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Unified complex variable solution for the effective transport properties of composites with a doubly-periodic array of fibers

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We present a complex variable method to evaluate the transverse effective transport properties of composites with a doubly-periodic array of fibers. The obtained complex variable solution is derived in a unified form for arbitrary doubly-periodic fiber arrays, and different fiber-matrix interfaces, i.e., perfect interface, contact resistance interface and coating. The present method can be seen as an extension of the method originated by Rayleigh only for symmetric fiber arrays. The limitation of Rayleigh's method is overcome by introducing a supplementary equation. Explicit formulae of the effective transport properties approximated to finite orders are obtained, which are written in a regular form for different fiber arrays, and reveal the reciprocal relations for symmetric fiber arrays. The validity and accuracy of the solution are verified by numerical examples and comparisons with existing methods.

1 Introduction

Effective transport properties such as thermal conductivity and dielectric constant, as well as their mathematically analogous properties such as elastic modulus, are key parameters characterizing the macroscopic characteristics of composites. Knowledge of such effective properties is of importance in designing and evaluation of composites [27].

For the fiber-reinforced composites, or more generally, an array of cylinders in matrix, the fiber distribution and the fiber-matrix interface are two important factors influencing the effective transport properties. Resistance interface and coating interface are two common models for investigating the influence of the real fiber-matrix interface. For investigating the influence of fiber distributions, the model of a composite with a periodic array of fibers is usually used.

The research into the effective properties of a medium with a periodic array of cylinders was initiated by Rayleigh [24], who presented the classic multipole method. Perrins et al. [23] extended Rayleigh's method to enable the calculation of the transport properties of circular cylinders in square arrays and in hexagonal arrays. Subsequently, Nicorovici [22] extended the method to the case of coated cylinders. Furthermore, Moosavi and Sarkomaa [20, 21] extended the method to the case of three-phase composite materials with interfacial resistance. However, due to the problems caused by a conditionally convergent sum and the difficulty in solving by real variable function, the extension of Rayleigh's method to consider general doubly-periodic fiber arrays has not been reported.

Besides above researches based on Rayleigh's method, there are also several other researches in different ways. Balagurov and Kashin [1], and Godin [5–7] developed a method based on the use of elliptic function, where general lattices are considered. Jiang et al. [11] developed a method by using Eshelby's equivalent inclusion concept integrated with the results from the doubly quasi-periodic Riemann boundary value problems. Rylko [26], Mityushev [17, 19] and Castro et al. [4] applied the method of functional equations to the cases of rectangular array of cylinders and the cases of imperfect interface between matrix and inclusions. Rodriguez-Ramos et al. [8, 14, 25] applied an asymptotic homogenization method. Lu and Lin [15] presented a boundary collocation scheme. Würkner et al. [28] developed a special procedure to handle the primary non-rectangular periodicity and imperfect interface, with common numerical homogenization techniques based on FE-models. The authors [30] also developed an eigenfunction expansion-variational method considering interfacial thermal contact resistance and coating.

These methods were verified by comparison with each other in numerical examples and with experimental results. However, as for giving explicit formulae of effective properties, there is a great deal of room for improvement. Explicit

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formulae with high accuracy are still highly desirable for designing of composites. This is the main motivation for this work.

This work aims to propose a unified complex variable solution for the effective transport properties of fiber composites, considering arbitrary doubly-periodic fiber distributions, and different fiber-matrix interfaces. The present method can be seen as an extension of the methods originated by Rayleigh only for symmetric fiber arrays. It inherits the advantages of Rayleigh's method in extracting explicit formulae. For the considered two-dimensional problems, complex variable method is powerful tool, by which the solution can be formulated in a concise form.

The problem of calculating the effective transport properties will be discussed here in the context of thermal conductivity. This work is organized as follows: First, basic equations of the problem are outlined and formulated in complex variables in Sect. 2. Then, a quasi-periodic complex potential in matrix is constructed and expanded into Laurent series in Sect. 3. In Sect. 4, linear equations about the unknown expansion coefficients are derived from the fiber-matrix interface conditions, and a supplementary equation making the solution complete is introduced. In Sect. 5, average fields are evaluated in complex variables, from which the coefficient equations are modified and solved in Sect. 6. In Sect. 7, effective conductivities are calculated by the average field method, also a series of explicit formulae of effective conductivities are given for different fiber arrays. Finally, the physics of the present modification is further discussed, and the validity of the present solution is verified by numerical examples in Sect. 8.

2 Basic equations and formulation of the problem

Consider a composite with a doubly-periodic array of fibers, whose transverse section is shown in Fig. 1a. For the problem of steady-state heat conduction with no internal generation, the heat flux q_j, temperature field T and temperature gradient H_k satisfy the following three equations [10]:

$$\text{Fourier's law}: q_j = -k_{jk} H_k, \tag{1a}$$

$$\text{Temperature gradient}: \ H_k = \nabla_k T, \tag{1b}$$

$$\text{Equilibrium equation}: \ \nabla_j q_j = 0, \tag{1c}$$

where k_{jk} is the thermal conductivity tensor. For isotropic materials, such as the matrix and fiber considered here, the thermal conductivity can be denoted by a scalar k , then the in-plane temperature field satisfies Laplace's equation [10]:

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} = 0. \tag{1d}$$

From (1a–d), in matrix and fibers, the temperature T, heat flux components $\{q_1, q_2\}$ and heat transfer rate Φ can be formulated by a complex potential $f(z)$:

$$q_1 - \mathrm{i}q_2 = -kf'(z), \tag{2a}$$

$$T = \frac{1}{2}[f(z) + \overline{f(z)}], \tag{2b}$$

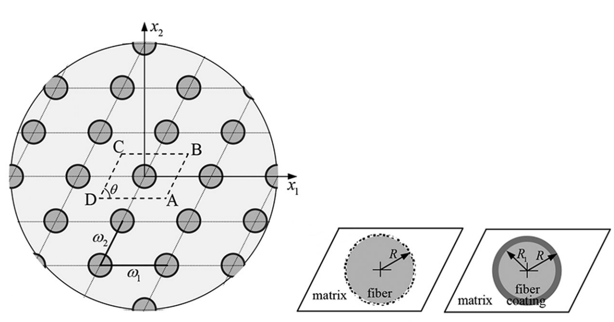

Fig. 1 Composite with a doubly-periodic array of fibers: (a) Transverse section and a unit cell ABCD; (b) Unit cell containing a fiber with resistance interface; (c) Unit cell containing a fiber with coating interface. The angle between the two fundamental complex periods ω_1 and ω_2 is θ.

$$\Phi = \frac{-k}{2\mathrm{i}}[f(z) - \overline{f(z)}]_{\mathrm{A}}^{\mathrm{B}}, \tag{2c}$$

where $\mathrm{i}^2 = -1$, $z = x_1 + \mathrm{i}x_2$ is a complex variable, the over bar denotes the complex conjugate, the prime denotes the derivative with respect to z, $[\cdot]_{\mathrm{A}}^{\mathrm{B}}$ denotes the difference of the values of the bracketed function from point A to point B.

In Sect. 3 and Sect. 4, we construct a complex potential in matrix. The temperature, gradient and flux expressed by it and its derivative satisfy the quasi-periodicity/periodicity condition, fiber-matrix interface condition and boundary condition of prescribed external gradient.

3 Construction and expansion of the complex potential in matrix

As shown in Fig. 1a, the space is occupied by matrix and a doubly-periodic array of fibers of radius R. Without loss of generality, unit area is assumed for the fundamental unit cell. The subscripts "m", "f", and "c" denote matrix, fiber, and coating, separately, in the following. An external gradient H_j^0 is applied to the matrix containing fibers. The internal gradient in the matrix subjected to the external gradient should be uniform if there is no fiber in matrix, so the complex potential in the matrix $f_{\mathrm{m}}(z)$ should include two parts:

$$f_{\mathrm{m}}(z) = A_0 z + f_{\mathrm{m}0}(z), \tag{3}$$

where the first part is corresponding to the external gradient, that is

$$A_0 = H_1^0 - \mathrm{i}H_2^0. \tag{4}$$

The second part in (3) is induced by the appearance of the fibers, which can be expressed by a summation:

$$f_{\mathrm{m}0}(z) = f_0(z) + \sum_{r,s} f_0(z - \omega_{rs}),\ r^2 + s^2 \neq 0, \tag{5}$$

where ω_{rs} denote the center locations of fibers in the doubly-periodic array. Introduce two fundamental complex periods of the array denoted by ω_1 and ω_2 as shown in Fig. 1a, then ω_{rs} are arbitrary complex periods of the doubly-periodic array, which can be expressed by

$$\omega_{rs} = r\omega_1 + s\omega_2,\ r^2 + s^2 \neq 0, \tag{6}$$

where r and s are integers.

The complex potential $f_0(z - \omega_{rs})$ is induced by the appearance of the fiber located at ω_{rs}, which approaches 0 when $z - \omega_{rs} \to \infty$. Thus $f_0(z)$ can be expanded as follows:

$$f_0(z) = \sum_{n=1}^{\infty} A_n \left(\frac{1}{z}\right)^n, \tag{7}$$

And then $f_{\mathrm{m}}(z)$ in (3) can be expressed as

$$f_{\mathrm{m}}(z) = A_0 z + \sum_{n=1}^{\infty} A_n \left(\frac{1}{z}\right)^n + \sum_{r,s}\sum_{n=1}^{\infty} A_n \left(\frac{1}{z - \omega_{rs}}\right)^n,\ r^2 + s^2 \neq 0, \tag{8}$$

where A_n are the unknown expansion coefficients.

Now we expand the complex potential into Laurent series. Note that

$$\left(\frac{1}{z - \omega_{rs}}\right)^n = \left(\frac{1}{-\omega_{rs}}\right)^n + \sum_{j=1}^{\infty} (-1)^j \mathrm{C}_{n+j-1}^j \left(\frac{1}{-\omega_{rs}}\right)^{n+j} z^j, \tag{9}$$

with

$$\mathrm{C}_{n+j-1}^j = \frac{(n+j-1)!}{j!(n-1)!}. \tag{10}$$

Then (5) can be rewritten as

$$f_{\mathrm{m}0}(z) = \sum_{n=1}^{\infty} A_n \left(\frac{1}{z}\right)^n + \sum_{j=1}^{\infty} (-1)^j \sum_{n=1}^{\infty} A_n \mathrm{C}_{n+j-1}^j \sum_{r,s} \left(\frac{1}{-\omega_{rs}}\right)^{n+j} z^j + \sum_{n=1}^{\infty} A_n \sum_{r,s} \left(\frac{1}{-\omega_{rs}}\right)^n. \tag{11}$$

After neglecting the constant, and due to the centrosymmetry of the fiber arrays, the complex potential in matrix is centrosymmetric ($f_{\mathrm{m}}(-z) = -f_{\mathrm{m}}(z)$), which can be written as

$$f_{\mathrm{m}}(z) = \sum_{n=1}^{\infty} A_{2n-1}\left(\frac{1}{z}\right)^{2n-1} + \sum_{n=1}^{\infty}\left(A_0\delta_{1,n} - \sum_{j=1}^{\infty} A_{2j-1}\mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2}\right)\cdot z^{2n-1}, \tag{12}$$

where $\delta_{1,n}$ is the Kronecker delta symbol, being unity for $n = 1$ and zero otherwise. The quantities $S_{2j+2n-2}$ in (12) are sums of the form

$$S_{2j+2n-2} = \sum_{r,s}\left(\frac{1}{\omega_{rs}}\right)^{2j+2n-2}, \ r^2 + s^2 \neq 0. \tag{13}$$

The constructed complex potential $f_{\mathrm{m}}(z)$ in (8) is quasi-periodic, which will be proved in Sect. 5.1. Therefore, the temperature gradient and heat flux fields expressed by its derivative satisfy the doubly-periodicity conditions. On the other hand, the heat flux and temperature in the matrix have to satisfy the interface condition between fiber and matrix. This interface condition is used to determine the unknown coefficients in (12).

4 Interface conditions and coefficient equations

Now three types of fiber-matrix interfaces, i.e., perfect interface, contact resistance interface and coating, are considered separately. Because of the doubly-periodicity, a unit cell with one fiber, i.e., the fiber located at the origin is considered without loss of generality, as shown in Fig. 1. The complex potential in the fiber region can be expanded into a Taylor series, while in the coating region as shown in Fig. 1c, it can be expanded into Laurent series. The conductivities of fiber, matrix and coating are denoted by k_{f}, k_{m}, and k_{c}, respectively, in the following.

4.1 Perfect interface

For the case of perfect interface, the heat transfer rate Φ and the temperature T across the fiber-matrix interfaces are continuous:

$$\Phi_{\mathrm{f}} = \Phi_{\mathrm{m}},\ T_{\mathrm{f}} = T_{\mathrm{m}},\ \text{at } |z| = R. \tag{14}$$

By substituting the complex potentials into (2a, b, c), and then into (14), the unknown coefficients in (12) satisfy the equations [30]:

$$\bar{A}_{2n-1} = \eta_{\mathrm{fm}} R^{2(2n-1)}\left[A_0\delta_{1,n} - \sum_{j=1}^{\infty} A_{2j-1}\mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2}\right],\ n = 1, 2, 3..., \tag{15}$$

with $\eta_{\mathrm{fm}} = (k_{\mathrm{m}} - k_{\mathrm{f}})/(k_{\mathrm{m}} + k_{\mathrm{f}})$.

4.2 Contact resistance interface

For the case of contact resistance interface as shown in Fig. 1b, the heat transfer rate Φ is continuous, while the temperature is discontinuous across the interface:

$$\Phi_{\mathrm{f}} = \Phi_{\mathrm{m}},\ q_{\mathrm{f}} = q_{\mathrm{m}} = -h(T_{\mathrm{f}} - T_{\mathrm{m}}),\ \text{at } |z| = R, \tag{16}$$

where h is the thermal contact conductance on the interface. By similar substitution, the unknown coefficients in (12) satisfy the equations [30]:

$$\bar{A}_{2n-1} = \frac{\eta_{\mathrm{fm}} + (2n-1)\beta(1-\eta_{\mathrm{fm}})}{1 + (2n-1)\beta(1-\eta_{\mathrm{fm}})} R^{2(2n-1)}\left[A_0\delta_{1,n} - \sum_{j=1}^{\infty} A_{2j-1}\mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2}\right],\ n = 1, 2, 3..., \tag{17}$$

with $\beta = k_{\mathrm{m}}/(2h\,R)$.

4.3 Coating interface

For the case of coating interface as shown in Fig. 1c, the heat transfer rate Φ and temperature T are continuous across the fiber-coating interfaces and coating-matrix interfaces:

$$\Phi_{\mathrm{f}} = \Phi_{\mathrm{c}},\ T_{\mathrm{f}} = T_{\mathrm{c}},\quad \text{at } |z| = R_1, \tag{18}$$

$$\Phi_{\mathrm{c}} = \Phi_{\mathrm{m}},\ T_{\mathrm{c}} = T_{\mathrm{m}},\quad \text{at } |z| = R. \tag{19}$$

Similarly, a series of substitutions yield the following coefficient equations [30]:

$$\bar{A}_{2n-1} = \frac{\eta_{\mathrm{cm}} + \eta_{\mathrm{fc}}(1+\xi)^{2-4n}}{1+\eta_{\mathrm{cm}}\eta_{\mathrm{fc}}(1+\xi)^{2-4n}} R^{2(2n-1)} \left[A_0 \delta_{1,n} - \sum_{j=1}^{\infty} A_{2j-1} \mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2} \right],\ n = 1, 2, 3..., \tag{20}$$

with

$$\eta_{\mathrm{fc}} = (k_{\mathrm{c}} - k_{\mathrm{f}})/(k_{\mathrm{c}} + k_{\mathrm{f}}),\ \eta_{\mathrm{cm}} = (k_{\mathrm{m}} - k_{\mathrm{c}})/(k_{\mathrm{m}} + k_{\mathrm{c}}),\ \xi = (R_1 - R)/R. \tag{21}$$

4.4 Unified coefficient equations

The coefficient equations (15), (17), (20) obtained from the three different interface conditions can be rewritten in a unified linear equation:

$$\bar{A}_{2n-1} = \eta_{2n-1} R^{2(2n-1)} \left[A_0 \delta_{1,n} - \sum_{j=1}^{\infty} A_{2j-1} \mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2} \right],\ n = 1, 2, 3..., \tag{22}$$

where

$$\eta_{2n-1} = \begin{cases} \eta_{\mathrm{fm}} & \text{perfect interface,} \\ \dfrac{\eta_{\mathrm{fm}} + (2n-1)\beta(1-\eta_{\mathrm{fm}})}{1+(2n-1)\beta(1-\eta_{\mathrm{fm}})} & \text{contact resistance,} \\ \dfrac{\eta_{\mathrm{cm}} + \eta_{\mathrm{fc}}(1+\xi)^{2-4n}}{1+\eta_{\mathrm{cm}}\eta_{\mathrm{fc}}(1+\xi)^{2-4n}} & \text{coating.} \end{cases} \tag{23}$$

By an appropriate truncation and solution of (22), the unknown coefficients are obtained. However, the sum S_2 is conditionally convergent, which is dependent on the summation sequence. To avoid such a problem, a supplementary equation is needed. Due to the periodicity, average gradient $\langle H_j \rangle$ over any unit cell should be equal to the average gradient over the whole transverse section of the composite, and that should be equal to the prescribed external gradient H_j^0 in order to satisfy the boundary condition. Therefore,

$$\langle H_1 \rangle + \mathrm{i}\,\langle H_2 \rangle = H_1^0 + \mathrm{i} H_2^0 = \bar{A}_0. \tag{24}$$

Before applying the supplementary equation (24), the average gradient $\langle H_j \rangle$ should be expressed by the unknown coefficients in (22).

5 Evaluation of the average fields in complex variables

Now we calculate the average gradient $\langle H_j \rangle$ and average flux $\langle q_j \rangle$ over a unit cell, in order to express the supplementary equation (24) and to calculate the effective conductivities.

5.1 Average flux over a unit cell

Consider an arbitrary unit cell ABCD of unit area in Fig. 1a. Complex coordinates at the three corners A, B, and C satisfy

$$z_{\mathrm{B}} = z_{\mathrm{A}} + \omega_2,\ z_{\mathrm{C}} = z_{\mathrm{B}} - \omega_1. \tag{25}$$

According to (2c), the heat transfer rate Φ across boundaries AB and BC can be written as

$$\Phi^{\mathrm{AB}} = k_{\mathrm{m}}\ \mathrm{Im}[f_{\mathrm{m}}(z_{\mathrm{B}}) - f_{\mathrm{m}}(z_{\mathrm{A}})],\ \Phi^{\mathrm{BC}} = k_{\mathrm{m}}\ \mathrm{Im}[f_{\mathrm{m}}(z_{\mathrm{C}}) - f_{\mathrm{m}}(z_{\mathrm{B}})]. \tag{26}$$

By application of the Green theorem and the periodicity of the flux field, the average flux over the unit cell in complex form can be expressed by

$$\langle q_1\rangle + i\,\langle q_2\rangle = \Phi^{AB}\omega_1 + \Phi^{BC}\omega_2. \tag{27}$$

From the expression of the complex potential $f_{\mathrm{m}}(z)$ in (8), and considering

$$\sum_{r,s}\left(\frac{1}{z_{\mathrm{A}}-\omega_{rs}}\right)^n = \sum_{r,s}\left(\frac{1}{z_{\mathrm{B}}-\omega_{rs}}\right)^n = \sum_{r,s}\left(\frac{1}{z_{\mathrm{C}}-\omega_{rs}}\right)^n,\ n \geq 3. \tag{28}$$

One obtains that

$$\begin{aligned} f_{\mathrm{m}}(z_{\mathrm{B}}) - f_{\mathrm{m}}(z_{\mathrm{A}}) &= A_0\omega_2 + A_1\sum_{r,s}\left(\tfrac{1}{z_{\mathrm{B}}-\omega_{rs}} - \tfrac{1}{z_{\mathrm{A}}-\omega_{rs}}\right),\\ f_{\mathrm{m}}(z_{\mathrm{C}}) - f_{\mathrm{m}}(z_{\mathrm{B}}) &= A_0(-\omega_1) + A_1\sum_{r,s}\left(\tfrac{1}{z_{\mathrm{C}}-\omega_{rs}} - \tfrac{1}{z_{\mathrm{B}}-\omega_{rs}}\right). \end{aligned} \tag{29}$$

According to the Weierstrass ς function

$$\zeta(z) = \frac{1}{z} + \sum_{r,s}\left(\frac{1}{z-\omega_{rs}} + \frac{1}{\omega_{rs}} + \frac{z}{\omega_{rs}^2}\right),\ r^2+s^2 \neq 0, \tag{30}$$

and the sum S_2 defined in (13), one obtains that

$$\begin{aligned} f_{\mathrm{m}}(z_{\mathrm{B}}) - f_{\mathrm{m}}(z_{\mathrm{A}}) &= A_0\omega_2 + A_1[2\zeta(\tfrac{\omega_2}{2}) - S_2\omega_2],\\ f_{\mathrm{m}}(z_{\mathrm{C}}) - f_{\mathrm{m}}(z_{\mathrm{B}}) &= A_0(-\omega_1) + A_1[-2\zeta(\tfrac{\omega_1}{2}) + S_2\omega_1]. \end{aligned} \tag{31}$$

Note that the unit cell ABCD is arbitrary, thus above derivation has, in fact, proved the quasi-periodicity of the constructed potential $f_{\mathrm{m}}(z)$ in (8).

By substituting (31) into (26) and then into (27), one obtains that

$$\langle q_1\rangle + i\,\langle q_2\rangle = k_{\mathrm{m}}\bar{A}_0 - k_{\mathrm{m}}(\pi A_1 + \bar{S}_2\bar{A}_1 - \bar{\varepsilon}\bar{A}_1), \tag{32}$$

where

$$\varepsilon = \frac{1}{2\mathrm{i}}\left[2\varsigma\left(\frac{\omega_2}{2}\right)\bar{\omega}_1 - 2\varsigma\left(\frac{\omega_1}{2}\right)\bar{\omega}_2\right]. \tag{33}$$

It can be seen that the average flux is related to the conditionally convergent sum S_2. The values of ε for some typical arrays are listed in the Table A1 in Appendix.

5.2 Relation of average gradient to average flux

Consider again the unit cell containing a fiber as shown in Fig. 1a. The area and the boundary of the unit cell are denoted by V and ∂V, respectively. The boundaries of the unit cell are always assumed to be located at the matrix. From (1b, c) the average fields can be calculated by

$$\langle q_i\rangle - k_{\mathrm{m}}\langle H_i\rangle = \frac{1}{V}\left(\int_V q_i\mathrm{d}V - k_{\mathrm{m}}\int_V H_i\mathrm{d}V\right) = \frac{1}{V}\int_V\left[\nabla_j(x_iq_j) - k_{\mathrm{m}}\nabla_iT\right]\mathrm{d}V. \tag{34}$$

By application of the Green theorem,

$$\langle q_i\rangle - k_{\mathrm{m}}\langle H_i\rangle = \frac{k_{\mathrm{m}}}{V}\oint_{\partial V}(x_iH_j - \delta_{ij}T)n_j\mathrm{d}S. \tag{35}$$

By substituting (2a) and (2b) into (35) and noting that

$$z = x_1 + \mathrm{i}x_2,\ \mathrm{d}z = \mathrm{d}x_1 + \mathrm{i}\mathrm{d}x_2,\ \mathrm{d}\bar{z} = \mathrm{d}x_1 - \mathrm{i}\mathrm{d}x_2, \tag{36}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}S} = n_1,\ \frac{\mathrm{d}x_1}{\mathrm{d}S} = -n_2, \tag{37}$$

one obtains the average fields expressed by the potential in matrix:

$$\langle q_1\rangle + \mathrm{i}\,\langle q_2\rangle - k_{\mathrm{m}}(\langle H_1\rangle + \mathrm{i}\,\langle H_2\rangle) = \frac{k_{\mathrm{m}}}{V}\frac{\mathrm{i}}{2}\left\{\oint_{\partial V}\left[\overline{f'_{\mathrm{m}}(z)}z\mathrm{d}\bar{z} + \overline{f_{\mathrm{m}}(z)}\mathrm{d}z\right] + \oint_{\partial V}\left[f_{\mathrm{m}}(z) - f'_{\mathrm{m}}(z)z\right]\mathrm{d}z\right\}. \tag{38}$$

For the first loop integration, we find

$$\mathrm{d}[\overline{f_{\mathrm{m}}(z)}z] = \overline{f'_{\mathrm{m}}(z)}z\mathrm{d}\bar{z} + \overline{f_{\mathrm{m}}(z)}\mathrm{d}z. \tag{39}$$

Therefore,

$$\oint_{\partial V} [\overline{f'_{\mathrm{m}}(z)}z\mathrm{d}\bar{z} + \overline{f_{\mathrm{m}}(z)}\mathrm{d}z] = 0. \tag{40}$$

For the second loop integration, according to the Residue theorem,

$$\oint_{\partial V} [f_{\mathrm{m}}(z) - f'_{\mathrm{m}}(z)z]\mathrm{d}z = 2\pi \mathrm{i} \cdot 2A_1. \tag{41}$$

If $V = 1$, (38) can be rewritten as

$$\langle q_1 \rangle + \mathrm{i}\,\langle q_2 \rangle - k_{\mathrm{m}}(\langle H_1 \rangle + \mathrm{i}\,\langle H_2 \rangle) = -k_{\mathrm{m}} \cdot 2\pi A_1. \tag{42}$$

It is seen from (32) and (42) that the average fields are only related to the first coefficient A_1 in the coefficient equation (22).

6 Modification and solution of the coefficient equations

From (32) and (42), the supplementary equation (24) can be expressed by the unknown coefficient:

$$S_2 A_1 = \pi \bar{A}_1 + \varepsilon A_1. \tag{43}$$

By substituting (43) into the coefficient equations (22), we obtain the modified coefficient equations without S_2:

$$\begin{cases} \bar{A}_1 = \frac{\eta_1 R^2}{1+\lambda\eta_1}\left[A_0 - A_1\varepsilon - \sum\limits_{j=2}^{\infty} A_{2j-1}\mathrm{C}_{2j-1}^{1} S_{2j}\right], & n = 1, \\ \bar{A}_{2n-1} = -\eta_{2n-1} R^{2(2n-1)} \sum\limits_{j=1}^{\infty} A_{2j-1}\mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2}, & n \geq 2. \end{cases} \tag{44}$$

Note that the fiber volume fraction (total volume fraction of fiber and coating for the case of coating interface) $\lambda = \pi R^2$, if $V = 1$. Introduce vector and matrix notations, and apply a truncation to order N,

$$\mathbf{A} = \{A_{2n-1}\}, \quad n = 1, 2, 3, \cdots N, \tag{45a}$$

$$\mathbf{A}_0 = \frac{\lambda\eta_1}{\pi(1+\lambda\eta_1)}\{A_0 \delta_{1,n}\}, \quad n = 1, 2, 3, \cdots N, \tag{45b}$$

$$\mathbf{M} = [M_{nj}], \quad n, j = 1, 2, 3, \cdots N, \tag{45c}$$

$$M_{nj} = \begin{cases} \dfrac{\lambda\eta_1}{\pi(1+\lambda\eta_1)}\varepsilon, & n = 1, j = 1, \\ \dfrac{\lambda\eta_1}{\pi(1+\lambda\eta_1)}\mathrm{C}_{2j-1}^{1} S_{2j}, & n = 1, j \geq 2, \\ \eta_{2n-1}\left(\dfrac{\lambda}{\pi}\right)^{2n-1} \mathrm{C}_{2j+2n-3}^{2n-1} S_{2j+2n-2}, & n \geq 2. \end{cases} \tag{45d}$$

Then the coefficient equations can be rewritten as

$$\bar{\mathbf{A}} = \mathbf{A}_0 - \mathbf{M}\mathbf{A}. \tag{46}$$

The coefficient vector can be solved as

$$\mathbf{A} = (\mathbf{I} - \bar{\mathbf{M}}\mathbf{M})^{-1}(\bar{\mathbf{A}}_0 - \bar{\mathbf{M}}\mathbf{A}_0) = (\mathbf{I} - \bar{\mathbf{M}}\mathbf{M})^{-1}\bar{\mathbf{A}}_0 - (\mathbf{I} - \bar{\mathbf{M}}\mathbf{M})^{-1}\bar{\mathbf{M}}\mathbf{A}_0. \tag{47}$$

Let us define

$$\mathbf{P} = (\mathbf{I} - \bar{\mathbf{M}}\mathbf{M})^{-1}, \ \mathbf{Q} = (\mathbf{I} - \bar{\mathbf{M}}\mathbf{M})^{-1}\bar{\mathbf{M}}. \tag{48}$$

Then

$$\begin{cases} A_1 = \dfrac{\lambda\eta_1}{\pi(1+\lambda\eta_1)}(P_{11}\bar{A}_0 - Q_{11}A_0), \\ A_2 = \dfrac{\lambda\eta_1}{\pi(1+\lambda\eta_1)}(P_{21}\bar{A}_0 - Q_{21}A_0), \\ \qquad\vdots \\ A_N = \dfrac{\lambda\eta_1}{\pi(1+\lambda\eta_1)}(P_{N1}\bar{A}_0 - Q_{N1}A_0). \end{cases} \tag{49}$$

For orthotropic fiber arrays, including rectangular arrays and rhombic arrays, S_{2m} $(m \geq 2)$ and ε are real, thus $\bar{\mathbf{M}} = \mathbf{M}$ in (48), and

$$\mathbf{P} + \mathbf{Q} = (\mathbf{I} - \mathbf{M})^{-1},\ \mathbf{P} - \mathbf{Q} = (\mathbf{I} + \mathbf{M})^{-1}. \tag{50}$$

After the expansion coefficients of the potential are determined by (49), the temperature, gradient and flux fields in matrix are finally solved. It can be seen that this solution is unified for arbitrary doubly-periodic fiber arrays, and can be reduced into the solution for the case of orthotropic fiber arrays.

7 Effective conductivities

The effective conductivities $\langle k_{jk}\rangle$ are calculated by using the average field method:

$$\langle q_j\rangle = -\langle k_{jk}\rangle \langle H_k\rangle\,,\ j, k = 1, 2. \tag{51}$$

By applying (24), Eq. (51) can be rewritten in the following complex variable form:

$$\langle q_1\rangle + \mathrm{i}\,\langle q_2\rangle = \frac{1}{2}\left(\langle k_{11}\rangle + \langle k_{22}\rangle\right)\bar{A}_0 + \frac{1}{2}\left(\langle k_{11}\rangle - \langle k_{22}\rangle + 2\mathrm{i}\,\langle k_{12}\rangle\right)A_0. \tag{52}$$

From (24), (42) and (49),

$$\langle q_1\rangle + \mathrm{i}\,\langle q_2\rangle = k_{\mathrm{m}}\left(1 - 2\frac{\lambda\eta_1}{1+\lambda\eta_1}P_{11}\right)\bar{A}_0 + 2k_{\mathrm{m}}\frac{\lambda\eta_1}{1+\lambda\eta_1}Q_{11}A_0. \tag{53}$$

Comparing (52) with (53), it can be obtained that

$$\begin{aligned} &\frac{1}{2}(\langle k_{11}\rangle + \langle k_{22}\rangle) = k_{\mathrm{m}}(1 - 2\frac{\lambda\eta_1}{1+\lambda\eta_1}P_{11}), \\ &\frac{1}{2}(\langle k_{11}\rangle - \langle k_{22}\rangle + 2\mathrm{i}\,\langle k_{12}\rangle) = 2k_{\mathrm{m}}\frac{\lambda\eta_1}{1+\lambda\eta_1}Q_{11}, \end{aligned} \tag{54a}$$

or

$$\begin{aligned} &\langle k_{11}\rangle + \mathrm{i}\,\langle k_{12}\rangle = k_{\mathrm{m}}\left[1 - 2\frac{\lambda\eta_1}{1+\lambda\eta_1}(P_{11} - Q_{11})\right], \\ &\langle k_{22}\rangle - \mathrm{i}\,\langle k_{12}\rangle = k_{\mathrm{m}}\left[1 - 2\frac{\lambda\eta_1}{1+\lambda\eta_1}(P_{11} + Q_{11})\right], \end{aligned} \tag{54b}$$

where P_{11} and Q_{11} are given by (48). When an appropriate truncation to an enough high order for (45) is applied, (54) gives numerical results of desired accuracy. When a truncation to finite order N is applied, explicit formulae of the effective conductivities of order N are obtained, which are given for different fiber arrays in the following. The geometric parameters S_4, S_6, and ε in the explicit formulae can be calculated by (A2), (A3), and (A9), which are also listed in Table A1 for some typical arrays. The material parameters η_{2n-1} $(n = 1, 2, 3 \cdots N)$ in the explicit formulae are given by (23).

7.1 General doubly-periodic array

For general doubly-periodic fiber arrays, the effective conductivities are anisotropic in general. By a square truncation to order N, the formula for the effective conductivities in complex form is obtained as:

$$\begin{cases} \langle k_{11}\rangle + \mathrm{i}\,\langle k_{12}\rangle = k_{\mathrm{m}}\dfrac{\pi^2 c_N^2 + \pi(2\pi b_N - \varepsilon b_N - \bar{\varepsilon}\bar{b}_N)c_N\eta_1\lambda - (a_N^2 - |b_N|^2)(\pi^2 - 2\pi\bar{\varepsilon} + |\varepsilon|^2)\eta_1^2\lambda^2}{\pi^2 c_N^2 + \pi(2\pi a_N - \varepsilon b_N - \bar{\varepsilon}\bar{b}_N)c_N\eta_1\lambda + (a_N^2 - |b_N|^2)(\pi^2 - |\varepsilon|^2)\eta_1^2\lambda^2}, \\ \langle k_{22}\rangle - \mathrm{i}\,\langle k_{12}\rangle = k_{\mathrm{m}}\dfrac{\pi^2 c_N^2 - \pi(2\pi b_N + \varepsilon b_N + \bar{\varepsilon}\bar{b}_N)c_N\eta_1\lambda - (a_N^2 - |b_N|^2)(\pi^2 + 2\pi\bar{\varepsilon} + |\varepsilon|^2)\eta_1^2\lambda^2}{\pi^2 c_N^2 + \pi(2\pi a_N - \varepsilon b_N - \bar{\varepsilon}\bar{b}_N)c_N\eta_1\lambda + (a_N^2 - |b_N|^2)(\pi^2 - |\varepsilon|^2)\eta_1^2\lambda^2}, \end{cases} \tag{55}$$

where a_N, b_N, and c_N are from the coefficient equation (46) for order N. When $N = 1$, $a_1 = 1$, $b_1 = 0$, $c_1 = 1$, the first order formula is

$$\begin{cases} \langle k_{11}\rangle + \mathrm{i}\,\langle k_{12}\rangle = k_{\mathrm{m}}\dfrac{\pi^2 - (\pi^2 - 2\pi\bar{\varepsilon} + |\varepsilon|^2)\eta_1^2\lambda^2}{\pi^2(1+\eta_1\lambda)^2 - |\varepsilon|^2\eta_1^2\lambda^2}, \\ \langle k_{22}\rangle - \mathrm{i}\,\langle k_{12}\rangle = k_{\mathrm{m}}\dfrac{\pi^2 - (\pi^2 + 2\pi\bar{\varepsilon} + |\varepsilon|^2)\eta_1^2\lambda^2}{\pi^2(1+\eta_1\lambda)^2 - |\varepsilon|^2\eta_1^2\lambda^2}. \end{cases} \tag{56}$$

When $N = 3$,

$$\begin{aligned} a_3 = 1 &- \frac{3}{\pi^4}|S_4|^2\eta_1\eta_3\lambda^4 - \frac{5}{\pi^6}|S_6|^2(20\eta_3^2 + \eta_1\eta_5)\lambda^6 - \frac{270}{\pi^8}|S_4|^4\eta_3\eta_5\lambda^8 \\ &- \frac{396900}{121\pi^{10}}|S_4|^2|S_6|^2\eta_5^2\lambda^{10} + \frac{5}{\pi^{12}}|9S_4^3 - 10S_6^2|^2\eta_1\eta_3^2\eta_5\lambda^{12} \\ &+ \frac{648675}{121\pi^{14}}|S_4|^4|S_6|^2\eta_1\eta_3\eta_5^2\lambda^{14} + \frac{2025}{121\pi^{16}}|S_4(33S_4^3 - 140S_6^2)|^2\eta_3^2\eta_5^2\lambda^{16}, \end{aligned} \tag{57a}$$

$$\begin{aligned} b_3 = \frac{30}{\pi^7}S_6\bar{S}_4^2\eta_1\eta_3^2\lambda^7 &+ \frac{90}{\pi^9}|S_4|^2 S_4\bar{S}_6\eta_1\eta_3\eta_5\lambda^9 + \frac{3150}{11\pi^{11}}S_4|S_6|^2\bar{S}_6\eta_1\eta_5^2\lambda^{11} \\ &+ \frac{2250}{121\pi^{17}}S_4\bar{S}_6(33S_4^3 - 140S_6^2)(18\bar{S}_4^3 + 11\bar{S}_6^2)\eta_1\eta_3^2\eta_5^2\lambda^{17}, \end{aligned} \tag{57b}$$

$$\begin{aligned} c_3 = 1 &- \frac{6}{\pi^4}|S_4|^2\eta_1\eta_3\lambda^4 - \frac{10}{\pi^6}|S_6|^2(10\eta_3^2 + \eta_1\eta_5)\lambda^6 \\ &+ \frac{9}{\pi^8}|S_4|^4\eta_3(\eta_1^2\eta_3 - 30\eta_5)\lambda^8 + \frac{30}{121\pi^{10}}|S_4S_6|^2(121\eta_1^2\eta_3 - 13230\eta_5)\eta_5\lambda^{10} \\ &+ \frac{5}{\pi^{12}}(2|9S_4^3 - 10S_6^2|^2\eta_1\eta_3^2\eta_5 + 5|S_6|^4\eta_1^2\eta_5^2)\lambda^{12} + \frac{1297350}{121\pi^{14}}|S_4|^4|S_6|^2\eta_1\eta_3\eta_5^2\lambda^{14} \\ &+ \frac{2025}{121\pi^{16}}|S_4(33S_4^3 - 140S_6^2)|^2\eta_3^2\eta_5^2\lambda^{16} - \frac{2500}{121\pi^{18}}|S_6(18S_4^3 + 11S_6^2)|^2\eta_1^2\eta_3^2\eta_5^2\lambda^{18}. \end{aligned} \tag{57c}$$

7.2 Orthotropic array

For orthotropic fiber arrays, including rectangular arrays and rhombic arrays, the transverse effective conductivities are orthotropic in general. From (13) and (33), when the fiber array is rotated by $\pi/2$, the fiber array parameters (ε, S_4, S_6) should be replaced by ($-\varepsilon$, S_4, $-S_6$) in the calculation, correspondingly. Therefore, the effective conductivity in direction 2 can be obtained from the formula of the one in direction 1, just by replacing the fiber array parameters. In addition, the effective conductivities in two principle directions satisfy the reciprocal relations [12,23]. The explicit formula can be written in a regular form as follows:

$$\begin{cases} \langle k_{11}\rangle = \langle k_{11}(\varepsilon, S_4, S_6, \eta_{2n-1})\rangle = k_{\mathrm{m}}\dfrac{c_N(1 - \eta_1\lambda + \frac{\varepsilon}{\pi}\eta_1\lambda) + d_N\eta_1\lambda^4}{c_N(1 + \eta_1\lambda + \frac{\varepsilon}{\pi}\eta_1\lambda) + d_N\eta_1\lambda^4} \\ \langle k_{22}\rangle = \langle k_{11}(-\varepsilon, S_4, -S_6, \eta_{2n-1})\rangle = \dfrac{k_{\mathrm{m}}^2}{\langle k_{11}(\varepsilon, S_4, S_6, -\eta_{2n-1})\rangle} \\ \langle k_{12}\rangle = 0 \end{cases}, \quad n = 1, 2, 3, \cdots, N, \tag{58}$$

where c_N and d_N are from square truncation of the coefficient equation (46) for order N. When $N = 1$, $c_1 = 1$, $d_1 = 0$, the first order explicit formula is obtained as

$$\langle k_{11}\rangle = k_{\mathrm{m}}\frac{1-\eta_1\lambda+\frac{\varepsilon}{\pi}\eta_1\lambda}{1+\eta_1\lambda+\frac{\varepsilon}{\pi}\eta_1\lambda}. \tag{59}$$

It considers the orthotropy of the effective conductivities by introducing the parameter ε, which is a modification of the classical estimation [2, 3, 13] . When $N = 4$,

$$\begin{aligned} c_4 = 1 &+ \frac{10}{\pi^3}S_6\eta_3\lambda^3 + \frac{630}{11\pi^5}S_4S_6\eta_5\lambda^5 + \frac{360}{\pi^7}S_4^2S_6\eta_7\lambda^7 - \frac{45}{11\pi^8}S_4(33S_4^3 - 140S_6^2)\eta_3\eta_5\lambda^8 \\ &+ \frac{360000}{121\pi^{10}}S_4^2S_6^2\eta_3\eta_7\lambda^{10} - \frac{1260}{1859\pi^{12}}(3564S_4^6 - 20520S_4^3S_6^2 + 6875S_6^4)\eta_5\eta_7\lambda^{12} \\ &- \frac{1800}{224939\pi^{15}}S_6(5524497S_4^6 - 17863650S_4^3S_6^2 + 5823125S_6^4)\eta_3\eta_5\eta_7\lambda^{15}, \end{aligned} \tag{60a}$$

$$\begin{aligned} d_4 = &-\frac{3}{\pi^4}S_4^2\eta_3 - \frac{5}{\pi^6}S_6^2\eta_5\lambda^2 - \frac{9}{7\pi^8}S_4^4\eta_7\lambda^4 - \frac{50}{11\pi^9}S_6(18S_4^3 + 11S_6^2)\eta_3\eta_5\lambda^5 \\ &- \frac{76590}{77\pi^{11}}S_4^4S_6\eta_3\eta_7\lambda^7 + \frac{270}{143\pi^{13}}S_4^2S_6(93S_4^3 - 770S_6^2)\eta_5\eta_7\lambda^9 \\ &+ \frac{135}{143143\pi^{16}}S_4^2(5489649S_4^6 - 17017560S_4^3S_6^2 - 11294500S_6^4)\eta_3\eta_5\eta_7\lambda^{12}. \end{aligned} \tag{60b}$$

7.3 Square array

For a square fiber array and a hexagonal fiber array, the transverse effective conductivities are equal in two orthogonal directions. For square array,

$$\langle k_{12}\rangle = 0,\ \langle k_{11}\rangle = \langle k_{22}\rangle = k_{\mathrm{m}}\frac{c_N(1-\eta_1\lambda)+d_N\eta_1\lambda^4}{c_N(1+\eta_1\lambda)+d_N\eta_1\lambda^4}. \tag{61}$$

When the order $N = 1$, it can be obtained that $c_1 = 1$, $d_1 = 0$. The obtained first order explicit formula is the classical Maxwell-Garnett estimation [16]:

$$\langle k_{11}\rangle = \langle k_{22}\rangle = k_{\mathrm{m}}\frac{1-\eta_1\lambda}{1+\eta_1\lambda}, \tag{62}$$

which is also derived in [2, 3, 13] for different fiber-matrix interfaces. This first order formula is without consideration of the fiber distribution. For order $N = 6$,

$$\begin{aligned} c_6 = 1 &- \frac{135}{\pi^8}S_4^4\eta_3\eta_5\lambda^8 - \frac{4860}{169\pi^{12}}S_4^6(84\eta_5\eta_7 + 5\eta_3\eta_9)\lambda^{12} - \frac{280665}{289\pi^{16}}S_4^8(55\eta_7\eta_9 + 7\eta_5\eta_{11})\lambda^{16} \\ &- \frac{1154736}{\pi^{20}}S_4^{10}\eta_9\eta_{11}\lambda^{20} + \frac{36260531897625}{8254129\pi^{24}}S_4^{12}\eta_3\eta_5\eta_7\eta_9\lambda^{24} \\ &+ \frac{6455729148660}{48841\pi^{28}}S_4^{14}\eta_3\eta_5\eta_9\eta_{11}\lambda^{28} + \frac{16086782983751865}{14115049\pi^{32}}S_4^{16}\eta_5\eta_7\eta_9\eta_{11}\lambda^{32}, \end{aligned} \tag{63a}$$

$$\begin{aligned} d_6 = &-\frac{3}{\pi^4}S_4^2\eta_3 - \frac{9}{7\pi^8}S_4^4\eta_7\lambda^4 - \frac{324}{1859\pi^{12}}S_4^6\eta_{11}\lambda^8 \\ &+ \frac{6124815}{1183\pi^{16}}S_4^8\eta_3\eta_5\eta_7\lambda^{12} + \frac{10225198215}{3760757\pi^{20}}S_4^{10}\eta_3(55\eta_7\eta_9 + 7\eta_5\eta_{11})\lambda^{16} \\ &+ \frac{6561}{90795419\pi^{24}}S_4^{12}(73689035\eta_5\eta_7 + 47682447552\eta_3\eta_9)\eta_{11}\lambda^{20} \\ &+ \frac{430381943244}{341887\pi^{28}}S_4^{14}\eta_7\eta_9\eta_{11}\lambda^{24} - \frac{665772408441299316000 15}{31041773415653\pi^{36}}S_4^{18}\eta_3\eta_5\eta_7\eta_9\eta_{11}\lambda^{32}. \end{aligned} \tag{63b}$$

It can be seen that, c_N and d_N contain only the terms λ^l which have l divisible by four.

7.4 Hexagonal array

For a hexagonal array,

$$\langle k_{12}\rangle = 0,\ \langle k_{11}\rangle = \langle k_{22}\rangle = k_{\mathrm{m}}\frac{c_N(1-\eta_1\lambda)+d_N\eta_1\lambda^6}{c_N(1+\eta_1\lambda)+d_N\eta_1\lambda^6}. \tag{64}$$

The obtained first order explicit formula for $N=1$ is also the classical estimation [2,3,13]. For order $N=8$,

$$\begin{aligned} c_8 = 1 &- \frac{787500}{169\pi^{12}}S_6^4\eta_5\eta_7\lambda^{12} - \frac{1750000}{4693\pi^{18}}S_6^6(572\eta_7\eta_{11}+35\eta_5\eta_{13})\lambda^{18} \\ &- \frac{87621187500}{2197\pi^{24}}S_6^8\eta_{11}\eta_{13}\lambda^{24} + \frac{692590665118515625000}{48387275053\pi^{36}}S_6^{12}\eta_5\eta_7\eta_{11}\eta_{13}\lambda^{36}, \end{aligned} \tag{65a}$$

$$\begin{aligned} d_8 = &-\frac{5}{\pi^6}S_6^2\eta_5 - \frac{625}{1859\pi^{12}}S_6^4\eta_{11}\lambda^6 + \frac{111861752187500}{113415731\pi^{24}}S_6^8\eta_5\eta_7\eta_{11}\lambda^{18} \\ &+ \frac{1723435365312500}{8724287\pi^{30}}S_6^{10}\eta_5\eta_{11}\eta_{13}\lambda^{24}. \end{aligned} \tag{65b}$$

It can be seen that, c_N and d_N contain only the terms λ^l which have l divisible by six.

It is worth noting that in formulae (57), (60), (63), and (65) the high-order terms of λ cannot be neglected for the accuracy of the explicit formulae.

8 Discussions and numerical examples

8.1 The conditionally convergent sum S_2

The sum S_2 is conditionally convergent as mentioned above. It is this point which had led to the limitation and questioning of the validity of the methods originated by Rayleigh. Though Perrins et al. [23] as well as Moosavi and Sarkomaa [20]

Table 1 For perfect interface, variations of the dimensionless effective conductivities with the truncation order N for different fiber arrays, and the comparison with the numerical results obtained by EEVM [30] and Perrins et al. [23]. For material parameters, refer to (23) with $k_{\mathrm{f}}/k_{\mathrm{m}} = 50$.

N	Hexagonal Array $\lambda=0.8$ $\lambda_{\lim}=0.9069$	Square Array $\lambda=0.7$ $\lambda_{\lim}=0.7854$	Rectangular Array $\frac{\lvert\omega_2\rvert}{\omega_1}=\frac{1}{2}$, $\lambda=0.36$ $\lambda_{\lim}=0.3927$		General Doubly Periodic $\frac{\mathrm{Im}\omega_2}{\omega_1}=\frac{1}{2}, \theta=\frac{\pi}{3}$, $\lambda=0.48$ $\lambda_{\lim}=0.5236$		
	$\frac{\langle k_{11}\rangle}{k_{\mathrm{m}}}=\frac{\langle k_{22}\rangle}{k_{\mathrm{m}}}$	$\frac{\langle k_{11}\rangle}{k_{\mathrm{m}}}=\frac{\langle k_{22}\rangle}{k_{\mathrm{m}}}$	$\frac{\langle k_{11}\rangle}{k_{\mathrm{m}}}$	$\frac{\langle k_{22}\rangle}{k_{\mathrm{m}}}$	$\frac{\langle k_{11}\rangle}{k_{\mathrm{m}}}$	$\frac{\langle k_{22}\rangle}{k_{\mathrm{m}}}$	$\frac{\langle k_{12}\rangle}{k_{\mathrm{m}}}$
1	7.64407	5.10778	1.66993	3.50961	2.69362	3.79330	0.943634
2	7.64407	6.18007	1.68932	4.02376	2.90209	4.32367	1.22573
3	8.21305	6.29151	1.68987	4.15497	2.94546	4.45468	1.29986
4	8.25787	6.32727	1.68988	4.18703	2.95635	4.48683	1.31857
6	8.25927	6.33552	1.68988	4.19710	2.95970	4.49690	1.32437
8	8.25999	6.33590	1.68988	4.19779	2.95993	4.49759	1.32476
10	8.26001	6.33592	1.68988	4.19784	2.95995	4.49764	1.32479
EEVM [30]	8.26001	6.33592	1.68988	4.19765	2.95995	4.49768	1.32481
Perrins et al. [23]	8.2600	6.3359	–	–	–	–	–

modified Rayleigh's method and extended the method to consider hexagonal array and orthotropic arrays, it is necessary to explain further the physics and mathematics of the method for the general doubly-periodic arrays.

The value of the sum S_2 is dependent on the summation sequence. Eisenstein [29] introduced a summation sequence as follows:

$$S_2^* = \lim_{N\to\infty} \sum_{s=-N}^{N} \left(\lim_{M\to\infty} \sum_{r=-M}^{M} \frac{1}{\omega_{rs}^2} \right),\ r^2 + s^2 \neq 0. \tag{66}$$

The sum S_2^* can be expressed by Weierstrass ς function [29]:

$$S_2^* = \frac{2}{\omega_2} \varsigma\left(\frac{\omega_1}{2}\right). \tag{67}$$

From (33), (A7), and (67), we find

$$\varepsilon = S_2^* - \frac{\bar{\omega}_1}{\omega_1}\pi. \tag{68}$$

As discussed in Sect. 3 and in Sect. 5, in order to make sure the average gradient over the unit cell is equal to the external gradient (24), the value of S_2 should satisfy (43). Therefore, from (43) and (68),

$$S_2 = S_2^* + \left(\frac{\bar{A}_1}{A_1} - \frac{\bar{\omega}_1}{\omega_1}\right)\pi. \tag{69}$$

Table 2 For contact resistance interface, variations of the dimensionless effective conductivities with the truncation order N for different fiber arrays, and the comparison with the numerical results obtained by EEVM [30] and Lu and Lin [15]. For material parameters, refer to (23) with $k_f/k_m = 1001$, $\beta = 4.995\times10^{-3}$.

N	Hexagonal Array $\lambda = 0.8$ $\lambda_{lim} = 0.9069$	Square Array $\lambda = 0.7$ $\lambda_{lim} = 0.7854$	Rectangular Array $\frac{\lvert\omega_2\rvert}{\omega_1} = \frac{\sqrt{3}}{2}$, $\lambda = 0.62$ $\lambda_{lim} = 0.6802$		General Doubly Periodic $\frac{\mathrm{Im}\omega_2}{\omega_1} = \frac{\sqrt{3}}{2}$, $\theta = \frac{5\pi}{12}$, $\lambda = 0.66$ $\lambda_{lim} = 0.7290$		
	$\frac{\langle k_{11}\rangle}{k_m} = \frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{11}\rangle}{k_m} = \frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{11}\rangle}{k_m}$	$\frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{11}\rangle}{k_m}$	$\frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{12}\rangle}{k_m}$
1	8.20000	5.34483	3.88350	5.34695	4.16683	5.62170	0.666908
2	8.20000	6.52913	3.60232	6.43325	4.38226	6.45305	0.775892
3	8.82902	6.64249	3.60237	6.61906	4.43359	6.64675	0.862522
4	8.87065	6.67684	3.60286	6.66858	4.43406	6.68391	0.866679
6	8.87174	6.68386	3.60292	6.68037	4.43496	6.69323	0.869527
8	8.87230	6.68413	3.60293	6.68096	4.43499	6.69365	0.869641
10	8.87231	6.68414	3.60293	6.68099	4.43499	6.69367	0.869646
EEVM [30]	8.87231	6.68414	3.60293	6.68099	4.43499	6.69367	0.869646
Lu and Lin [15]	8.90664	6.68414	–	–	–	–	–

For symmetric fiber arrays which are placed symmetric about the x-axis, when the external gradient H_i^0 is also symmetric about the x-axis, the expansion coefficient A_1 is real, then Eq. (69) is rewritten as

$$S_2 = S_2^* + \left(1 - \frac{\bar{\omega}_1}{\omega_1}\right)\pi. \tag{70}$$

Furthermore, if the first period ω_1 is placed along the x-axis, ω_1 is real, and then Eq. (70) is reduced into $S_2 = S_2^*$. In this case, the summation is over a so-called "needle"-shaped region [23]. Therefore, the existing solutions by setting $S_2 = S_2^*$ are only applicable in the case of symmetric fiber arrays.

In other words, for general doubly-periodic arrays, in order to satisfy the boundary condition of prescribed external gradient (24), the value of the sum S_2 must satisfy Eq. (69). However, equation (69) contains an expansion coefficient A_1 to be determined, thus for this case the sum S_2 can only be determined by combing Eq. (69) and the coefficient equations (22). Therefore, for the case of general doubly-periodic array, the modification introduced in Sect. 6 is necessary.

8.2 Validity and accuracy of the present solution

In order to verify the validity and accuracy of the present solution, convergence analysis and comparison with other methods are made.

Different fiber arrays, including hexagonal array, square array, rectangular arrays, rhombic arrays and general doubly-periodic arrays, are considered. The corresponding geometric parameters are listed in Table A1. The material properties are chosen from the references for convenience of comparison. The predicted dimensionless effective conductivities for three fiber-matrix interfaces, i.e., perfect interface, contact resistance interface and coating interface, are listed in Tables 1–3, respectively. The variations of the predictions with the truncation order N are shown. The limiting volume fraction of fiber, λ_{lim}, for each fiber array is also given in the tables. With the fiber volume fraction, λ, approaching the limiting value,

Table 3 For coating interface, variations of the dimensionless effective conductivities with the truncation order N for different fiber arrays, and the comparison with the numerical results obtained by EEVM [30] and Lu and Lin [15]. For material parameters, refer to (23) with $k_f/k_m = 1/100$, $k_c/k_m = 990.5$, $\xi = 0.1$.

N	Hexagonal Array $\lambda = 0.8$ $\lambda_{\text{lim}} = 0.9069$	Square Array $\lambda = 0.7$ $\lambda_{\text{lim}} = 0.7854$	Rhombic Array $\frac{\text{Im}\omega_2}{\omega_1} = 1$, $\lambda = 0.7$ $\lambda_{\text{lim}} = 0.7854$		General Doubly Periodic $\frac{\text{Im}\omega_2}{\omega_1} = 1$, $\theta = \frac{5\pi}{12}$, $\lambda = 0.7$ $\lambda_{\text{lim}} = 0.7854$		
	$\frac{\langle k_{11}\rangle}{k_m} = \frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{11}\rangle}{k_m} = \frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{11}\rangle}{k_m}$	$\frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{11}\rangle}{k_m}$	$\frac{\langle k_{22}\rangle}{k_m}$	$\frac{\langle k_{12}\rangle}{k_m}$
1	8.22428	5.35492	6.47373	4.61585	5.97169	4.94750	0.452284
2	8.22428	6.63206	6.90262	4.68798	6.80287	5.44000	0.342860
3	8.92889	6.77930	7.22368	4.76057	7.05683	5.50599	0.454821
4	8.98960	6.82609	7.25461	4.76175	7.08825	5.51496	0.438090
6	8.99155	6.83743	7.26463	4.76176	7.09915	5.51601	0.440132
8	8.99255	6.83797	7.26515	4.76177	7.09963	5.51602	0.440131
10	8.99259	6.83800	7.26517	4.76177	7.09965	5.51602	0.440128
EEVM [30]	8.99259	6.83800	7.26517	4.76177	7.09965	5.51602	0.440128
Lu and Lin [15]	8.99004	6.83800	–	–	–	–	–

the required truncation order N increases. In the tables, λ, are chosen to obtain convergent results to the accuracy shown in the tables within $N = 10$.

It can be seen from Tables 1–3 that the present predictions are in good agreement with the results by the eigenfunction expansion-variational method (EEVM) [30], the extended Rayleigh's method by Perrins et al. [23] and the boundary collocation method by Lu and Lin [15]. Note that Perrins et al. [23] and Lu and Lin [15] considered only the hexagonal array and square array. The eigenfunction expansion-variational method is available for arbitrary periodic arrays, but it gives explicit formulae of effective conductivities only for the hexagonal array and square array. The present method gives a series of explicit formulae with moderate length for arbitrary periodic arrays. In the explicit formulae, the truncation order N is up to 3 for general doubly-periodic arrays (formulae (55) and (57)), $N = 4$ for orthotropic arrays (formulae (58) and (60)), $N = 6$ for the square array (formulae (61) and (63)) and $N = 8$ for the hexagonal array (formulae (64) and (65)). From the comparison in Tables 1–3, high accuracy of the explicit formulae is observed.

9 Conclusions

A complex variable method is developed to solve the problem of steady-state heat conduction of composites with a doubly periodic array of fibers. The present method can be seen as an extension of Rayleigh's method to consider general doubly-periodic fiber array and different fiber-matrix interfaces. The limitation of Rayleigh's method caused by the conditionally convergent sum is overcome by introducing a complementary equation.

Average fields are expressed by the first expansion coefficient of the complex potential, with the aid of Residue theorem, Green theorem and Elliptic function theory. Then unified complex variable solution of the effective conductivities is obtained in a concise form, for arbitrary doubly-periodic fiber arrays and different fiber-matrix interfaces. The validity of the present solution is verified by comparing with other methods in the numerical examples.

By applying appropriate truncation to finite order, a series of explicit formulae of effective conductivities are given. The explicit formulae for different fiber arrays are written in a regular form, which reveals the reciprocal relations for conductivities. High accuracy of the explicit formulae is shown in the numerical examples.

Appendix

Evaluation of the sums S_{2m} and ε

Let us first evaluate the sums S_{2m} $(m \geq 2)$, which are defined by

$$S_{2m} = \sum_{r,s} \left(\frac{1}{r\omega_1 + s\omega_2} \right)^{2m}, \; r^2 + s^2 \neq 0, \; m \geq 2. \tag{A1}$$

The sums S_4 and S_6 can be calculated by following rapidly convergent series [18]:

$$S_4 = \frac{1}{60} \left(\frac{\pi}{\omega_1} \right)^4 \left(\frac{4}{3} + 320 \sum_{m=1}^{\infty} \frac{m^3 h^{2m}}{1 - h^{2m}} \right), \tag{A2}$$

$$S_6 = \frac{1}{140} \left(\frac{\pi}{\omega_1} \right)^6 \left(\frac{8}{27} - \frac{448}{3} \sum_{m=1}^{\infty} \frac{m^5 h^{2m}}{1 - h^{2m}} \right), \tag{A3}$$

with $h = \mathrm{e}^{(\omega_2/\omega_1)\pi \mathrm{i}}$. All the other higher-order sums can be evaluated by using the recurrence formulae [9, 18]:

$$S_{2m} = \frac{3}{(4m^2 - 1)(m - 3)} \sum_{r=2}^{m-2} (2r - 1)(2m - 2r - 1) S_{2r} S_{2m-2r}, \; m \geq 4. \tag{A4}$$

Table A1 Numerical results of S_4, S_6, and ε calculated by formulae (A2), (A3), and (A9) for some typical arrays. The area of the unit cell shown in Fig. 1a is always set to be 1, and the first period is always set to be real ($\omega_1 = \alpha$) without loss of generality.

Type of arrays	Parameters	S_4	S_6	ε
General arrays $\omega_1 = \alpha$, $\omega_2 = \frac{1}{\alpha \tan\theta} + \frac{1}{\alpha}\mathrm{i}$	$\alpha = 1, \theta = \pi/3$	$1.31600 - 0.439787\mathrm{i}$	$3.66164 + 0.798631\mathrm{i}$	$0.278184 + 0.068192\mathrm{i}$
	$\alpha = 1, \theta = 5\pi/12$	$2.03960 + 0.960266\mathrm{i}$	$2.3647 - 1.87494\mathrm{i}$	$0.165674 - 0.146324\mathrm{i}$
	$\alpha = \sqrt{2\sqrt{3}/3}$, $\theta = \pi/4$	$2.74007 - 1.32525\mathrm{i}$	$-0.54364 + 2.57047\mathrm{i}$	$-0.489415 + 0.224841\mathrm{i}$
	$\alpha = \sqrt{2\sqrt{3}/3}$, $\theta = 5\pi/12$	$1.74903 + 1.69162\mathrm{i}$	$1.40321 - 2.94776\mathrm{i}$	$-0.322068 - 0.295271\mathrm{i}$
	$\alpha = \sqrt{2}, \theta = \pi/3$	$-2.52763 + 4.23714\mathrm{i}$	$6.68110 - 0.209052\mathrm{i}$	$-0.899597 - 1.54389\mathrm{i}$
	$\alpha = \sqrt{2}, \theta = 5\pi/12$	$3.76139 + 6.51491\mathrm{i}$	$-13.2745\mathrm{i}$	$-2.59684 - 1.49929\mathrm{i}$
Rhombic arrays $\omega_1 = \alpha$, $\omega_2 = \frac{\alpha}{2} + \frac{1}{\alpha}\mathrm{i}$	$\alpha = 1$	1.21069	3.83332	0.294899
	$\alpha = \sqrt{2\sqrt{3}/3}$ (Hexagonal array)	0	3.80815	0
	$\alpha = \sqrt{2}$ (Rotated square array)	−3.15121	0	0
Rectangular arrays $\omega_1 = \alpha$, $\omega_2 = \frac{1}{\alpha}\mathrm{i}$	$\alpha = 1$ (Square array)	3.15121	0	0
	$\alpha = \sqrt{2\sqrt{3}/3}$	3.37869	−1.99103	−0.592671
	$\alpha = \sqrt{2}$	8.66583	−16.2489	−3.43759

Then the expressions of some higher-order sums by S_4 and S_6 are

$$
\begin{aligned}
S_8 &= \frac{3}{7} S_4^2, \\
S_{10} &= \frac{5}{11} S_4 S_6, \\
S_{12} &= \frac{1}{143}(18 S_4^3 + 25 S_6^2), \\
S_{14} &= \frac{30}{143} S_4^2 S_6.
\end{aligned}
\tag{A5}
$$

Let us now evaluate ε, which is defined by

$$
\varepsilon = \frac{1}{2\mathrm{i}} \left[2\varsigma\left(\frac{\omega_2}{2}\right)\bar{\omega}_1 - 2\varsigma\left(\frac{\omega_1}{2}\right)\bar{\omega}_2 \right]. \tag{A6}
$$

According to the Weierstrass ς function [29],

$$
2\varsigma\left(\frac{\omega_1}{2}\right)\omega_2 - 2\varsigma\left(\frac{\omega_2}{2}\right)\omega_1 = 2\pi\mathrm{i}, \tag{A7}
$$

and $\varsigma(\frac{\omega_1}{2})$ can be calculated by the following series [18, 19, 26]:

$$\varsigma\left(\frac{\omega_1}{2}\right) = \frac{\omega_1}{2}\left(\frac{\pi}{\omega_1}\right)^2\left(\frac{1}{3} - 8\sum_{m=1}^{\infty}\frac{mh^{2m}}{1-h^{2m}}\right). \tag{A8}$$

Then

$$\varepsilon = \left(\frac{\pi}{\omega_1}\right)^2\left(\frac{1}{3} - 8\sum_{m=1}^{\infty}\frac{mh^{2m}}{1-h^{2m}}\right) - \frac{\bar{\omega}_1}{\omega_1}\pi. \tag{A9}$$

Let us consider some particular arrays. Due to the symmetry of arrays, some special properties can be obtained:

- For orthotropic arrays including rectangular arrays and rhombic arrays, $S_{2m} = \bar{S}_{2m}$, $\varepsilon = \bar{\varepsilon}$, thus S_{2m} and ε are real;
- For a square array, $S_{2m} = \mathrm{i}^{2m} S_{2m}$ and $\varepsilon = \mathrm{i}^2\varepsilon$, thus besides S_{2m} and ε are real, $S_{2m} = 0$ for $2m \neq 4l$ $(m \geq 2, l \geq 1)$ and $\varepsilon = 0$;
- For a hexagonal array, $S_{2m} = (\mathrm{e}^{\mathrm{i}\pi/3})^{2m} S_{2m} = (\mathrm{e}^{\mathrm{i}2\pi/3})^{2m} S_{2m}$ and $\varepsilon = (\mathrm{e}^{\mathrm{i}\pi/3})^2\varepsilon = (\mathrm{e}^{\mathrm{i}2\pi/3})^2\varepsilon$, thus besides S_{2m} and ε are real, $S_{2m} = 0$ for $2m \neq 6l$ $(m \geq 2, l \geq 1)$ and $\varepsilon = 0$.

Numerical results of S_4, S_6, and ε calculated by formulae (A2), (A3), and (A9) for some typical arrays are listed in Table A1. Note that the sums S_{2m} and ε are related to the elliptic functions, thus they can also be evaluated directly in some standard math software [5].

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