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Letters

Multi-scale Equations for Incompressible Turbulent Flows

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Abstract The short-range property of interactions between scales in incompressible turbulent flow was examined. Some formulae for the short-range eddy stress were given. A concept of resonant-range interactions between extremely contiguous scales was introduced and some formulae for the resonant-range eddy stress were also derived. Multi-scale equations for the incompressible turbulent flows were proposed.

Key words turbulence, incompressible flow, interactions between scales, multi-scale equations.

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1 Introduction

Turbulent flow contains a wide range of time- and space-scales. The interactions between different scales play a key role in the evolution of turbulent flow. In the traditional theory of turbulence, eddy-viscosity was introduced a century ago by J. Boussinesq and developed later by G. I. Taylor and L. Prandtl, and they claimed that the interactions are mainly between widely separated scales[1,2]. This is so-called long-range interactions between scales. However, it is generally believed that the dominant interactions are between contiguous, rather than widely separated, scales[1]. This may be called short-range interactions between scales. Both the "direct interaction" theory[2] presented by R. Kraichnan and the numerical inference acquired through the analysis of direct numerical simulation databases for channel turbulent flow by J. Domaradzki *et al*[3] confirmed that the interactions are mainly between contiguous wave numbers. The aim of this paper is to extend the multiscale model of turbulence[4] and to confirm further short-range property of interactions between scales, which is applied to space-average analysis of turbulence and to deduce multi-scale equations for the incompressible turbulent flows.

2 Short-Range Interactions between Scales in Turbulence

Starting from the space-average Navier-Stokes (NS) equations for the incompressible flows, we prove the interactions being mainly between contiguous, rather than widely separated, scales and derive expressions of short-range turbulent (or call eddy, the same below) stress and then introduce a concept of resonant-range interactions between extremely contiguous scales and deduce expressions of resonant-range eddy stress. The space-average NS equations for the incompressible flow can be written as

$$\frac{\partial U_{cj}}{\partial x_j}=0, \quad (2.1.1)$$

$$\frac{\partial U_{ci}}{\partial t}+U_{cj}\frac{\partial U_{ci}}{\partial x_j}=-\frac{\partial p_c}{\partial x_i}+\frac{1}{Re}\frac{\partial^2 U_{ci}}{\partial x_j\partial x_j}-\frac{\partial F_{ci}}{\partial x_j} \quad (i=1,2,3), \quad (2.1.2)$$

$$\frac{\partial e_{ct}}{\partial t}+U_{cj}\frac{\partial e_{ct}}{\partial x_j}+U_{cj}\frac{\partial p_c}{\partial x_j}=-\frac{\partial E_c}{\partial x_j}-\frac{\partial P_c}{\partial x_j}+\frac{1}{Re}\frac{\partial}{\partial x_j}[U_{ci}\tau_{ji}(U_{cj})]+\frac{1}{Re}\frac{\partial \Pi_c}{\partial x_j}+\frac{1}{Re\,Pr\,(\gamma-1)M_\infty^2}\frac{\partial^2 T_c}{\partial x_j\partial x_j}, \quad (2.1.3)$$

where

$$(U_{ci},p_c,T_c,e_{ct})=V_c^{-1}\int(u_i,p,T,e_t)\mathrm{d}v, \quad V_c=\Delta x_c\Delta y_c\Delta z_c, \quad (2.2)$$

$$F_{ci}=F_{ci}(u_i,U_{ci})=V_c^{-1}\int(u_j-U_{cj})(u_i-U_{ci})\mathrm{d}v, \quad (2.3)$$

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$$E_c = E_c(u_j, e_{ct}) = V_c^{-1}\int (u_j - U_{cj})(e_t - e_{ct})\mathrm{d}v, \tag{2.4}$$

$$P_c = P_c(u_j, p_c) = V_c^{-1}\int (u_j - U_{cj})(p - p_c)\mathrm{d}v, \tag{2.5}$$

$$\Pi_c = \Pi_c(u_i, \tau_{ji}(U_{cj})) = V_c^{-1}\int (u_i - U_{ci})[\tau_{ji}(u_j) - \tau_{ji}(U_{cj})]\,\mathrm{d}v, \tag{2.6}$$

$$\tau_{ji}(u_j) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),\ \tau_{ji}(U_{ci}) = \mu\left(\frac{\partial U_{ci}}{\partial x_j} + \frac{\partial U_{ci}}{\partial x_i}\right), \tag{2.7}$$

$Re = \rho U_\infty L/\mu$; $Pr = \mu C_p/k$; $\gamma = C_p/C_v$; $M_\infty = U_\infty/a_\infty$; x_i, the time t, the velocity u_i, the pressure p, the temperature T and the total energy e_t are normalized with reference to the boundary characteristic length L, L/U_∞, U_∞, ρU_∞^2, T_∞ and U_∞^2, where the subscript ∞ denotes the free stream conditions; $e_t = C_v T + \frac{1}{2}u_i u_i$ is the total energy; τ_{ij} is the viscous stress. Since the solutions u_i, p and e_t of the NS equations and the solutions U_{ci}, p_c and e_{ct} of the space-average NS equations are continuous and differentiable, the following expressions can be deduced from the definition (2.3) of eddy stress.

$$F_{ci} = \frac{1}{12}\left(\frac{\partial u_j}{\partial x}\frac{\partial u_i}{\partial x}\Delta x_c^2 + \frac{\partial u_j}{\partial y}\frac{\partial u_i}{\partial y}\Delta y_c^2 + \frac{\partial u_j}{\partial z}\frac{\partial u_i}{\partial z}\Delta z_c^2\right) + O(\Delta x_c^4), \tag{2.8}$$

$$F_{ci}(u_i, U_{ci}) = F_{fi}(u_i, U_{fi}) + F_{cfi}(U_{fi}, U_{ci}) + O(\Delta x_f^2 \Delta x_c^2), \tag{2.9}$$

where

$$(U_{fc}, p_f, T_f, e_{ft}) = V_f^{-1}\int (u_i, p, T, e_t)\mathrm{d}v,\quad V_f = \Delta x_f \Delta y_f \Delta z_f \tag{2.10}$$

and $V_f < V_c$, $\Delta x_f < \Delta x_c$, $\Delta y_f < \Delta y_c$, $\Delta z_f < \Delta z_c$ (for short, $\Delta x_f < \Delta x_c$, the same below). Suppose that without losing generality, the side-length of the volume elements (cuboids) V_c and V_f satisfy $\Delta x_f/\Delta x_c = \Delta y_f/\Delta y_c = \Delta z_f/\Delta z_c$, then we deduce from Eq. (2.9)

$$F_{fi} = V_f^{-1}\int (u_j - U_{fj})(u_i - U_{fi})\mathrm{d}v = \frac{\Delta x_f^2}{\Delta x_c^2}F_{ci} + O(\Delta x_f^2 \Delta x_c^2), \tag{2.14}$$

$$F_{cfi} = V_c^{-1}\int (U_{fj} - U_{cj})(U_{fi} - U_{ci})\mathrm{d}v = \left(1 - \frac{\Delta x_f^2}{\Delta x_c^2}\right)F_{ci} + O(\Delta x_f^2 \Delta x_c^2). \tag{2.15}$$

F_{fi} (F_{ci}) represents the eddy stress of the whole scale range with scales $\Delta x < \Delta x_f$ ($\Delta x < \Delta x_c$) acting on the scale range with scales $\Delta x > \Delta x_f$ ($\Delta x > \Delta x_c$), and F_{cfi} represents the eddy stress of the contiguous scales ranging from Δx_f to Δx_c acting on the scale range with scales $\Delta x > \Delta x_c$. From Eqs. (2.14) and (2.15) we know that F_{fi} is only $\Delta x_f^2/\Delta x_c^2$ of F_{ci} and that F_{cfi} is $(1 - \Delta x_f^2 \Delta x_c^{-2})$ of F_{ci}. When $\Delta x_f/\Delta x_c$ equals 2^{-1}; 3^{-1} and 5^{-1}, F_{fi}/F_{ci} equals to 0.25, 0.11 and 0.04 respectively; F_{cfi}/F_{ci} equals 0.75, 0.89 and 0.96, respectively. Therefore, one may deduce that the interactions between scales $\Delta x > \Delta x_c$ and $\Delta x < \Delta x_c$ are mainly short-range ones between scales $\Delta x > \Delta x_c$ and the contiguous scales ranging from Δx_f to Δx_c, where Δx_f should equal $(0.2-0.5)\Delta x_c$. Since the space-average velocities U_{ci} and U_{fi} are continuous and differentiable, the differential formula for the short-range eddy stress F_{cfi} can be deduced from its integral expression (2.15).

$$F_{cfi}^d = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial U_{fi}}{\partial x}\Delta x_c^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial U_{fi}}{\partial y}\Delta y_c^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial U_{fi}}{\partial z}\Delta z_c^2\right) + O(\Delta x_c^4). \tag{2.16}$$

Through similar operations, some expressions similar to Eqs. (2.15) and (2.16) can be obtained. These expressions give the integral and differential formulae for the short-range eddy heat transfer E_{cf}, the short-range eddy pressure-power P_{cf} and the short-range eddy stress-power, or say, dissipation Π_{cf}. They are respectively

$$E_{cf}(U_{fj}, e_{ct}) = V_c^{-1}\int (U_{fj} - U_{cj})(e_{ft} - e_{ct})\mathrm{d}v + O(\Delta x_f^2 \Delta x_c^2), \tag{2.17}$$

$$E_{cf}^d = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial e_{ft}}{\partial x}\Delta x_c^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial e_{ft}}{\partial y}\Delta y_c^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial e_{ft}}{\partial z}\Delta z_c^2\right) + O(\Delta x_c^4), \tag{2.18}$$

$$P_{cf}(U_{fj}, p_c) = V_c^{-1}\int (U_{fj} - U_{cj})(p_f - p_c)\mathrm{d}v + O(\Delta x_f^2 \Delta x_c^2), \tag{2.19}$$

$$P_{cf}^d = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial p_f}{\partial x}\Delta x_c^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial p_f}{\partial y}\Delta y_c^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial p_f}{\partial z}\Delta z_c^2\right) + O(\Delta x_c^4), \tag{2.20}$$

$$\Pi_{cf}(U_{fi}, \tau_{ji}(U_{cj})) = \frac{1}{V_x}\int (U_{fi} - U_{ci})[\tau_{ji}(U_{fj}) - \tau_{ji}(U_{cj})]\,\mathrm{d}v + O(\Delta x_f^2 \Delta x_c^2), \tag{2.21}$$

$$\Pi_{cf}^d = \frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial \tau_{ji}(U_{fj})}{\partial x}\Delta x_c^2 + \frac{\partial U_{fi}}{\partial y}\frac{\partial \tau_{ji}(U_{fj})}{\partial y}\Delta y_c^2 + \frac{\partial U_{fi}}{\partial z}\frac{\partial \tau_{ji}(U_{fj})}{\partial z}\Delta z_c^2\right). \tag{2.22}$$

Discussion The short-range interactions between scales imply that as to the space-average analysis of turbulent flow, it would be best to adopt a multi-scale model, at least a two-scale model. In addition, an inspiration acquired from all the differential formulae (2.16), (2.18), (2.20) and (2.22) of the short-range interactions is that we should introduce a concept of resonant-range interactions between scales, which define the interactions between scales $\Delta x > \Delta x_c$ and the small scales being smaller than Δx_c but extremely near it. According to the definitions of the space-average velocities we know that the U_{fi} tends to U_{ci} as the Δx_f tends to Δx_c, Therefore, the differential formula of the resonant-range eddy stress can be deduced directly from the formula (2.16) of the short-range stress.

$$F^d_{xci} = \frac{1}{12}\left(\frac{\partial U_{cj}}{\partial x}\frac{\partial U_{ci}}{\partial x}\Delta x_c^2 + \frac{\partial U_{cj}}{\partial y}\frac{\partial U_{ci}}{\partial y}\Delta y_c^2 + \frac{\partial U_{cj}}{\partial z}\frac{\partial U_{ci}}{\partial z}\Delta z_c^2\right) + O(\Delta x_c^4). \quad (2.23)$$

Similarly, for the scale Δx_f, the differential formulae of the resonant-range eddy stress F^d_{cci}, the resonant-range eddy heat transfer E^d_{ff}, the resonant-range eddy pressure power P^d_{ff} and the resonant-range eddy dissipation Π^d_{ff} are respectively

$$F^d_{ffi} = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial U_{fi}}{\partial x}\Delta x_f^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial U_{fi}}{\partial y}\Delta y_f^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial U_{fi}}{\partial z}\Delta z_f^2\right) + O(\Delta x_f^4), \quad (2.24)$$

$$E^d_{ff} = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial e_{ft}}{\partial x}\Delta x_f^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial e_{ft}}{\partial y}\Delta y_f^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial e_{ft}}{\partial z}\Delta z_f^2\right) + O(\Delta x_f^4), \quad (2.25)$$

$$P^d_{ff} = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial p_f}{\partial x}\Delta x_f^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial p_f}{\partial y}\Delta y_f^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial p_f}{\partial z}\Delta z_f^2\right) + O(\Delta x_f^4), \quad (2.26)$$

$$\Pi^d_{ff} = \frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial \tau_{ji}(U_{fj})}{\partial x}\Delta x_f^2 + \frac{\partial U_{fi}}{\partial y}\frac{\partial \tau_{ji}(U_{fj})}{\partial y}\Delta y_f^2 + \frac{\partial U_{fi}}{\partial z}\frac{\partial \tau_{ji}(U_{fj})}{\partial z}\Delta z_f^2\right). \quad (2.27)$$

3 Multi-scale Equations for Incompressible Turbulent Flows

Dividing beforehand the resolved scales into two or more scale-ranges and utilizing all the integral and differential formulae of the short-and resonant-range interactions given in the above section, we can obtain multi-scale equations of turbulence. Consider the case of two scale-ranges, in which the resolved scale-range $(\Delta x_f, 1)$ are divided into small scale-one $(\Delta x_f, \Delta x_c)$ and large scale-one $(\Delta x_c, 1)$. The large-scale equations governing the average motions of the large scale-range are

$$\frac{\partial U_{ci}}{\partial x_i} = 0, \quad (3.1.1)$$

$$\frac{\partial U_{ci}}{\partial t} + U_{cj}\frac{\partial U_{ci}}{\partial x_j} = -\frac{\partial p_c}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 U_{ci}}{\partial x_j \partial x_j} - \frac{\partial F_{cfi}}{\partial x_j} \quad (i = 1, 2, 3), \quad (3.1.2)$$

$$\frac{\partial e_{ct}}{\partial t} + U_{cj}\frac{\partial e_{ct}}{\partial x_j} = -U_{cj}\frac{\partial p_c}{\partial x_j} - \frac{\partial E_{cf}}{\partial x_j} - \frac{\partial P_{cf}}{\partial x_j} + \frac{1}{Re}\frac{\partial}{\partial x_j}[U_{ci}\tau_{ji}(U_{cj})] + \frac{1}{Re}\frac{\partial \Pi_{cf}}{\partial x_j} + \frac{1}{Re\,Pr(\gamma-1)M_\infty^2}\frac{\partial^2 T_c}{\partial x_j \partial x_j}. \quad (3.1.3)$$

The small-scale equations governing the fluctuation motions of the small-scale (or, say, fine-grid) average quantities relating to the large scale (coarse-grid) average ones are as follows:

$$\frac{\partial}{\partial x_i}(U_{fi} - U_{ci}) = 0, \quad (3.2.1)$$

$$\frac{\partial}{\partial t}(U_{fi} - U_{ci}) + (U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(U_{fi} - U_{ci}) = -\frac{\partial}{\partial x_i}(p_f - p_c) - U_{cj}\frac{\partial}{\partial x_j}(U_{fi} - U_{ci}) - (U_{fj} - U_{cj})\frac{\partial U_{ci}}{\partial x_j} + \frac{1}{Re}\frac{\partial^2 (U_{fi} - U_{ci})}{\partial x_j \partial x_j} + \frac{\partial F_{cfi}}{\partial x_j} - \frac{\partial F^d_{ffi}}{\partial x_j} \quad (i = 1, 2, 3), \quad (3.2.2)$$

$$\frac{\partial}{\partial t}(e_{ft} - e_{ct}) + (U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(e_{ft} - e_{ct}) = -U_{cj}\frac{\partial}{\partial x_j}(e_{ft} - e_{ct}) - (U_{fj} - U_{cj})\frac{\partial e_{ct}}{\partial x_j} - (U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(p_f - p_c) - (U_{fj} - U_{cj})\frac{\partial p_c}{\partial x_j} - U_{cj}\frac{\partial}{\partial x_j}(p_f - p_c) + \frac{1}{Re}\frac{\partial}{\partial x_j}[U_{fi}\tau_{ji}(U_{fj}) - U_{ci}\tau_{ji}(U_{fj})] + \frac{\partial E_{cf}}{\partial x_j} - \frac{\partial F^d_{ff}}{\partial x_j} + \frac{\partial P_{cf}}{\partial x_j} - \frac{\partial P^d_{ff}}{\partial x_j} - \frac{\partial \Pi_{cf}}{\partial x_j} + \frac{\partial \Pi^d_{ff}}{\partial x_j} + \frac{1}{Re\,Pr(\gamma-1)M_\infty^2}\frac{\partial^2 (T_f - T_c)}{\partial x_j \partial x_j}, \quad (3.3)$$

where $(U_{ci}, p_c, T_c, e_{ct})$ and $(U_{fi}, p_f, T_f, e_{ft})$ are defined in Eqs. (2.2) and (2.10). Both the integral and differential formulae of the short-range interactions F_{cfi}, E_{cf}, P_{cf} and Π_{cf} can be used and are given in the formulae (2.15)—(2.22), respectively. The differential formulae F^d_{ffi}, E^d_{ff}, P^d_{ff} and Π^d_{ff} expressing the resonant-range interactions are given in Eqs. (2.24)—(2.27). In general, Δx_f is consistent with the filtered scale in the large eddy simulations (LES), and suppose $\Delta x_c \cong (2\sim 5)\Delta x_f$. The multi-scale equations with scale-ranges being more than two can be similarly deduced. The large-small scale (LSS) equations (3.1) and (3.2) can be used to determine the ten unknown quantities U_{ci}, U_{fi} $(i=1, 2, 3)$, P_c, P_f, e_{ct} (or T_c) and e_{ft} (or T_f). Therefore, the LSS equations (3.1) and (3.2) are approximately closed and do not contain any empirical constants or relations. And the following conclusions can be reached: 1) the nonlinear dynamics of the resolved large scales $\Delta x > \Delta x_c$ are governed mainly by their interactions with the resolved small scales in the range $\Delta x_c > \Delta x > \Delta x_f$ and much smaller unresolved scales $\Delta x < \Delta x_f$ have negligible effects on the resolved large scales $\Delta x > \Delta x_c$, which are neglected; 2) The dynamics of the resolved small scales in the range $\Delta x_c > \Delta x > \Delta x_f$ are largely governed by their interactions with the resolved large scales $\Delta x > \Delta x_c$ and much smaller unresolved scales $\Delta x < \Delta x_f$ have secondary effects on the resolved small scales, which are approximated by the resonant-range eddy stress *etc*. It should be noted that the above conclusions agree with those obtained through the numerical analysis of direct numerical simulation (DNS) databases for the incompressible channel flow by J. Domaradzki *et al*[3]. The other conclusion given by the LSS equations (3.1) and (3.2) is that the fluctuation motions of the resolved short-range small scales ranging from Δx_f to Δx_c relating to the large scales $\Delta x > \Delta x_c$ are caused mainly by the resolved large scales $\Delta x > \Delta x_c$.

A brief comparison of the multi-scale equations (3.1) and (3.2) with the traditional LES equations is as follows. In the former the unresolved small scales $\Delta x < \Delta x_f$ act only on the resolved small scales in the range $\Delta x_c > \Delta x > \Delta x_f$; and in the latter the unresolved small scales $\Delta x < \Delta x_f$ act on the whole resolved scales $\Delta x > \Delta x_f$. Therefore, as to detecting the nonlinear interactions between contiguous scales and their effects, the former gains dominance over the latter. In addition, the unresolved small scales $\Delta x < \Delta x_f$ contain still a wide range of time- and space-scales, therefore, any formulae expressing their interactions with the resolved small scales are certainly imperfect. Perhaps it is another choice to use empirical sub-grid scale (SGS) model instead of the formulae of the resonant-range interactions.

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