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SURFACE WAVE PATTERNS AND INSTABILITY IN A VERTICALLY OSCILLATING CIRCULAR CYLINDRICAL VESSEL*

JIAN Yong-jun
The First Institute of Oceanography, State Oceanic Administration, Qingdao 266061, China
Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, China,
e-mail: jianyongjun@yahoo.com.cn
E Xue-quan
Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, China
FENG Liu-lin
Hohhot Commanding School, Public Security and Frontier Defence Army, Hohhot 010051, China

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ABSTRACT: The natural frequency of surface wave, which has been derived from a vertically oscillating circular cylindrical vessel in inviscid fluid, was modified by considering the influence of surface tension and weak viscosity. Many flow patterns were found at different forced frequencies by numerical computation. In addition, the nonlinear amplitude equation derived in inviscid fluid was modified by adding viscous damping and the unstable regions were determined by stability analysis.

KEY WORDS: vertically forced oscillation, flow patterns, amplitude equation, surface tension, viscous damping

1. INTRODUCTION

The hydrodynamic instability of vertically driven surface waves was first studied experimentally by Faraday[1]. He realized that these surface waves have a frequency equal to one half of the exciting one and belong to subharmonic resonance. Benjamin and Ursell[2] demonstrated that the linear behaviour of inviscid surface waves can be understood in terms of a Mathieu oscillator. The Faraday resonance was an ideal model to study nonlinear pattern formation, bifurcation, and chaos. A review on this subject was given by Miles and Henderson[3]. Many problems associated with the Faraday instability have been solved in inviscid fluids. Miles[4-5] has studied nonlinear effects of this problem adopting a variational approach in inviscid fluids. However, until now, no reasonable nonlinear theory for strongly damped surface waves has been established. Difficulties arise due to the interplay of intricate nonlinear boundary conditions at the free surface and the external excitation, which makes the problem non-autonomous.

In the last two decades, many flow patterns, which include hexagons, triangles, twelvefold quasi-periodic pattern, two-dimensional quasi-crystal, supper-lattice patterns, etc., were observed in experiment with one or two-frequency drive[6-8].

Faraday's instability has been extensively investigated for weakly viscous fluids in confined and extended systems[9], secondary instabilities and transition to spatio-temporal chaos[10], and turbulence[11].

E et al.[12-14] carried out the flow visualization experimentally on surface wave patterns in a circular cylindrical vessel by vertical external vibrations. They obtained many elegant pictures of free surface patterns in wider range of driven frequencies, and most of them have not been reported before.

Recently, Jian el al.[15-17] proposed a mathematical formulation associated with the flowing visualization of Refs. [12-14], from which the second order free surface elevations and their contours were obtained by two-time scale singular perturbation expansion. Due to ignoring surface tension and viscid dissipation in their theoretical analysis, there existed large discrepancies in the forced frequency between theoretical and experimental re-

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sults although the numerical contours of free surface waves agreed well with those of the experimental visualization.

In this paper, we present some flow patterns obtained in a theory including the effect of surface tension and weak viscosity. In addition, the nonlinear amplitude equation is modified by adding viscous damping and its instability is demonstrated.

2. MATHEMATICAL DESCRIPTION

We still consider surface waves excited by the vertical motion of a circular cylindrical basin filled with fluid. All parameters and the choice of the coordinate system are the same as in Ref.[15].

Similarly, when surface tension is considered, we can obtain the following velocity potential and free surface elevation:

$$\varphi(r,\theta,z,t,\tau)=\varepsilon\varphi_1+\varepsilon^2\varphi_2,$$

$$\eta(r,\theta,t,\tau)=\varepsilon\eta_1+\varepsilon^2\eta_2 \tag{1}$$

The first order velocity potential $\varphi_1(r,\theta,z,t,\tau)$ and free surface elevation $\eta_1(r,\theta,t,\tau)$ are

$$\varphi_1=J_m(\lambda r)\cosh[(\lambda z+h/R)]\circ[p(\tau)e^{i\Omega t}+\bar{p}(\tau)e^{-i\Omega t}]\cos m\theta \tag{2}$$

$$\eta_1=\frac{\lambda}{i\Omega}J_m(\lambda r)\sinh(\lambda h/R)\circ[p(\tau)e^{i\Omega t}-\bar{p}(\tau)e^{-i\Omega t}]\cos m\theta \tag{3}$$

where the meaning of all parameters are the same as in Ref.[15].

Moreover, the dispersion relationship is

$$\Omega^2=\lambda_{mn}\tanh(\lambda_{mn}h/R)\circ(1+\frac{\Gamma}{\rho}\lambda_{mn}{}^2)=\Omega_{mn}{}^2 \tag{4}$$

where Ω is the natural frequency of the surface wave, Γ is the surface tension coefficient, and ρ is the mass density.

The second order velocity potential $\varphi_2(r,\theta,t,\tau)$ and free surface elevation $\eta_2(r,\theta,t,\tau)$ can be expressed as

$$\varphi_2(r,\theta,z,t,\tau)=[X_1(r,z)+X_2(r,z)\cos(2m\theta)]\circ[p^2(\tau)e^{2i\Omega t}-\bar{p}^2(\tau)e^{-2i\Omega t}] \tag{5}$$

$$\eta_2(r,\theta,t,\tau)=[Y_1(r)+Y_2(r)\cos(2m\theta)]\circ[p^2(\tau)e^{2i\Omega t}+\bar{p}^2(\tau)e^{-2i\Omega t}] \tag{6}$$

where the detail expressions $Y_1(r)$, $Y_2(r)$, $X_1(r,z)$ and $X_2(r,z)$ can be found in Ref.[17] and they are omitted in this paper.

With the so-called solvability condition, a nonlinear amplitude equation can be written as

$$i\frac{dp(\tau)}{d\tau}=M_1p^2(\tau)\bar{p}(\tau)+M_2e^{2i\sigma\tau}\bar{p}(\tau) \tag{7}$$

where i is the unit of imaginary number, and M_1 and M_2 are constants. Although the form of Eq.(7) is equivalent to Eq.(65) in Ref.[15], the former includes the effect of surface tension.

In fact, the damping will appear in actual physical system due to the viscous dissipation of the fluid. Jian[18] obtained a analytical expression of damping coefficient in weak viscous fluid by dividing whole fluid fields into outer potential region and inner boundary layer region. The detailed expression of damping coefficient β can be found in Ref.[18].

Henderson[19] pointed out that the real and the imaginary parts of the damping coefficient β mean damping and "frequency shift" respectively. The damping causes the attenuation of the surface wave, while the "frequency shift" changes the natural frequency of the surface wave.

By taking the effect of weak viscosity into account, the dispersive relation Eq.(4) can be modified by adding viscous damping and it becomes to

$$\hat{\Omega}=\Omega-\beta_2 \tag{8}$$

where β_2 is the imaginary part of the viscous damping coefficient β and its detailed expression is

$$\beta_1=\left\{\frac{\lambda[\sinh(2\lambda h/R)+2\lambda h/R]}{8\Omega\cosh^2(\lambda h/R)}+\frac{\lambda^2}{4\Omega\cosh^2(\lambda h/R)}+\frac{\lambda^2\Omega}{2(\lambda^2-m^2)}\right\}\sqrt{\frac{2\nu}{\Omega}} \tag{9}$$

3. INSTABILITY ANALYSIS

Substituting the real part β_1 of the damping coefficient β into the left hand of nonlinear evolution Eq. (7), we obtain the modified amplitude evolution equation

$$i\left(\frac{d}{d\tau}+\beta_1\right)p(\tau)=M_1p^2(\tau)\bar{p}(\tau)+M_2e^{2i\sigma\tau}\bar{p}(\tau) \quad (10)$$

where σ, β_1, M_1 and M_2 are real numbers.

For the convenience of solving the modified amplitude Eq. (10), we make a transformation for a unknown function $p(\tau)$, letting

$$q(\tau)=p(\tau)e^{-i\sigma\tau} \quad (11)$$

and then Eq. (10) becomes

$$i\frac{dq(\tau)}{d\tau}=-i\beta_1q(\tau)+\sigma q(\tau)+M_1q^2(\tau)\bar{q}(\tau)+M_2\bar{q}(\tau) \quad (12)$$

It can be shown that the stable property of the amplitude $p(\tau)$ and $q(\tau)$ is equivalent. Divide the unknown variable into real and imaginary parts, and the amplitude Eq. (12) yields the following nonlinear ordinary differential equations

$$\frac{dq_1(\tau)}{d\tau}=-\beta_1q_1(\tau)+(\sigma-M_2)q_2(\tau)+M_1q_2(\tau)\circ[q_1^2(\tau)+q_2^2(\tau)] \quad (13)$$

$$\frac{dq_2(\tau)}{d\tau}=-\beta_1q_2(\tau)-(\sigma+M_2)q_1(\tau)-M_1q_1(\tau)\circ[q_1^2(\tau)+q_2^2(\tau)] \quad (14)$$

The instability analysis includes linear instability and nonlinear instability. Linear instability means that when the forced energy exceeds a threshold value, the surface wave will appear at the free surface. While nonlinear instability is associated with whether the surface waves will lost their stability, namely the so called "secondary instability".

By linear instability analysis, it is easy to prove that when the condition

$$M_2^2>\sigma^2+\beta_1^2 \quad (15)$$

is satisfied, the surface wave appears at the free surface. However, the free surface keeps plane if

$$M_2^2<\sigma^2,\ \text{or}\ \sigma^2<M_2^2<\sigma^2+\beta_1^2 \quad (16)$$

As to the nonlinear instability, the derivative with respect to time equals to zero in Eq. (12), the equilibrium solution yields

$$i\beta_1q(\tau)=\sigma q(\tau)+M_1q^2(\tau)\bar{q}(\tau)+M_2\bar{q}(\tau) \quad (17)$$

Letting $q_0=a_0e^{i\theta}\neq 0$ (where a_0 is a real number) is an equilibrium solution of Eq. (17), and taking q_0 into Eq. (17), we have

$$a_0=\left[\frac{-\sigma\pm\sqrt{M_2^2-\beta_1^2}}{M_1}\right]^{1/2},\quad \sin 2\theta=-\frac{\beta_1}{M_2} \quad (18)$$

Supposing $q'(\tau)$ is an infinitesimal disturbance associated to the equilibrium solution q_0, taking disturbed expression $q_1(\tau)=q'(\tau)+q_0$ into Eq. (12), and ignoring the nonlinear term of infinitesimal disturbance, we can rewrite the disturbed equation as

$$i\frac{dq'(\tau)}{d\tau}=-i\beta_1q'(\tau)+\sigma q'(\tau)+M_2\bar{q}'(\tau)+M_1[q_0^2\bar{q}'(\tau)+2|q_0|^2\bar{q}'(\tau)] \quad (19)$$

We separate Eq. (19) into real and imaginary parts, let $q'(\tau)=A_1(\tau)+iA_2(\tau)$, and take it into Eq. (19), and the following ordinary differential equations will be satisfied.

$$\frac{dA_1}{d\tau}=-\beta_1\left[1+\frac{-\sigma\pm\sqrt{M_2^2-\beta_1^2}}{M_2}\right]A_1+\left[(\sigma-M_2)+(-\sigma\pm\sqrt{M_2^2-\beta_1^2})\circ\left(2\pm\frac{\sqrt{M_2^2-\beta_1^2}}{M_2}\right)\right]A_2 \quad (20)$$

$$\frac{dA_2}{d\tau}=\left[-(\sigma+M_2)-(-\sigma\pm\sqrt{M_2^2-\beta_1^2})\circ\left(2\pm\frac{\sqrt{M_2^2-\beta_1^2}}{M_2}\right)\right]A_1-$$

$$\beta_1[1-\frac{-\sigma\pm\sqrt{M_2^2-\beta_1^2}}{M_2}]A_2 \qquad (21)$$

The eigenfunction of Eqs. (20) and (21) can be given as

$$(\delta+\beta_1)^2=-4(M_2^2-\frac{5}{4}\beta_1^2\mp\sigma\sqrt{M_2^2-\beta_1^2}) \qquad (22)$$

It can be shown that, whenever $M_1 > 0$ or $M_1 < 0$, if the condition $M_2^2 > \sigma^2+\beta_1^2$ is obeyed, the formed surface waves is stable.

4. COMPUTATIONAL RESULTS

4.1 *Mode selection of the surface waves*

If the forced amplitude is prescribed, different forced frequencies will produce different surface wave modes. When the forced amplitude is 11.4μm and forced frequencies are 8.73, 10.92 and 17.85Hz, the contours of the free surface eveva-tions and corresponding three-dimensional surface determined by Eq. (1), are plotted in Fig. 1 at the time $t = 215.9$s. In Fig. 1, the effects of surface tension and weak viscosity have been included, and the meaning of solid lines, dased lines and parametrical couple (m, n) are the same as in Ref. [15].

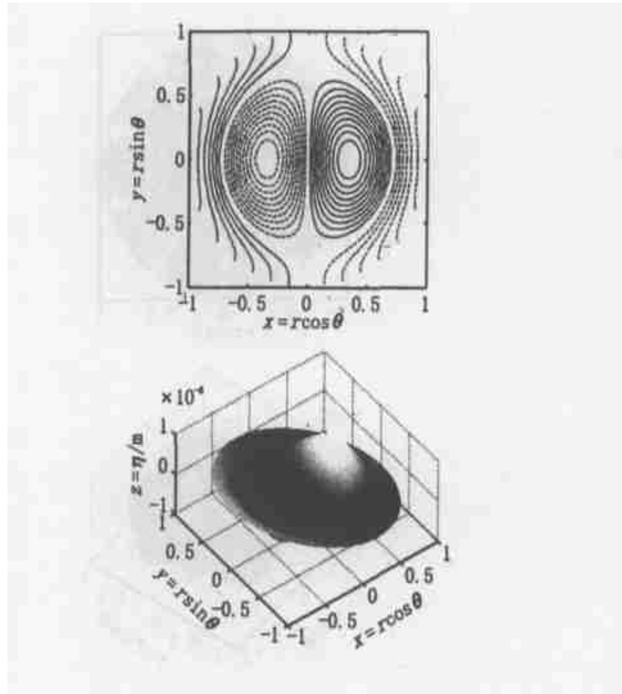
It can be shown that the shapes of the excited modes of the surface waves become more and more complex with increasing forced frequency.

4.2 *Unstable regions and nonlinear evolution of the modified amplitude equation*

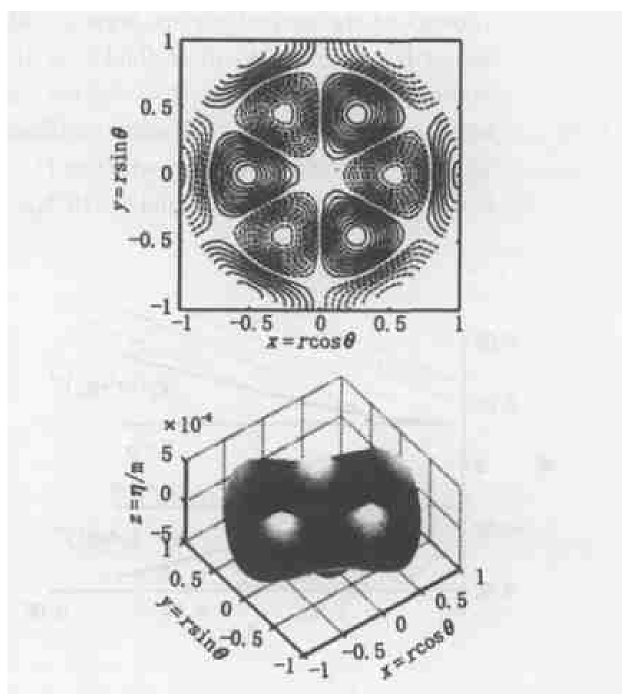
The results of instability are illustrated in Fig. 2 and the unstable regions are determined by Eq. (15). The shaded regions in Fig. 2 are unstable regions. When the parameters enter these regions, the surface wave can be excited due to the first instability.

The evolutions of the amplitude Eqs. (13) and (14) with time and the phase-plane trajectory are depicted in Fig. 3. When the parameters satisfy the first condition of Eq. (16), the stable surface wave can not be formed. This situation is plotted in Figs. 3(a) and 3(b).

It can be easily seen from Fig. 3 that the amplitude decreases gradually and tends to zero eventually under the prescribed initial conditions. This indicates that the external driven energy can not overcome the viscous dissipation, and the stable surface wave can not be formed.



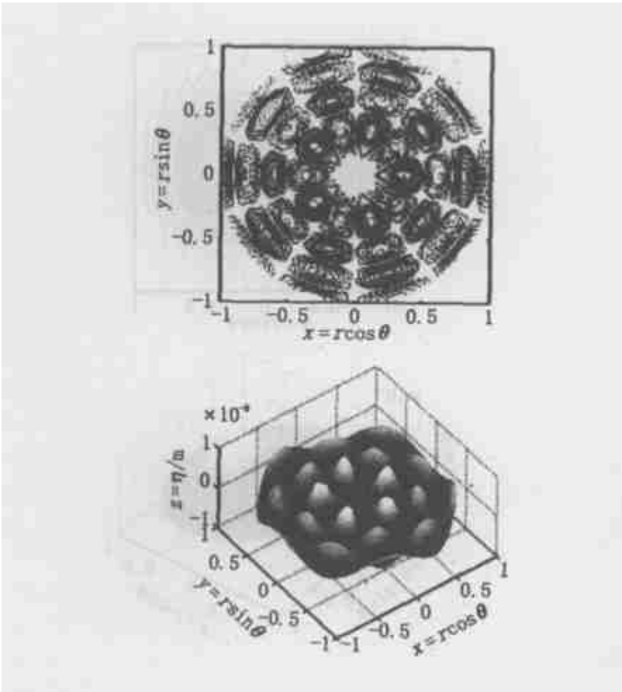
1(a) The forced frequency $f = 8.73$ Hz, and the mode of surface wave is (1, 2)



1(b) The forced frequency $f = 10.92$ Hz, the mode of surface wave is (3, 2)

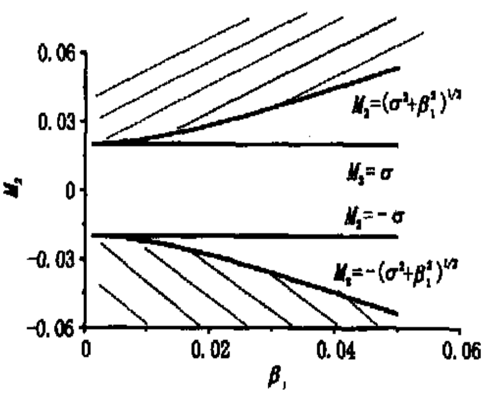
Similarly, when the parameters satisfy the second condition of Eq. (16), the stable surface wave still can not be formed. However, when the parameters yield instability condition (15), the instability will happen and the surface wave will appear.

Figure 4 illustrates the evolution of the ampli-



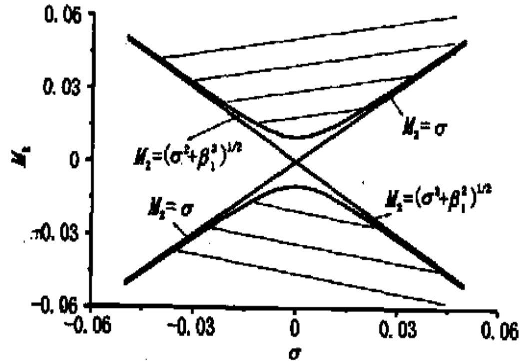
1(c) The forced frequency $f = 17.85$ Hz, the mode of surface wave is (5, 4)

Fig. 1 Contours (up) and three-dimensional modes (down) of the excited surface wave at different forced frequencies (depth of fluid $h = 1.0$ cm, radius of circular cylinder $R = 7.5$ cm, forcing amplitude $A = 11.4\mu$m, viscosity coefficient $\nu = 10^{-6}$ m^2/s, surface tension coefficient $\Gamma =$ 0.072716N/m, density of fluid $\rho = 10^3$Kg/m^3)



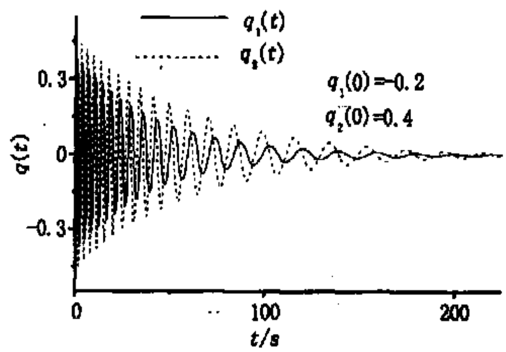
2(a) Unstable region determined by damping coefficient β_1 and excited coefficient M_2($\sigma = 0.02$)

tude with time and corresponding phase-plane trajectory. It can be seen from Fig.4 that the amplitude tends to a constant and a fixed point with the development of time from Figs. 4(a) and 4(b) respectively. In this condition, stable surface wave is formed.

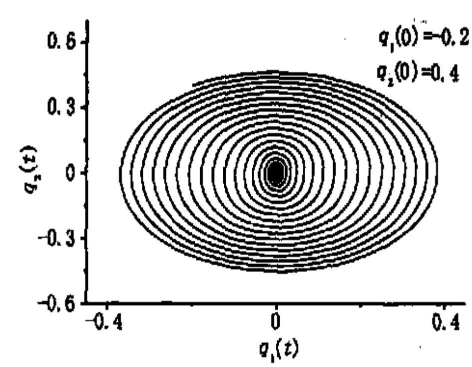


2(b) Unstable region determined by frequency difference coefficient σ and excited coefficient M_2($\beta_1 = 0.01$)

Fig. 2 Unstable region determined by damping coefficient β_1, frequency difference coefficient σ and excited coefficient M_2



3(a) Evolutions of the amplitude with time



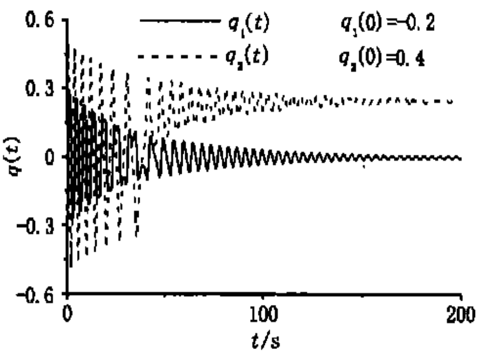
3(b) Phase-plane trajectories

Fig. 3 The evolution of the amplitude with time and phase-plane trajectories
($\sigma = 0.5$, $M_1 = 10$, $M_2 = 0.4$, $\beta_1 = 0.02$)

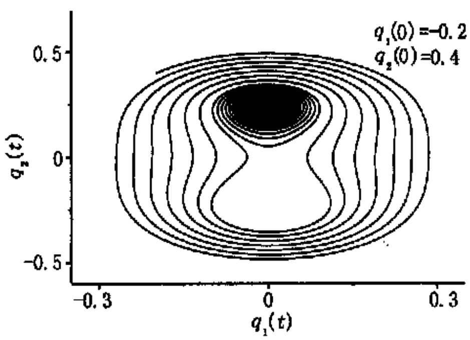
5. CONCLUSIONS

From above analysis, the following results can be drawn:

(1) The method of two-time scale expansion is effective to resolve the problem of considering surface tension and weak viscosity in vertically oscil-



4(a) Evolutions of the amplitude with time



4(b) Phase-plane trajectories

Fig. 4 Evolution of amplitude with time and phase-plane trajectories
($\sigma = 0.4$, $M_1 = 10$, $M_2 = 1$, $\beta_1 = 0.02$)

lating circular cylindrical container.

(2) The analytical expression of the damping coefficient is obtained, and the nonlinear amplitude equation is modified by it.

(3) An unstable condition of appearing surface wave is determined and the critical curve is obtained analytically. Moreover, the analytical results are testified by numerical computation.

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Biography: JIAN Yong-jun (1974-), Male, Doctor