cos\varphi_i\cdot[\pm\sin\varphi_i-M_0\mathrm{ch}\varphi_r]$$

Since the sufficient and necessary condition for shock instability is $\mathrm{Im}(\Omega_0)>0$, and from

$$\Omega_0=\frac{\sqrt{M_0^2-1}}{\bar{C}_r^2+\bar{C}_i^2}\cdot(\bar{C}_i+i\bar{C}_r),\ \ \bar{C}_r=\mathrm{ch}\varphi_r\cos\varphi_i$$

the sufficient and necessary condition for shock instability transforms into $\bar{C}_r>0$, i.e. $\mathrm{ch}\varphi_r\cdot\cos\varphi_i>0$, i.e. $\cos\varphi_i>0$. Therefore, we have both

$$\mathrm{Im}\left(\frac{l_0^{(1)}}{k},\ \frac{l_0^{(2)}}{k}\right)<0 \quad \text{and}$$

$$\mathrm{Im}\left(\frac{l_{10}}{k}\right)=-\frac{1}{M_0}\frac{\sqrt{M_0^2-1}}{\bar{C}_r^2+\bar{C}_i^2}\mathrm{ch}\varphi_r\cdot\cos\varphi_i<0$$

i.e. as long as the shock is unstable, the disturbed physical quantities in front of the shock have to be zero. So far, we have proved that as long as the shock is unstable and $\theta_r>0$ as well as $-\pi/2<\theta_i<\pi/2$, we still have

$$F=A_0=B_0=E_0=F_0=G_0=0$$

even when a complex ω is conerned now.

Following the example in Section II and III, we obtain the same dispersion relation:

$$(M\mathrm{ch}\theta-1)(f\mathrm{ch}^2\theta+g\mathrm{ch}\theta+h)=0$$

where the coefficients f, g and h have the same expression as previous. From the expression of $\mathrm{ch}\theta$ and $\mathrm{sh}\theta$, it is easy to see that under the condition $\theta_r>0$, the C_r and S_r share the same sign, and when one is zero, the another will be zero too.

We now give a criterion to divide unstable case from stable case for the second instability mode when $j^2(\partial V/\partial P)_H$ varies in the region $(-\infty,\ \infty)$. The criterion is:

(1) If one of the two roots satisfies $C_i=0$ and $C_r>1$, or $C_i\neq 0$ and $C_r>0$, then the shock wave is unstable;

(2) If one of the two roots satisfies $C_i=0$ and $C_r=1$, or $C_i\neq 0$ and $C_r=0$, then

(A) as $\Gamma=0$, the shock wave is marginal stable;

(B) as $\Gamma=\infty$, the shock wave is most unstable;

(3) If it does not belong to the above two cases, then the shock wave is stable.

We now discuss the shock wave instability based on the dispersion relation using the above criterion. The first mode is an instability one and is still the same as previous:

$$\mathrm{ch}\theta=1/M \qquad (C_i=0,\ C_r>1)$$

For the quadric equation of $\mathrm{ch}\theta$ in the dispersion relation, we divide the value of $j^2(\partial V/\partial P)_H$ on the real axis into five regions and will discuss them individually. From $g^2-4fh=0$, we obtain

$$\Delta=j^2\left(\frac{\partial V}{\partial P}\right)_H=Q_1=-1+\frac{(1-M^2)W/W_0}{1+\sqrt{(1-M^2)(1-W/W_0)}}$$

$$\Delta=j^2\left(\frac{\partial V}{\partial P}\right)_H=Q_2=-1+\frac{(1-M^2)W/W_0}{1-\sqrt{(1-M^2)(1-W/W_0)}}$$

Obviously, we have

$$-1<Q_1<Q_2<1+2M$$

When $\Delta = j^2(\partial V/\partial P)_H \in (Q_1, Q_2)$, because $f>0$, so $g^2-4fh<0$. the two roots of qudratic equation are a pair of conjugate complex number, whose $C_{r+}=C_{r-}<0$ and $C_i \neq 0$. According to the above criterion, the shock wave is stable in this region. When Δ is outside the region (Q_1, Q_2), we have two roots of real number ($C_i=0$), and from the requirement of instability: $\mathrm{ch}\theta>1$; we get the prerequisite for the presence of instability: $\Delta<-1$ or $\Delta>1+2\cdot M$. Hence there is no shock wave instability in the regions $-1<\Delta\leqslant Q_1$ and $Q_2\leqslant\Delta<1+2M$.

Table 1

$\Delta=j^2\left(\frac{\partial V}{\partial P}\right)_H$	the first root $(\mathrm{ch}\theta)_+=C_{r+}+iC_{i+}$	the second root $(\mathrm{ch}\theta)_-=C_{r-}+iC_{i-}$	situation of stability
$\Delta<-1$	$C_{i+}=0,\ C_{r+}>1$	$C_{i-}=0,\ C_{r-}<-1$	unstable (U)
$(1+\Delta)\to 0^-$	$C_{i+}=0,\ C_{r+}\to -1/M$	$C_{i-}=0, C_{r-}\to\infty,\ \Gamma\to 0$	marginal stable (mS)
$\Delta=-1$	$C_{i+}=0\ \ C_{r+}=-1/M$	none	
$-1<\Delta\leqslant Q_1$	$C_{i+}=0,\ C_{r+}<1$	$C_{i-}=0,\ C_{r-}<-1$	stable (S)
$Q_1<\Delta<Q_2$	$C_{i+}\neq 0,\ C_{r+}<0$	$C_{i-}\neq 0,\ C_{r+}=C_{r-}<0$	stable (S)
$Q_2\leqslant\Delta<1+2M$	$C_{i+}=0,\ \ C_{r+}<1$	$C_{i-}=0,\ \ C_{r-}<1$	stable (S)
$\Delta=1+2M$	$C_{i+}=0, C_{r+}=1,\ \Gamma=+\infty$	$C_{i-}=0,\ \ C_{r-}<-1$	most unstable (MU)
$\Delta>1+2M$	$C_{i+}=0,\ C_{r+}>1$	$C_{i-}=0,\ C_{r-}<-1$	unstable (U)

We now discuss the performance of roots in the vincinity of $\Delta=-1$ and $\Delta=1+2M$. As $(\Delta+1)\to 0^-$, $C_{r+}\to -1/M$ and $C_{r-}\to+\infty$, the latter corresponds to $\Gamma\to 0$, i.e. marginal stability. However at $\Delta=-1$, there is only one root: $C_r=-1/M$. As $(\Delta-1-2M)\to 0^+$, $C_{r+}\to 1^+$ and $C_{r-}<-1$, the former corresponds to $\Gamma\to+\infty$, hence $\Delta=1+2M$ is the most unstable point (in the meaning of maximum instability growth rate). At $\Delta=1+2M$, $C_{r+}=1$ and $C_{r-}<-1$, the former also corresponds to $\Gamma=+\infty$.

VI. Conclusions

1. The shock wave instability in a channel with a rectangular cross section has been discussed and solved strictly in mathematics.

First, two sets of general solutions of perturbation equations are obtained. One set corresponds to upstream and the other to downstream of shock front, and each set contains five constants, representing five kinds of disturbances correspondingly, to be determined.

Secondly, three kinds of boundary conditions are given. The first one demands that the amplitudes of all disturbed physical quantities should approach to zero as $x\to\pm\infty$. Since it is assumed that only the shock wave instability, instead of stability, is concerned, we might assume that $\omega = i\gamma$ at the beginning where γ is the growth rate of instability ($\gamma \geqslant 0$). Based on this assumption and the first kind of boundary conditions, it could be proved that all of the five undetermined constants for upstream set should be zero, i.e. there are no disturbances upstream, and for downstream set, one of the two acoustic disturbances should be vanished, i. e. one of the other five undetermined constants should be zero. The second kind of boundary conditions demands that the normal disturbance velocity should be zero at the solid wall of channel, so that only a particular set of discrete values should be assigned to the wave number of disturbances. The third kind of boundary conditions consists of five conservation equations for disturbed physical quantities that should be valid at the disturbed shock front.

Thirdly, the remained four undetermined constants for downstream set and a constant representing the amplitude of disturbed shock are determined by the five conservation equations, then a dispersion relation is obtained. Based on this dispersion relation, it is easy to check that positive real γ does exist (so it is consistent with the previous assumption), and easy to derive the corresponding criteria for instability.

Fourthly, to generalize our results, a complex ω with its imaginary part being positive real number is also assumed instead of being assumed a pure imaginary number at the beginning. The problem is treated in a similar way, and the same dispersion relation is also obtained. Its roots as well as the instability criteria obtained are also the same as before. It means that ω could be proved to be a pure imaginary number as long as only the shock instability is concerned.

2. After analysing the obtained dispersion relation, we find that there are two modes, instead of one, for shock instability in a channel with a rectangular cross section. One mode is an absolute instability and it is a newly discovered one. This mode has nothing to do with the thermodynamical properties of gases (such as $j^2(\partial V/\partial P)_H$), and its growth rate is $\gamma = -\boldsymbol{W}\cdot\boldsymbol{k}$ ($W<0$) or $\Gamma = M$. It represents a type of short-wave-instability, and shows that a weak shock might be more unstable than a strong shock under certain circumstances.

The instability criteria derived from second mode are

$$j^2\left(\frac{\partial V}{\partial P}\right)_H < -1 \quad \text{or} \quad j^2\left(\frac{\partial V}{\partial P}\right)_H > 1+2M$$

they are closely related with the thermodynamical and flow properties of gases, and have almost the same form as those from Dýakov[2] and Swan-Fowles[3], Moreover, an expression of instability growth rate is further given in this paper. It shows that the second instability mode is also a short-wave-instability, and ω is really a pure imaginary number. In addition, we further point out that at $j^2(\partial V/\partial P)_H = -1^-$, the shock wave is neutrally stable; and at $j^2(\partial V/\partial P)_H = 1+2M$, it is most unstable.

3. When $j^2(\partial V/\partial P)_H$ varies along the real number axis, the variation of the situation of second mode is shown in table.

4. When the width $2a$ and height $2b$ of the cross section of a channel approach infinity, then the results derived in this paper degenerate into the case with an infinite shock front. In comparison with the results in [2, 3], the improvement in this paper are: 1) the instability problem of shock wave with an infinite shock front is strictly solved from a mathematical view; 2) two instability modes, instead of one, are obtained; 3) the ω of the second instability mode is proved to be a pure imaginary number as long as only shock instability is concerned;

i.e. the disturbed physical quantities grow exponentially as long as instability occurs.

5. In comparison with [1], a geometrical extension from two to three dimensional case, a channel with a rectangular cross section, is realized. In addition, a discussion about the variation of the situation of second mode is supplemented in the table, which shows that $j^2(\partial V/\partial P)_H = -1^-$ is the point of neutral stability and $j^2(\partial V/\partial P)_H = 1+2M$ is the most unstable point.

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