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· *Letters* ·

Multi-scale Equations for Compressible Turbulent Flows

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Abstract The short-range property of interactions between scales in the compressible turbulent flow was examined. An estimation of the short-range scale scope and some formulae for the short-range eddy stress and heat transfer *etc*. were given. A concept of resonant-range interactions between extremely contiguous scales was introduced and some formulae for the resonant-range eddy stress and heat transfer *etc*. were also given. Multi-scale equations for the compressible turbulent flows were presented. The multi-scale equations are approximately closed and do not contain any empirical constants. The compressibility effects on turbulence are determined by the Farve averaged variables and the nonlinear relationships between the Farve- and physical-averaged variables.

Key words turbulence, compressible flow, interactions between scales, multi-scale equations.
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The multi-scale approach to computing the incompressible turbulent flows proposed by the paper[1, 2] is extended into the case of compressible turbulent flows.

1 Short-Range Interactions between Scales in Compressible Turbulent Flows

Starting from the space-average Navier-Stokes (NS) equations for the compressible flows, we examine the short-range property of interactions between scales and derive formulae of the short-range stress and then introduce a concept of resonant-range interactions between extremely contiguous scales and deduce formulae of the resonant-range eddy stress. The space averaged NS equations for the compressible flows can be written as

$$\frac{\partial \rho_c}{\partial t}+\frac{\partial U_{\rho cj}}{\partial x_j}=0, \tag{1.1}$$

$$\frac{\partial U_{\rho ci}}{\partial t}+\frac{\partial}{\partial x_j}(U_{ci}U_{\rho cj})=-\frac{\partial p_c}{\partial x_i}+\frac{1}{Re_\infty}\frac{\partial \tau_{ij}(U_{ci})}{\partial x_j}-\frac{\partial F_{ci}}{\partial x_j}\quad (i=1,2,3), \tag{1.2}$$

$$\frac{\partial e_{t\rho c}}{\partial t}+\frac{\partial}{\partial x_j}(U_{cj}e_{t\rho c})=-\frac{\partial}{\partial x_j}(U_{cj}p_c)+\frac{1}{Re_\infty}\frac{\partial}{\partial x_j}[U_{ci}\tau_{ij}(U_{ci})]-\frac{\partial E_c}{\partial x_j}-\frac{\partial P_c}{\partial x_j}+\frac{1}{Re_\infty}\frac{\partial \Pi_c}{\partial x_j}+\frac{\mu}{Re_\infty Pr(\gamma-1)M_\infty^2}\frac{\partial^2 T_c}{\partial x_j\partial x_j}, \tag{1.3}$$

$$p_c M_\infty^2=\rho_c T_c+\Phi_{TC}, \tag{1.4}$$

where

$$(\rho_c, U_{ci}, U_{\rho ci}, p_c, T_c, e_{tc}, e_{t\rho c})=V_c^{-1}\int(\rho, u_i, \rho u_i, p, T, e_t, \rho e_t)\mathrm{d}v\quad (V_c=\Delta x_c\Delta y_c\Delta z_c) \tag{2}$$

$$\tau_{ij}(U_{ci})=\mu\left(\frac{\partial U_{ci}}{\partial x_j}+\frac{\partial U_{cj}}{\partial x_i}-\frac{2}{3}\frac{\partial U_{cj}}{\partial x_j}\delta_{ij}\right), \tag{3}$$

$$F_{ci}=F_{ci}(u_i, U_{\rho ci})=V_c^{-1}\int(u_i-U_{ci})(\rho u_j-U_{\rho cj})\mathrm{d}v, \tag{4}$$

$$E_c=E_c(u_j, e_{t\rho c})=V_c^{-1}\int(u_j-U_{cj})(\rho e_t-e_{t\rho c})\mathrm{d}v, \tag{5}$$

$$P_c=P_c(u_j, p_c)=V_c^{-1}\int(u_j-U_{cj})(p-p_c)\mathrm{d}v, \tag{6}$$

$$\Pi_c=\Pi_c(u_i, \tau_{ij}(U_{ci}))=V_c^{-1}\int(u_i-U_{ci})[\tau_{ij}(u_i)-\tau_{ij}(U_{ci})]\mathrm{d}v, \tag{7}$$

$$\Phi_{TC}=\Phi_{TC}(\rho, T_c)=V_c^{-1}\int(\rho-\rho_c)(T-T_c)\mathrm{d}v, \tag{8}$$

$Re_\infty=\rho_\infty U_\infty L/\mu_\infty$; $Pr=\mu C_p/k$; $\gamma=C_p/C_v$; $M_\infty=U_\infty/a_\infty$; t, x_j, u_j, ρ, p, T, e_t, C_v and μ are made dimensionless with reference to L/U_∞, L, U_∞, ρ_∞, $\rho_\infty U_\infty^2$, T_∞, U_∞^2, R and μ_∞, respectively; the subscript ∞ denotes the free stream conditions; The total energy $e_t=C_vT+\frac{1}{2}u_iu_i$. The fluctuations of both the thermal conductivity k and the coefficient of viscosity μ are neglected in the space averaged equations (1.2) and (1.3). By similar considerations and operations made in the multi-scale analysis for incompress-

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ible turbulent flows[1], we can verify the short-range properties of interactions between scales in the compressible turbulent flow and give a reasonable estimation of the short-range scale scope and obtain integral and differential formulae of the short-range eddy stress F_{cfi}, the short-range eddy heat transfer E_{cf}, the short-range eddy pressure-power P_{cf}, the short-range eddy dissipation Π_{cf} and the short-range density-temperature correlation Φ_{cf} as

$$F_{cfi}=F_{cfi}(U_{fi},U_{\rho ci})=V_c^{-1}\int(U_{fi}-U_{ci})(U_{\rho fj}-U_{\rho cj})\mathrm{d}v, \tag{9.1}$$

$$F_{cfi}^d=\frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial U_{\rho fj}}{\partial x}\Delta x_c^2+\frac{\partial U_{fi}}{\partial y}\frac{\partial U_{\rho fj}}{\partial y}\Delta y_c^2+\frac{\partial U_{fi}}{\partial z}\frac{\partial U_{\rho fj}}{\partial z}\Delta z_c^2\right); \tag{9.2}$$

$$E_{cf}=E_{cf}(U_{fj},e_{t\rho c})=V_c^{-1}\int(U_{fj}-U_{cj})(e_{t\rho f}-e_{t\rho c})\mathrm{d}v, \tag{10.1}$$

$$E_{cf}^d=\frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial e_{t\rho f}}{\partial x}\Delta x_c^2+\frac{\partial U_{fj}}{\partial y}\frac{\partial e_{t\rho f}}{\partial y}\Delta y_c^2+\frac{\partial U_{fj}}{\partial z}\frac{\partial e_{t\rho f}}{\partial z}\Delta z_c^2\right); \tag{10.2}$$

$$P_{cf}=P_{cf}(U_{fj},p_c)=V_c^{-1}\int(U_{fj}-U_{cj})(p_f-p_c)\mathrm{d}v, \tag{11.1}$$

$$P_{cf}^d=\frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial P_f}{\partial x}\Delta x_c^2+\frac{\partial U_{fj}}{\partial y}\frac{\partial p_f}{\partial y}\Delta y_c^2+\frac{\partial U_{fj}}{\partial z}\frac{\partial p_f}{\partial z}\Delta z_c^2\right); \tag{11.2}$$

$$\Pi_{cf}=\Pi_{cf}(U_{fi},\tau_{ij}(U_{ci}))=V_c^{-1}\int(U_{fi}-U_{ci})[\tau_{ij}(U_{fj})-\tau_{ij}(U_{cj})]\,\mathrm{d}v, \tag{12.1}$$

$$\Pi_{cf}^d=\frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial\tau_{ij}(U_{fi})}{\partial x}\Delta x_c^2+\frac{\partial U_{fi}}{\partial y}\frac{\partial\tau_{ij}(U_{fi})}{\partial y}\Delta y_c^2+\frac{\partial U_{fi}}{\partial z}\frac{\partial\tau_{ij}(U_{fi})}{\partial z}\Delta z_c^2\right); \tag{12.2}$$

$$\Phi_{Tcf}=\Phi_{Tcf}(\rho_f,T_c)=V_c^{-1}\int(\rho_f-\rho_c)(T_f-T_c)\mathrm{d}v, \tag{13.1}$$

$$\Phi_{Tcf}^d=\frac{1}{12}\left(\frac{\partial\rho_f}{\partial x}\frac{\partial T_f}{\partial x}\Delta x_c^2+\frac{\partial\rho_f}{\partial y}\frac{\partial T_f}{\partial y}\Delta y_c^2+\frac{\partial\rho_f}{\partial z}\frac{\partial T_f}{\partial z}\Delta z_c^2\right). \tag{13.2}$$

We may introduce an idea of resonant-range interaction between extremely contiguous scales in turbulence as proposed in the multi-scale equations for the incompressible turbulent flows[1,2] and deduce the differential formulae of the resonant-range eddy stress F_{ffi}^d, the resonant-range eddy heat transfer E_{ff}^d, the resonant-range eddy pressure power P_{ff}^d, the resonant-range eddy dissipation Π_{ff}^d and the resonant-range density-temperature correlation Φ_{ff}^d as

$$F_{ffi}^d=\frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial U_{\rho fj}}{\partial x}\Delta x_f^2+\frac{\partial U_{fi}}{\partial y}\frac{\partial U_{\rho fj}}{\partial y}\Delta y_f^2+\frac{\partial U_{fi}}{\partial z}\frac{\partial U_{\rho fj}}{\partial z}\Delta z_f^2\right), \tag{14}$$

$$E_{ff}^d=\frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial e_{t\rho f}}{\partial x}\Delta x_f^2+\frac{\partial U_{fj}}{\partial y}\frac{\partial e_{t\rho f}}{\partial y}\Delta y_f^2+\frac{\partial U_{fj}}{\partial z}\frac{\partial e_{t\rho f}}{\partial z}\Delta z_f^2\right), \tag{15}$$

$$P_{ff}^d=\frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial p_f}{\partial x}\Delta x_f^2+\frac{\partial U_{fj}}{\partial y}\frac{\partial p_f}{\partial y}\Delta y_f^2+\frac{\partial U_{fj}}{\partial z}\frac{\partial p_f}{\partial z}\Delta z_f^2\right), \tag{16}$$

$$\Pi_{ff}^d=\frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial\tau_{ij}(U_{fi})}{\partial x}\Delta x_f^2+\frac{\partial U_{fi}}{\partial y}\frac{\partial\tau_{ij}(U_{fi})}{\partial y}\Delta y_f^2+\frac{\partial U_{fi}}{\partial z}\frac{\partial\tau_{ij}(U_{fi})}{\partial z}\Delta z_f^2\right), \tag{17}$$

$$\Phi_{Tff}^d=\frac{1}{12}\left(\frac{\partial\rho_f}{\partial x}\frac{\partial T_f}{\partial x}\Delta x_f^2+\frac{\partial\rho_f}{\partial y}\frac{\partial T_f}{\partial y}\Delta y_f^2+\frac{\partial\rho_f}{\partial z}\frac{\partial T_f}{\partial z}\Delta z_f^2\right), \tag{18}$$

where

$$(\rho_f,U_{fi},U_{\rho fi},p_f,T_f,e_{tf},e_{t\rho f})=V_f^{-1}\int(\rho,u_i,\rho u_i,p_f,T,e_t,\rho e_t)\mathrm{d}v \tag{19}$$

and $V_f=\Delta x_f\Delta y_f\Delta z_f$. The F_{cfi} indicates the eddy stress of the short-range scales ranging from Δx_f to Δx_c acting on the large scales $\Delta x>\Delta x_c$. An estimation of the short-range scale scope should be $\Delta x_f=(0.2-0.5)\Delta x_c$. The meanings of E_{cf}, P_{cf}, Π_{cf} and Φ_{cf} are similar to that of F_{cfi}. The F_{ffi}^d indicates the eddy stress of the resonant-range scales being smaller than but near extremely Δx_f acting on the small scales in the range $\Delta x_f<\Delta x<\Delta x_c$. The meanings of E_{ff}^d, P_{ff}^d, Π_{ff}^d and Φ_{ff}^d are similar to that of F_{ffi}^d. The short-range property of interactions between scales imply that for the space-average analysis of compressible flows, it would be best to adopt a multi-scale model, at least a two-scales model. We consider two-scales model.

2 Multi-scale Equations for Compressible Turbulent Flows

Dividing in prior the resolved scale-range (Δx_f, 1) into small scale-one (Δx_f, Δx_c) and large scale-one (Δx_c, 1). The large-scale equations governing the motion of the large scale-range are

$$\frac{\partial\rho_c}{\partial t}+\frac{\partial U_{\rho cj}}{\partial x_j}=0, \tag{20.1}$$

$$\frac{\partial U_{\rho ci}}{\partial t}+\frac{\partial}{\partial x_j}(U_{ci}U_{\rho cj})=-\frac{\partial p_c}{\partial x_i}+\frac{1}{Re_\infty}\frac{\partial \tau_{ij}(U_{ci})}{\partial x_j}-\frac{\partial F_{cfi}}{\partial x_j}\quad (i=1,2,3), \tag{20.2}$$

$$\frac{\partial e_{\rho tc}}{\partial t}+\frac{\partial}{\partial x_j}(U_{cj}e_{\rho tc}+p_cU_{cj})=\frac{1}{Re_\infty}\frac{\partial}{\partial x_j}[U_{ci}\tau_{ij}(U_{ci})]-\frac{\partial E_{cf}}{\partial x_j}-\frac{\partial P_{cf}}{\partial x_j}+\frac{\partial \Pi_{cf}}{\partial x_j}+\frac{\mu}{Re_\infty Pr(\gamma-1)M_\infty^2}\frac{\partial^2 T_c}{\partial x_j\partial x_j}, \tag{20.3}$$

$$p_cM_\infty^2=\rho_cT_c+\Phi_{Tcf}(\rho_f, T_c), \tag{20.4}$$

$$U_{\rho ci}=\rho_cU_{ci}+\Phi_{ucf}(U_{fi}, \rho_c)\quad (i=1,2,3), \tag{20.5}$$

$$e_{\rho tc}=\rho_ce_{tc}+\Phi_{ecf}(\rho_f, e_{tc}). \tag{20.6}$$

The small-scale equations governing the fluctuation motions of the small-scale (or say fine-grid) averaged variables relating to the large scale (coarse-grid) averaged ones are as follows:

$$\frac{\partial}{\partial t}(\rho_f-\rho_c)+\frac{\partial}{\partial x_j}(U_{\rho fj}-U_{\rho cj})=0, \tag{21.1}$$

$$\frac{\partial}{\partial t}(U_{\rho fi}-U_{\rho ci})+\frac{\partial}{\partial x_j}[(U_{fi}-U_{ci})(U_{\rho fj}-U_{\rho cj})]=-\frac{\partial}{\partial x_i}(p_f-p_c)-\frac{\partial}{\partial x_j}[U_{ci}(U_{\rho fj}-U_{\rho cj})]-\frac{\partial}{\partial x_j}[U_{\rho cj}(U_{fi}-U_{ci})]+\frac{1}{Re_\infty}\frac{\partial}{\partial x_j}[\tau_{ij}(U_{fi})-\tau_{ij}(U_{ci})]+\frac{\partial F_{cfi}}{\partial x_j}-\frac{\partial F^d_{ffi}}{\partial x_j},\quad i=1,2,3, \tag{21.2}$$

$$\frac{\partial}{\partial t}(e_{\rho tf}-e_{\rho tc})+\frac{\partial}{\partial x_j}\left\{(U_{fj}-U_{cj})[(e_{\rho tf}-e_{\rho tc})+(p_f-p_c)]\right\}=-\frac{\partial}{\partial x_j}\left\{U_{cj}[(e_{\rho tf}-e_{\rho tc})+(p_f-p_c)]+(U_{fj}-U_{cj})(e_{\rho tc}-p_c)\right\}+\frac{1}{Re_\infty}\frac{\partial}{\partial x_j}[U_{fi}\tau_{ij}(U_{fi})-U_{ci}\tau_{ij}(U_{ci})+\frac{\partial E_{cf}}{\partial x_j}]-\frac{\partial F^d_{ff}}{\partial x_j}+\frac{\partial P_{cf}}{\partial x_j}-\frac{\partial P^d_{ff}}{\partial x_j}-\frac{\partial \Pi_{cf}}{\partial x_j}+\frac{\partial \Pi^d_{ff}}{\partial x_j}+\frac{1}{Re_\infty Pr(\gamma-1)M_\infty^2}\frac{\partial^2(T_f-T_c)}{\partial x_j\partial x_j}, \tag{21.3}$$

$$(p_f-p_c)M_\infty^2=\rho_fT_f-\rho_cT_c-\Phi_{Tcf}+\Phi^d_{Tff}, \tag{21.4}$$

$$U_{\rho fi}-U_{\rho ci}=\rho_fU_{fi}-\rho_cU_{ci}-\Phi_{ucf}(\rho_f, T_c)+\Phi^d_{uff}(\rho_f, T_f)\quad (i=1,2,3), \tag{21.5}$$

$$e_{\rho tf}-e_{\rho ci}=\rho_fe_{tf}-\rho_ce_{tc}-\Phi_{ecf}(U_{fi}, \rho_c)+\Phi^d_{eff}(U_{fi}, \rho_f), \tag{21.6}$$

where

$$\Phi_{ucf}=V_c^{-1}\int(U_{fi}-U_{ci})(\rho_f-\rho_c)\mathrm{d}v,\ i=1,2,3, \tag{22.1}$$

$$\Phi^d_{ucf}=\frac{1}{12}\left(\frac{\partial U_{fi}}{\partial x}\frac{\partial \rho_f}{\partial x}\Delta x_c^2+\frac{\partial U_{fi}}{\partial y}\frac{\partial \rho_f}{\partial y}\Delta y_c^2+\frac{\partial U_{fi}}{\partial z}\frac{\partial \rho_f}{\partial z}\Delta z_c^2\right),\quad i=1,2,3; \tag{22.2}$$

$$\Phi_{ecf}=V_c^{-1}\int(\rho_f-\rho_c)(e_{tf}-e_{tc})\mathrm{d}v, \tag{23.1}$$

$$\Phi^d_{ecf}=\frac{1}{12}\left(\frac{\partial \rho_f}{\partial x}\frac{\partial e_{tf}}{\partial x}\Delta x_c^2+\frac{\partial \rho_f}{\partial y}\frac{\partial e_{tf}}{\partial y}\Delta y_c^2+\frac{\partial \rho_f}{\partial z}\frac{\partial e_{tf}}{\partial z}\Delta z_c^2\right). \tag{23.2}$$

Φ_{ucf} and Φ_{ecf} are the short-range velocity-density and energy-density correlations, respectively. Then we can obtain differential-formulae of Φ^d_{uff} and Φ^d_{eff} by using Δx_f, Δy_f and Δz_f instead of Δx_c, Δy_c and Δz_c, respectively in the formulae (22.2) and (23.2). Φ^d_{uff} and Φ^d_{eff} can be called the resonant-range velocity-density and energy-density correlations, respectively. The large-small scale (LSS) equations (20) and (21), in which the eight equations are actually the relationships between the eight Farve space-averged variables ($U_{\rho ci}$, $U_{\rho fi}$, $e_{t\rho c}$, $e_{t\rho f}$) and the common space-averaged variables (U_{ci}, U_{fi}, e_{tc}, e_{tf}), can be used to determine twenty unknown quantities (ρ_c, U_{ci}, $U_{\rho ci}$, p_c, T_c, $e_{t\rho c}$) and (ρ_f, U_{fi}, $U_{\rho fi}$, p_f, T_f, $e_{t\rho f}$). Therefore, the LSS equations are approximately closed and do not contain any empirical constants or relations. Compared with the large-small scale (LSS) equations for the incompressible turbulent flows[1], the LSS equations (20) and (21) increase eight Farve averaged variables and the eight nonlinear relationships between the Farve- and space-averaged variables. The compressibility effects on turbulence are determined by the above-stated eight nonlinear relationships and the eight Farve averaged variables, which are reduced to the eight space averaged variables in common used when the density ρ equals to constant. From the LSS equations (20) and (21) we know that the nonlinear dynamics of the resolved large scales $\Delta x>\Delta x_c$ are governed mainly by their interactions with the resolved small scales in the range $\Delta x_c>\Delta x>\Delta x_f$ ($\Delta x_f=0.5-0.2\Delta x_c$) and much smaller unresolved scales $\Delta x<\Delta x_f$ have negligible effect on the resolved large scales $\Delta x>\Delta x_c$, which are neglected; and that the dynamics of the resolved small scales in the range $\Delta x_c>\Delta x>\Delta x_f$ are largely governed by their interactions with the large scales $\Delta x>\Delta x_c$ and much smaller unresolved scales $\Delta x<\Delta x_f$ have secondary effects on the resolved small scales in the range $\Delta x_c>\Delta x>\Delta x_f$, which are approximated by the resonant-range eddy stress *etc*.; and that the fluctuation motions of the resolved small scales in the range $\Delta x_c>\Delta x>\Delta x_f$ relating to the resolved large scales $\Delta x>\Delta x_c$ are caused mainly by the large scales $\Delta x > \Delta x_c$. The above conclusions are agreement with those given by the multiscale equations the direct numerical simulation (DNS) for the incompressible turbulent flows[1-4]. The numerical inference acquired through the analysis of DNS databases for the channel incompressible turbulent flows by Domaradzki *et al*[4] confirmed that the nonlinear dynamics of the resolved modes with wave number $k < k_1$ are governed by their interactions with a limited range of modes with wave number not exceeding $2k_1$ and much smaller scales have a negligible effect on the resolved ones and that the nonlinear dynamics of the modes with wave number ranging from k_1 to $2k_1$ are largely determined by their interactions with the resolved scales with wave number $k < k_1$. In addition, the unresolved small scales $\Delta x < \Delta x_f$ contain still a wide range of time- and space-scales, hence, any formulae expressing their interactions with the resolved small scales in the range $\Delta x_c > \Delta x > \Delta x_f$ are certainly imperfect. Perhaps it is other choice to use empirical sub-grid scale (SGS) model instead of those formulae of the resonant-range interactions.

In addition, by similar deductions made in the above-stated large-small-scales, *i.e.* two scales equations we can obtained three-scales equations for the compressible turbulent flow. The three-scales equations are also approximately closed and do not contain any empirical constants or relations. It should be mentioned that in the vicinity of shock wave the law of interaction between scales in the compressible turbulent flow is not clear and so the multiscale model is not suitable to shock waves.

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