A NONLOCAL THEORY FOR BRITTLE FRACTURE

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ABSTRACT: In this paper, a nonlocal theory of fracture for brittle materials has been systematically developed, which is composed of the nonlocal elastic stress fields of Griffith cracks of mode-I, II and III, the asymptotic forms of the stress fields at the neighborhood of the crack tips, and the maximum tensile stress criterion for brittle fracture. As an application of the theory, the fracture criteria of cracks of mode-I, II, III and mixed mode I-II, I-III are given in detail and compared with some experimental data and the theoretical results of minimum strain energy density factor.

KEY WORDS: brittle fracture, nonlocal elasticity, maximum tensile stress criterion

I. INTRODUCTION

Due to the prominent contributions of Griffith. Orowan and Sih, the theory of brittle fracture has been crowned with a great success in predicting macro fracture of brittle materials. Due to the development of advanced materials such as ceramics composites and especially the investigation of their toughening mechanism, the phenomenal theory of fracture has been studied from the view of the micro mechanism of crack extension. This trend defies the current theory in two aspects. The classical continuum mechanics needs to be extended to incorporate the effect of micro structure of materials, and the environmental parameters around a crack tip such as stress intensity factor and J-integral need to be connected with the local state of stress or strain on a micro scale. The nonlocal continuum theory[1,2] developed in 1970's can meet to some extent the first need, since this generalized continuum theory bridges the gap between the classical continuum theory and the lattice dynamics. The present paper is devoted to the second need, i.e., to develop a new theory of fracture for brittle materials within the framework of nonlocal elasticity.

The analysis of a crack problem with nonlocal elasticity is originated by Eringen[3-5], one of the founders of the nonlocal continuum theory. He and his coworkers have obtained two significant results on the nonlocal stress distribution of Griffith cracks of mode-I, II and III. No singularity appears in the nonlocal stresses at a crack tip, and if the maximum stress near a crack tip is adopted as the critical parameter of crack extension, the known experiments on brittle fracture can be explained from the cleavage mechanism of breaking the coherence of atoms. Yu and Zheng[6] extended the nonlocal elastic analysis of a crack to nonlocal elasto-plastic analysis. Though these results may be used for a new approach to the problems of brittle fracture, the numerical solutions of stress fields are unsatisfactory in practical applications and insufficient to treat a crack of mixed-mode. Recently, the author[7] has worked out an explicit form of nonlocal stress field of a mode-I crack. Following section 2 in which the nonlocal elasticity is briefly introduced, we focus on the explicit solution of Griffith cracks of mode-II and III in section 3. For the sake of integrity, the solution of mode-I Griffith crack is included. Based on the results, the asymptotic nonlocal stress fields near the crack tips are derived in section 4. In section 5 the maximum tensile stress criterion for brittle fracture is proposed and used to deal with some important crack problems.

II. NONLOCAL THEORY OF ELASTICITY

Nonlocal theory of elasticity[1,2] is an extension of classical (local) theory of elasticity by

taking into account the effect of long range force in deformed materials. For an infinitesimal deformation of isotropic and homogeneous materials, the equilibrium equations of nonlocal elasticity are identical to the one of classical elasticity.

$$t_{ij,j} + f_i = \rho \ddot{u}_i \tag{1}$$

where $\ddot{u}_i$ is displacement components, t_{ij} nonlocal stress, f_i body force and ρ mass density. Here we employ as usual the superposed dot and a comma to indicate partial derivatives with respect to time and the rectangular coordinate x_i respectively and repeated indices mean a summation.

Nonlocal theory of elasticity is characterized by its constitutive relation as follows

$$t_{ij}(\mathbf{x}) = \int_v \alpha(|x'-x|, \varepsilon) [\lambda e_{kk}(x')\delta_{ij} + 2\mu e_{ij}(x')] \, dv(x') \tag{2}$$

where λ and μ are the Lame constants, δ_{ij} is the Kronecker δ, v the material domain,

$$e_{ij} = \frac{1}{2}(u_{i'j} + u_{j'i}) \tag{3}$$

the strain, $\alpha(|x'-x|), \varepsilon)$ the constitutive function called the nonlocal elastic kernel, ε a parameter and $|x'-x|$ the distance from x' to x.

Relation (2) indicates that the stress at a point x depends on strains at all points in the material domain. This feature reflects the physical fact in lattice dynamics that an atom is interacted by another atom at a long distance. Naturally, it is expected that the interaction diminishes with the distance of atoms so that the nonlocal elastic kernel is an attenuation function.

On the other hand, if we denote the Cauchy stress in classical elasticity by σ_{ij}, constitutive relation (2) can be rewritten as

$$t_{ij} = \int_v \alpha(|x'-x|, \varepsilon) \sigma_{ij}(x') \, dv(x') \tag{4}$$

where

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{5}$$

is the classical Hooke's law. Hence the nonlocal elasticity takes the classical elasticity as a limit when

$$\alpha(|x'-x|, \varepsilon)|_{\varepsilon \to 0} = \delta(|x'-x|) \tag{6}$$

where $\delta(|x'-x|)$ is the Dirac function.

Though the nonlocal elastic kernel can be related to the interaction potential of atoms in lattice dynamics [2], the practical way of determining its form reasonably and reliably is not available up to now. In the published literature, such as in Eringen's paper[4], the kernel is selected as a trial attenuation function under condition (6) and then to determine the constant ε by comparing the wave dispersion relation of nonlocal elasticity resulted from the kernel with the one of lattice dynamics. In this way, several forms of the kernel have been proposed. In a previous paper[7] in which the nonlocal stress field of a mode-I Griffith crack was investigated, the author proposed

$$\alpha(|x'-x|, \varepsilon) = \frac{2\varepsilon^4}{\pi(|x'-x|^2 + \varepsilon^2)^3} \tag{7}$$

where $\varepsilon = a/2$, a is the atom distance in a perfect lattice. Since the kernel has been successfully used in treating a crack problem, we will still use the kernel in what follows.

III. NONLOCAL STRESS FIELDS OF GRIFFITH CRACKS

By a Griffith crack is meant a crack in an infinite plane which is subject

to a uniform force at the crack faces or at the infinite distance from the crack . Here we investigate three modes of a Griffith crack : mode-I which corresponds to a tension force , mode-II to a shear force and mode-III to an anti-plane shear force. For a Griffith crack , Eringen[4] showed that the field equations of nonlocal elasticity are the same as those of classical elasticity. Therefore , making use of Fourier transformation technique and the symmetry of the problems , we can find formal solutions of the nonlocal elastic stress fields :

$$
\left.
\begin{aligned}
t_{xx} &= -\frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\left(1+y'\frac{\partial}{\partial y'}\right)\frac{x'-t}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy' \\
t_{xy} &= -\frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\left(y'\frac{\partial}{\partial y'}\right)\frac{y'}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy' \\
t_{yy} &= -\frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\left(1-y'\frac{\partial}{\partial y'}\right)\frac{x'-t}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy'
\end{aligned}
\right\}\ \text{for mode-I} \quad (8)
$$

$$
\left.
\begin{aligned}
t_{xx} &= -\frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\left(2+y'\frac{\partial}{\partial y'}\right)\frac{y'}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy' \\
t_{xy} &= -\frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\left(1+y'\frac{\partial}{\partial y'}\right)\frac{x'-t}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy' \\
t_{yy} &= -\frac{b}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\left(y'\frac{\partial}{\partial y'}\right)\frac{y'}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy'
\end{aligned}
\right\}\ \text{for mode-II} \quad (9)
$$

$$
\left.
\begin{aligned}
t_{xz} &= \frac{\mu}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\frac{y'}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy' \\
t_{yz} &= -\frac{\mu}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-1}^{1}\alpha(|x'-x|,\varepsilon)\frac{x'-t}{(x'-t)^2+y'^2}u'(t)\,dt\,dx'\,dy'
\end{aligned}
\right\}\ \text{for mode-III} \quad (10)
$$

where $b=\mu/(1+v)$, v is the Poisson's ratio, $2l$ the length of a crack , vectors $x=xi+yj+zk$ and $x'=x'i+y'j+z'k$ are originated at the centre of a crack , x-axis lies on the crack line and y-axis is perpendicular to the crack line , $u(t)$ is the displacement of the crack surface , namely, $u(t)=u_y(x,0)$ for mode-I crack , $u(t)=u_x(x,0)$ for mode-II and $u(t)=u_z(x,0)$ for mode-III in which $|x|\leqslant 1$. With the nonlocal elastic kernel (7), (8)—(10) can be further integrated .

$$
\left.
\begin{aligned}
t_{xx} &= -\frac{b}{\pi}\int_{-1}^{1}\left(1+y\frac{\partial}{\partial y}\right)\frac{x-t}{(x-t)^2+y^{*2}}u'(t)\,dt \\
t_{xy} &= -\frac{b}{\pi}\int_{-1}^{1}\left(y\frac{\partial}{\partial y^*}\right)\frac{y^*}{(x-t)^2+y^{*2}}u'(t)\,dt \\
t_{yy} &= -\frac{b}{\pi}\int_{-1}^{1}\left(1-y^*\frac{\partial}{\partial y^*}\right)\frac{x-t}{(x-t)^2+y^{*2}}u'(t)\,dt
\end{aligned}
\right\}\ \text{for mode-I} \quad (11)
$$

$$
\left.
t_{xx} = \frac{b}{\pi}\int_{-1}^{1}\left(2+y\frac{\partial}{\partial y}\right)\frac{y}{(x-t)^2+y^{*2}}u'(t)\,dt
\right\}
$$

$$t_{xy} = -\frac{b}{\pi} \int_{-1}^{1} \left(1 + y\frac{\partial}{\partial y}\right) \frac{x-t}{(x-t)^2 + y^{*2}} u'(t)\,dt \; \Big\}$$

for mode-II   (12)

$$t_{yy} = -\frac{b}{\pi} \int_{-1}^{1} \left(y\frac{\partial}{\partial y^*}\right) \frac{y^*}{(x-t)^2 + y^{*2}} u'(t)\,dt \; \Big\}$$

$$t_{xz} = \frac{\mu}{\pi} \int_{-1}^{1} \frac{y}{(x-t)^2 + y^{*2}} u'(t)\,dt \; \Big\}$$

for mode-III   (13)

$$t_{yz} = -\frac{\mu}{\pi} \int_{-1}^{1} \frac{x-t}{(x-t)^2 + y^{*2}} u'(t)\,dt \; \Big\}$$

in which $y^* = \sqrt{y^2 + \varepsilon^2}$.

Generally speaking, displacement $u(t)$ in (11)—(13) are to be determined by stress conditions on crack surfaces. But up to now, the relevant nonlocal stress boundary condition is under investigation. The concept "uniform force loaded on a crack surface" is based on the classical Cauchy stress, rather than the nonlocal stress. Eringen[4] used the classical concept to solve $u(t)$ in (11)—(13) numerically. But Atkinson[8] pointed out that the Eringen's results are not convergent. In fact, the classical stress boundary condition is in general not compatible with the nonlocal continuum theory. In the auther's opinion, it is neccessary for the time being to keep the classical description of the boundary condition:

$$\sigma_{yy}(x,y)|_{y=0} = -p \qquad \text{for mode-I} \; \Big\rangle$$
$$\sigma_{xy}(x,y)|_{y=0} = -p \qquad \text{for mode-II} \; \Big\}$$
$$\sigma_{yz}(x,y)|_{y=0} = -p \qquad \text{for mode-III} \; \Big\rangle$$

(14)

where p is the uniform force. Without any ambiguity, tension force and shear force are denoted by p.

From (4) and (5), the relation between σ_{ij} and t_{ij} is

$$\sigma_{ij} = t_{ij}|_{\varepsilon \to 0} \tag{15}$$

Hence, displacement $u(t)$ must satisfy, by combining (11)—(15)

$$-\frac{b}{\pi} \int_{-1}^{1} \frac{u'(t)}{x-t}\,dt = -p \qquad \text{for mode-I, II} \; \Big\}$$

(16)

$$-\frac{\mu}{\pi} \int_{-1}^{1} \frac{u'(t)}{x-t}\,dt = -p \qquad \text{for mode-III} \; \Big\}$$

They are obviously identical to the classical results, that is,

$$u(x) = \frac{p}{b}\sqrt{1^2 - x^2} \qquad \text{for mode-I, II} \; \Big\}$$

(17)

$$u(x) = \frac{p}{\mu}\sqrt{1^2 - x^2} \qquad \text{for mode-III} \; \Big\}$$

Now by substituting (17) into (11)—(13), we find

$$t_{xx} = \left(1 + y\frac{\partial}{\partial y}\right)I \qquad t_{xy} = \left(y\frac{\partial}{\partial y^*}\right)J \qquad t_{yy} = \left(1 - y^*\frac{\partial}{\partial y^*}\right)I \qquad \text{for mode-I} \quad (18)$$

$$t_{xx} = -\left(2 + y\frac{\partial}{\partial y}\right)\left(\frac{y}{y^*}\,J\right) \qquad t_{xy} = \left(1 + y\frac{\partial}{\partial y}\right)I \qquad t_{yy} = \left(y\frac{\partial}{\partial y^*}\right)J \quad \text{for mode-II} \quad (19)$$

$$t_{xz}=-\left(1-\frac{\varepsilon}{2}\frac{\partial}{\partial\varepsilon}\right)\left(\frac{y}{y^*}J\right)\qquad t_{yz}=\left(1-\frac{\varepsilon}{2}\frac{\partial}{\partial\varepsilon}\right)I\qquad \text{for mode-III} \quad(20)$$

in which

$$I=p\left(\sqrt{\frac{T+S\sqrt{R}}{2R}}-1\right)\qquad J=\frac{\sqrt{2}\,pl^2\,xy^*}{\sqrt{R(T+S\sqrt{R})}}\qquad S=x^2+y^{*2}\left.\begin{array}{c}\\\\\end{array}\right\}$$

$$T=(x^2+y^{*2})^2+l^2(y^{*2}-x^2)\qquad R=(x^2+y^{*2})^2+2l^2(y^{*2}-x^2)+l^4 \qquad(21)$$

Those are the nonlocal stress fields of Griffith cracks in an explicit form, in which (18) has been reported in a previous paper[7].

In (18)—(20), $y^{*2}=y^2+\varepsilon^2>0$, $S>0$, $R>0$ and $T+S\sqrt{R}>0$. Hence, no singularity exists in the nonlocal stress fields.

IV. ASYMPTOTIC NONLOCAL STRESS FIELDS OF CRACKS

The explicit solution of nonlocal stress fields of Griffith cracks makes it possible to find out the asymptotic distribution of nonlocal stresses near the tip of a crack. To this end, we take such a pseudo-polar coordinate system at a crack tip $(x,y)=(1,0)$ as

$$\left.\begin{array}{c}x=1+r\cos\theta\\y=r\sin\theta\sin\varphi\\\varepsilon=r\sin\theta\cos\varphi\end{array}\right\}\qquad(22)$$

in which $0\leqslant\theta\leqslant\pi$, $-\frac{\pi}{2}\leqslant\varphi\leqslant\frac{\pi}{2}$ and only two of the three parameters (r,θ,φ) are independent since ε is a material constant.

From (22), $r=\sqrt{(x-1)^2+y^2+\varepsilon^2}$. Hence $r\ll1$ at the neighborhood of the tip of a macro crack where $|x-1|\ll1$, $|y|\ll1$ and $\varepsilon=a/2\ll1$.

Upon substituting (22) into (18)—(20) and neglecting the high order of r, we obtain

$$t_{xx}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin^2\varphi\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\left.\begin{array}{c}\\\\\\\end{array}\right.$$

$$t_{xy}=\frac{K_I}{\sqrt{2\pi r}}\sin\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\qquad\qquad\text{for mode-I}\quad(23)$$

$$t_{yy}=\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

$$t_{xx}=\frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(2+\cos^2\varphi+\sin^2\varphi\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\left.\begin{array}{c}\\\\\\\end{array}\right.$$

$$t_{xy}=\frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-\sin^2\varphi\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\qquad\qquad\text{for mode-II}\quad(24)$$

$$t_{yy}=\frac{K_{II}}{\sqrt{2\pi r}}\sin\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$t_{xz}=-\frac{K_{III}}{\sqrt{2\pi r}}\sin\varphi\sin\frac{\theta}{2}\left(1+\frac{1}{2}\cos^2\varphi-\frac{1}{2}\sin^2\varphi\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\left.\begin{array}{c}\\\\\\\end{array}\right\}\text{for mode-III}\quad(25)$$

$$t_{yz}=\frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1+\frac{1}{2}\cos^2\varphi\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)$$

where $K_i = p\sqrt{\pi l}$ ($i = $ I, II, III) is the stress intensity factor.

Now, we make the following significant observations:

(1) Since $r > 0$, stresses at a crack tip are finite but concentrated with the order of $\sqrt{1/a}$.

(2) The maximum stress occurs at a point near the crack tip. For example,

$$t_{yy\,max}^{I} = t_{yy}(x, y)\Big|_{\substack{x=1+0.2a \\ y=0}} = 0.873\,\frac{K_1}{\sqrt{\pi a/2}}$$

This is due to the physical fact that the stress fields are continuously distributed.

(3) Asymptotic stress fields (23)—(25) take the classical ones as their limits as $\varepsilon \to 0$ when $\varphi = \pi/2$ and (r, θ) becomes the polar coordinates.

(4) (23)—(25) are a slight modification of the classical stress fields on a scale of atom distance from the crack tip, because the effect of ε can be almost neglected when $(x-1)^2 + y^2 \gg \varepsilon^2$.

(5) Basing on (iv), we find that (23)—(25) can be directly derived by substituting the classical asymptotic stress fields into constitutive relation (4). This fact results from the attenuation behavior of interaction force of atoms. In this view, we believe that (23)—(25), except K_i, are valid for all cracks though they are obtained from the stress fields of Griffith cracks.

V. MAXIMUM TENSILE STRESS CRITERION FOR BRITTLE RUPTURE

In classical theory of fracture, the rupture criterion, such as the minimum potential by Griffith, the stress intensity factor by Irwin, or the minimum strain energy factor by Sih, is not related with the theoretical strength of a brittle material, since the stresses at a crack tip is infinite in the classical elasticity. Nonlocal elasticity provides finite values of stresses at a crack tip. Therefore, it is natural for us to put forward the following rupture criterion for a crack in brittle materials:

When the maximum tensile stress of a crack reaches the theoretical strength, the crack will extend along the direction normal to the maximum stress.

Since the maximum stress must be the first principal stress, the above criterion can be formulated as

$$\max_{x \in \Omega} (t_1(x)) = t_c \tag{26}$$

where Ω denotes the neighborhood domain near a crack tip, t_c the theoretical strength, $t_1(x)$ the first principal stress at point x, i.e., $t_1 \geqslant t_2 \geqslant t_3$, t_i ($i = 1, 2, 3$) is the principal stress.

Noting that the principal stress, t, and its direction, n, of the stress t_{ij} can be determined from

$$(t_{ij} - t\delta_{ij})n_j = 0 \tag{27}$$

where

$$n_i n_i = 1 \tag{28}$$

we can give the rupture criterion for a crack by substituting asymptotic stress fields (23)—(25) into (27), (28) and then into (26). In the following, we take some examples to evaluate the criterion.

1. Three Fundamental Cracks

For a mode-I crack,

$$\max_{x \in \Omega} (t_1(x)) = t_{yy}(x, y)\Big|_{\substack{x=1+0.2a \\ y=0}} = 0.873\,\frac{K_I}{\sqrt{\pi a/2}}$$

Hence, the rupture condition is

$$0.873\,\frac{K_I}{\sqrt{\pi a/2}} = t_c \tag{29}$$

and extending direction is on the x-axis .

Since theoretical strength t_c can be estimated by surface energy density γ,

$$t_c = \sqrt{\frac{2\mu\gamma}{(1+v)a}} \tag{30}$$

we find that condition (29) is less 1.5% than Griffith criterion $t_c = 0.886\, K_1/\sqrt{\pi a/2}$.

For a mode-II crack, the maximum stress is $0.885\, K_{II}/\sqrt{\pi a/2}$ and makes angle $23.6°$ to the crack line . Therefore, the rupture condition becomes

$$0.885\, \frac{K_{II}}{\sqrt{\pi a/2}} = t_c \tag{31}$$

and the extending direction is $90° - 23.6° = 66.4°$ to the crack line .

From (29) and (31), we find $K_{II}/K_1 = 0.980$. This ratio is in good agreement with 0.957 by Sih's theory for $v = 0.3$. But Sih's theory gives $82.3°$ for the extending angle which is larger than our result .

For a mode-III crack ,

$$\max_{x \in \Omega} (t_1(\mathbf{x})) = t_{yz}(x, y)\Big|_{\substack{x=1+0.2a \\ y=0}} = 0.719\, \frac{K_1}{\sqrt{\pi a/2}}$$

so that the rupture condition is

$$0.719\, \frac{K_{III}}{\sqrt{\pi a/2}} = t_c \tag{32}$$

and the extending direction is along the crack line .

By comparing (29) with (32), $K_{III}/K_1 = 1.214$. It is larger than $K_{III}/K_1 = (1-2v)^{0.5} < 1$ by Sih's theory, but almost the same as the experimental result $K_{III}/K_1 = 1.22$ given by Shah[9] .

Maximum tensile stress criterion (29) (31) (32) can also be written in the form of Irwin's, i .e .,

$$K_1 = K_{Ic} \qquad K_{II} = K_{IIc} \qquad K_{III} = K_{IIIc} \tag{33}$$

where

$$K_{Ic} = 1.44\, t_c \sqrt{a} \qquad K_{IIc} = 1.42\, t_c \sqrt{a} \qquad K_{IIIc} = 1.74\, t_c \sqrt{a} \tag{34}$$
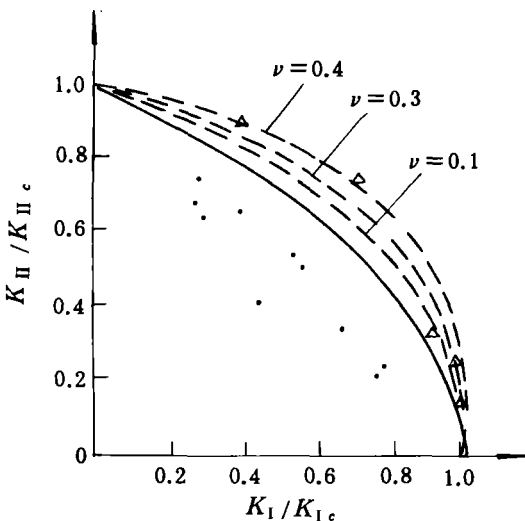
2. A Mixed I-II crack

Fig.1 Critical curve for mode I-II crack
—— Nonlocal theory — — — Sih's theory
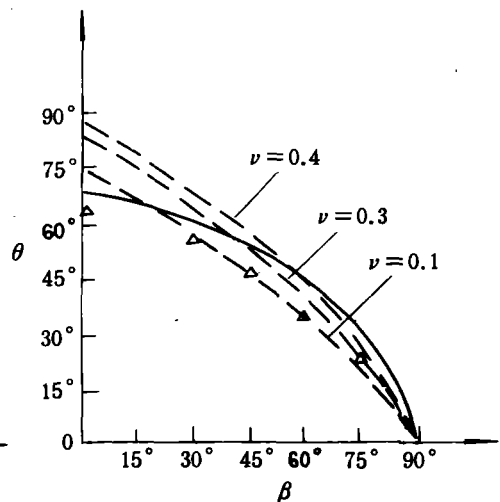· Experiment[9] Δ Experiment[10]

Fig.2 Extending directions for mode I-II crack
—— Nonlocal theory — — — Sih's theory
Δ Experiment[10]

When a crack is loaded simultaneously by a tension force and a shear force, the asymptotic stress field of the crack consists of the superposition of (23) and (24) because of the linearity of elasticity. In this case, the maximum stress is related linearly with both K_I and K_{II}. Hence, the rupture condition of a mixed I-II crack can be expressed, by noting (33), in the form of

$$F\left(\frac{K_I}{K_{Ic}}, \frac{K_{II}}{K_{IIc}}\right) = 1 \tag{35}$$

where F is a linear function of its variables.

Eq. (35) is plotted in Fig.1 in which the results of Sih's theory and experimental data by Shah[9] and Zhao[10] are also given. We find that the maximum stress criterion gives its results in the middle of the two sets of experimental data and under the curve of Sih's theory.

While determining the function F, the maximum stress direction is found for a given K_{II}/K_I or $\beta = \mathrm{tg}^{-1}(K_{II}/K_I)$. Let α denote the angle of the direction to the x axis. Then, the extending angle, defined by $\theta = 90° - \alpha$, is related with the load parameter β. Fig.2 shows the result of $\theta \sim \beta$ and its comparison with the counterparts by Sih's theory and Zhao's experimental data[10]. They are in general in agreement.

3. A Mixed I-III Crack

Analogous to the above discussion, the rupture condition of a mixed I-III crack can be represented by

$$H\left(\frac{K_I}{K_{Ic}}, \frac{K_{III}}{K_{IIIc}}\right) = 1 \tag{36}$$

Its detail is illustrated by Fig.3. Like Sih's results, the maximum stress condition gives its curve under the experimental data by Shah and Zhao. As to the extending direction, the maximum stress criterion predict the well-known fact that a mixed I-III crack extends along the crack line.
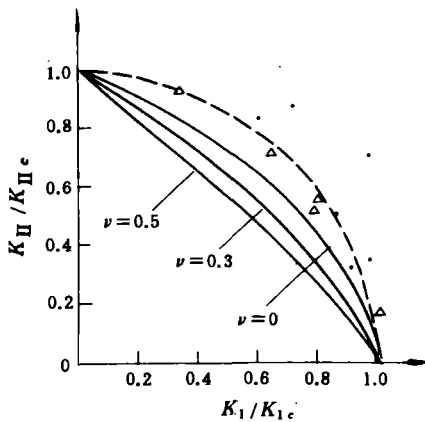
Fig.3 Critical curve for mode I-III crack
—— Nonlocal theory – – – Sih's theory
· Experiment[9] △ Experiment[10]

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