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The energy equality for weak solutions to the equations of non-Newtonian fluids

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ABSTRACT

In this short paper, we extend the result of Shinbrot (1974) to an incompressible fluid with shear dependent viscosity. It is shown that a weak solution to the equations of non-Newtonian fluids lying in a space $L^q(0,T;L^p)$ satisfies an energy equality, where $\frac{2r}{r-1} \le p \le \frac{2r}{r-2}$ and $\frac{1}{p}+\frac{1}{q} \le \frac{r-1}{r}$, if $r > 2$; $p \ge \frac{2r}{r-1}$ and $\frac{r-1}{p}+\frac{1}{q} = \frac{r-1}{2}$, if $\frac{2(n+1)}{n+2} < r \le 2$. In particular, our result implies that the weak solution must satisfy the energy equality when $r \ge \frac{3n+2}{n+2}$, which is consistent with the known fact.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \ge 2)$ be a domain with a sufficiently smooth boundary. We consider a non-Newtonian incompressible fluid which is governed by the following system

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \operatorname{div}\left(|D(\mathbf{u})|^{r-2}D(\mathbf{u})\right) + \nabla\pi = 0, & \text{in } \Omega\times(0,T), \\ \operatorname{div}\,\mathbf{u} = 0, & \text{in } \Omega\times(0,T), \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), & \text{in } \Omega. \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ denotes the unknown velocity of the fluid and π the pressure, and

$$D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right).$$

We first give the definition of a weak solution to (1). To this end, we denote by $C_0^\infty(\Omega)$ the space of smooth functions with compact support. Let $C_{0,\sigma}^\infty(\Omega) = \{\boldsymbol{\varphi} \in C_0^\infty(\Omega) | \nabla\cdot\boldsymbol{\varphi} = 0\}$. $L_\sigma^p(\mathbb{R}^3)$ denotes the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\cdot\|_p$. $\dot{H}_{0,\sigma}^{1,p}(\Omega)$ denotes the closure of $C_{0,\sigma}^\infty(\Omega)$ in the norm $\|\nabla\cdot\|_p$. We write

$$\|\mathbf{u}\|_{p,q} = \left(\int_0^T \|\mathbf{u}(t)\|_p^q dt\right)^{\frac{1}{q}}, \quad 1 \le q < \infty,$$

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and

$$\|\mathbf{u}(t)\|_{p,\infty} = ess \sup_{0<t<T} \|\mathbf{u}\|_p.$$

Definition 1. Let $r \geq \frac{2n}{n+2}$, $\mathbf{u}_0 \in L^2_\sigma(\Omega)$. A vector-valued function $\mathbf{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\dot{H}^{1,r}_{0,\sigma}(\Omega))$ is called a weak solution to (1) if the following identity

$$(\mathbf{u}_0, \boldsymbol{\varphi}(0)) + \int_0^T \left[(\mathbf{u}, \partial_t \boldsymbol{\varphi}) - (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\varphi}) - \left(|D(\mathbf{u})|^{r-2} D(\mathbf{u}), D(\boldsymbol{\varphi}) \right) \right] dt = 0 \tag{2}$$

holds for all $\boldsymbol{\varphi} \in C_0^\infty\left([0,T); C^\infty_{0,\sigma}(\Omega)\right)$. Here, one restricts $r \geq \frac{2n}{n+2}$ to make sure that the expression $\int_0^T (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\varphi})\, dt$ makes sense.

The existence of weak solutions of (1) is shown in [1,2] with the periodic boundary condition, and in [3] for the whole space. In [4], J. Wolf showed the existence of weak solutions with Dirichlet boundary condition for $r > \frac{2(n+1)}{n+2}$. Moreover, we know that a weak solution satisfies the global energy inequality

$$\|\mathbf{u}(t)\|_2^2 + 2\int_0^t \|D(\mathbf{u})(\tau)\|_r^r d\tau \leq \|\mathbf{u}_0\|_2^2, \quad t \geq 0.$$

A natural question is to consider the possible validity of the energy equality. For Newtonian fluids, i.e. $r = 2$, the pioneering results by Prodi [5] and Lions [6] concern the validity of energy equality for a weak solution such that

$$\mathbf{u} \in L^4(0,T,L^4(\Omega)).$$

Later, Shinbrot [7] enlarged the range of exponents proving that if a weak solution belongs to

$$\mathbf{u} \in L^q(0,T;L^p(\Omega)), \quad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \quad p \geq 4.$$

More results in this respect, the reader can see [8] and references therein. However, as far as I know, there is no result for the non-Newtonian fluids. In this paper, we will extend Shinbrot's result to the non-Newtonian fluids. Our main result is stated as follows.

Theorem 1. *Let $r > \frac{2(n+1)}{n+2}$, r' is the Hölder conjugate of r, $\mathbf{u}_0 \in L^2_\sigma(\Omega)$, and let $\mathbf{u}$ be a weak solution of (1). If $\mathbf{u} \in L^q(0,T;L^p(\Omega))$, where $2r' \leq p \leq \frac{2r}{r-2}$ and $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r'}$, if $r > 2$; $p \geq 2r'$ and $\frac{r-1}{p} + \frac{1}{q} \leq \frac{r-1}{2}$, if $r \leq 2$. Then $\mathbf{u}$ satisfies the energy equality*

$$\|\mathbf{u}(t)\|_2^2 + 2\int_0^t \|D(\mathbf{u})(t)\|_r^r d\tau = \|\mathbf{u}_0\|_2^2, \quad 0 \leq t < T. \tag{3}$$

Remark 1. Here, since the existence of weak solutions to (1) is still unsolved for $r \leq \frac{2(n+1)}{n+2}$, see [4], we have to restrict $r > \frac{2(n+1)}{n+2}$.

Remark 2. It is well known that the weak solution is strong and unique for $r \geq \frac{3n+2}{n+2}$, see for example [2], thus the weak solution satisfies automatically the energy equality. It is remarkable that our result is consistent with this fact. Actually, by virtue of Gagliardo–Nirenberg inequalities and Korn's inequality, one easily verifies

$$L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;\dot{H}^{1,r}_{0,\sigma}(\Omega)) \hookrightarrow L^{\frac{(n+2)r^2-2nr}{n}}(0,T;L^{2r'}_\sigma(\Omega)).$$

When $r \geq \frac{3n+2}{n+2}$, one can easily check that

$$\frac{1}{2r'} + \frac{n}{(n+2)r^2 - 2nr} \leq \frac{1}{r'},$$

which implies a weak solution of (1) must satisfy the energy equality for any $r \geq \frac{3n+2}{n+2}$.

2. The proof of Theorem 1

Firstly, we have the following property of weak solutions.

Lemma 2. *Let* $r > \frac{2(n+1)}{n+2}$, $\mathbf{u}_0 \in L^2_\sigma(\Omega)$, *and let* $\mathbf{u}$ *be a weak solution of* (1). *Then, after suitable redefinition of* $\mathbf{u}$ *on a set of values of* t *of one-dimensional measure zero, we have*

$$(\mathbf{u}(t), \boldsymbol{\varphi}(t)) = (\mathbf{u}_0, \boldsymbol{\varphi}(0)) + \int_0^t \Big[(\mathbf{u}, \partial_\tau \boldsymbol{\varphi}) - (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\varphi}) - \Big(|D(\mathbf{u})(\tau)|^{r-2} D(\mathbf{u})(\tau), D(\boldsymbol{\varphi})(\tau)\Big)\Big]\, d\tau = 0 \tag{4}$$

holds for all $\boldsymbol{\varphi} \in C_0^\infty\left([0,T); C^\infty_{0,\sigma}(\Omega)\right)$ *and all* $0 \le t < T$.

This lemma is completely similar with that of Lemma 2.1 in [9], see also [5,10,11], we omit the details here.

Lemma 3. *Let* $r > 1$, $\boldsymbol{\phi} \in L^p(\Omega)$, $\boldsymbol{\psi} \in \dot{H}^{1,r}_0(\Omega)$, $\boldsymbol{\chi} \in L^2(\Omega) \cap L^p(\Omega)$, *where*

$$\begin{aligned} 2r' \le p \le \frac{2r}{r-2}, &\quad \text{if } \ r > 2; \\ p \ge 2r', &\quad \text{if } \ 1 < r \le 2. \end{aligned} \tag{5}$$

Then

$$|(\boldsymbol{\phi}\cdot\nabla\boldsymbol{\psi}, \boldsymbol{\chi})| \le \|\boldsymbol{\phi}\|_p \|\nabla\boldsymbol{\psi}\|_r \|\boldsymbol{\chi}\|_2^\theta \|\boldsymbol{\chi}\|_p^{1-\theta}, \tag{6}$$

where

$$\theta = \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{2} - \frac{1}{p}} = \frac{2(p-q)}{q(p-2)},$$

and q *is defined by* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r'}$.

Proof. If $\boldsymbol{\chi} \in L^q(\Omega)$, then

$$|\,(\boldsymbol{\phi}\cdot\nabla\boldsymbol{\psi}, \boldsymbol{\chi})\,| \le \|\boldsymbol{\phi}\|_p \|\nabla\boldsymbol{\psi}\|_r \|\boldsymbol{\chi}\|_q.$$

From the assumption (5), one can easily check that $2 \le q \le p$. Hence we can use interpolation and write q as $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{p}$, which give us $\theta := \frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{2}-\frac{1}{p}} = \frac{2(p-q)}{q(p-2)}$. Thus,

$$|(\boldsymbol{\phi}\cdot\nabla\boldsymbol{\psi}, \boldsymbol{\chi})| \le \|\boldsymbol{\phi}\|_p \|\nabla\boldsymbol{\psi}\|_r \|\boldsymbol{\chi}\|_2^\theta \|\boldsymbol{\chi}\|_p^{1-\theta}.$$

Lemma 4. *Let* $r > 1$, *and*

$$\boldsymbol{\phi} \in L^{\tilde{q}}(0,T; L^p(\Omega)),\ \boldsymbol{\psi} \in L^r(0,T; \dot{H}^{1,r}_0(\Omega)),\ \boldsymbol{\chi} \in L^\infty(0,T; L^2(\Omega)) \cap L^{\tilde{q}}(0,T; L^p(\Omega)),$$

where p *satisfies the assumption* (5). *Then*

(i) *If* $r > 2$ *and* $\frac{1}{p} + \frac{1}{\tilde{q}} = \frac{1}{r'}$, *one has*

$$\left| \int_0^T (\boldsymbol{\phi}(t)\cdot\nabla\boldsymbol{\psi}(t), \boldsymbol{\chi}(t))\, dt \right| \le T^{\left(\frac{1}{r'}-\frac{1}{2}\right)\theta} \|\boldsymbol{\phi}\|_{p,\tilde{q}} \|\nabla\boldsymbol{\psi}\|_{r,r} \|\boldsymbol{\chi}\|_{2,\infty}^\theta \|\boldsymbol{\chi}\|_{p,\tilde{q}}^{\frac{q}{2}-1}, \tag{7}$$

(ii) *If* $1 < r < 2$ *and* $\frac{r-1}{p} + \frac{1}{\tilde{q}} = \frac{r-1}{2}$, *one has*

$$\left| \int_0^T (\boldsymbol{\phi}(t)\cdot\nabla\boldsymbol{\psi}(t), \boldsymbol{\chi}(t))\, dt \right| \le \|\boldsymbol{\phi}\|_{p,\tilde{q}} \|\nabla\boldsymbol{\psi}\|_{r,r} \|\boldsymbol{\chi}\|_{2,\infty}^\theta \|\boldsymbol{\chi}\|_{p,\tilde{q}}^{\frac{q}{2}-1}. \tag{8}$$

Proof. If $r > 2$ and $\frac{1}{p} + \frac{1}{\tilde{q}} = \frac{1}{r'}$, one can easily check that

$$\frac{1}{\tilde{q}} + \frac{1}{r} + \frac{1-\theta}{\tilde{q}} + \left(\frac{1}{r'} - \frac{1}{2}\right)\theta = 1,$$

then by Hölder's inequality, one can obtain (7) by Lemma 4. If $1 < r < 2$ and $\frac{1}{\tilde{q}} + \frac{r-1}{p} = \frac{r-1}{2}$, it is easy to know that

$$\frac{1}{\tilde{q}} + \frac{1}{r} + \frac{1-\theta}{\tilde{q}} = 1,$$

thus, one can get (8) by virtue of Lemma 4.

Now, using the above result, we can prove Theorem 1.

Proof of Theorem 1. The proof follows [7]. Let

$$(k_\varepsilon * \phi)(t) = \int_0^{t_0} k_\varepsilon(t-\tau)\phi(\tau)d\tau$$

be a mollifier, so that k_ε is C^∞, real-valued, nonnegative function, supported in $[-\varepsilon, \varepsilon]$, and integrating to unity. Let $\{\mathbf{u}_m\} \subset C_0^\infty([0,\infty); C_{0,\sigma}^\infty(\Omega))$ be a sequence converging to $\mathbf{u}$ in $L^2(0,T; L_\sigma^2(\Omega)) \cap L^r(0,T; \dot{H}_{0,\sigma}^{1,r}(\Omega)) \cap L^q(0,T; L^p(\Omega))$. Set $t = t_0$ and $\boldsymbol{\varphi} = k_\varepsilon * \mathbf{u}_m$ in (4). One obtains

$$\begin{aligned}
&\int_0^{t_0} k_\varepsilon(t_0 - t)(\mathbf{u}(t_0), \mathbf{u}_m(t))dt \\
&= \int_0^{t_0} k_\varepsilon(-t)\,(\mathbf{u}_0, \mathbf{u}_m(t)) + \int_0^{t_0}\int_0^{t_0} \partial_t k_\varepsilon(t-\tau)(\mathbf{u}(t), \mathbf{u}_m(\tau))d\tau dt \\
&\quad - \int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\left[(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t), \mathbf{u}_m(\tau)) + \left(|D(\mathbf{u})(t)|^{r-2}D(\mathbf{u})(t), D(\mathbf{u}_m)(\tau)\right)\right] d\tau dt.
\end{aligned} \tag{9}$$

From Lemma 3, we have

$$\begin{aligned}
&\left|\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\,(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t), \mathbf{u}_m(\tau) - \mathbf{u}(\tau))\,d\tau dt\right| \\
&\le \int_0^{t_0} \|\mathbf{u}(t)\|_p \|\nabla\mathbf{u}(t)\|_r dt \int_0^{t_0} \|\mathbf{u}_m(\tau) - \mathbf{u}(\tau)\|_2^\theta \|\mathbf{u}_m(\tau) - \mathbf{u}(\tau)\|_p^{1-\theta} d\tau.
\end{aligned}$$

Since $r' \le q$, we have

$$\int_0^{t_0} \|\mathbf{u}(t)\|_p \|\nabla\mathbf{u}(t)\|_r dt \le C(t_0)\|\mathbf{u}\|_{p,q}\|\nabla u\|_{r,r}.$$

On the other hand, since $\frac{1}{2} + \frac{1}{q} \le 1$, we have

$$\int_0^{t_0} \|\mathbf{u}_m(\tau) - \mathbf{u}(\tau)\|_2^\theta \|\mathbf{u}_m(\tau) - \mathbf{u}(\tau)\|_p^{1-\theta} d\tau \le C(t_0)\|\mathbf{u}_m - \mathbf{u}\|_{2,2}^\theta \|\mathbf{u}_m - \mathbf{u}\|_{p,q}^{1-\theta} \to 0, \quad \text{as } m \to \infty.$$

Thus, let $m \to \infty$ in (9), we have

$$\begin{aligned}
&\int_0^{t_0} k_\varepsilon(t_0 - t)(\mathbf{u}(t_0), \mathbf{u}(t))dt \\
&= \int_0^{t_0} k_\varepsilon(-t)\,(\mathbf{u}_0, \mathbf{u}(t)) + \int_0^{t_0}\int_0^{t_0} \partial_t k_\varepsilon(t-\tau)(\mathbf{u}(t), \mathbf{u}(\tau))d\tau dt \\
&\quad - \int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\left[(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t), \mathbf{u}(\tau)) + \left(|D(\mathbf{u})(t)|^{r-2}D(\mathbf{u})(t), D(\mathbf{u})(\tau)\right)\right] d\tau dt.
\end{aligned} \tag{10}$$

The term here involving the derivative vanishes if k is chosen to be even. We send ε to zero in the remaining terms. Because of the usual properties of mollifiers,

$$\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\left(|D(\mathbf{u})(t)|^{r-2}D(\mathbf{u})(t), D(\mathbf{u})(\tau)\right)d\tau dt \to \int_0^{t_0}\|D(\mathbf{u})(t)\|_p^p dt.$$

In addition,

$$\int_0^{t_0} k_\varepsilon(t_0-t)(\mathbf{u}(t_0),\mathbf{u}(t))dt = \int_0^{\varepsilon} k_\varepsilon(t)\left(\mathbf{u}(t_0),\mathbf{u}(t_0-t)\right)dt,$$

since as a function of t, $\mathbf{u}$ is continuous in the weak topology of $L^2_\sigma(\Omega)$ (see [4] Theorem 1.3 for example), we have

$$\int_0^{t_0} k_\varepsilon(t_0-t)(\mathbf{u}(t_0),\mathbf{u}(t))dt = \int_0^{\varepsilon} k_\varepsilon(t)[\|\mathbf{u}(t_0)\|_2^2+o(1)]dt \to \frac{1}{2}\|\mathbf{u}(t_0)\|_2^2.$$

Similarly,

$$\int_0^{t_0} k_\varepsilon(-t)(\mathbf{u}_0,\mathbf{u}(t))dt \to \frac{1}{2}\|\mathbf{u}_0\|_2^2.$$

Finally, we consider the nonlinear term in (10). We have

$$\begin{aligned}&\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\left(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),\mathbf{u}(\tau)\right)d\tau dt - \int_0^{t_0}\left(\mathbf{u}(t)\nabla\mathbf{u}(t),\mathbf{u}(t)\right)dt\\ &= \int_0^{t_0}\left(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),(k_\varepsilon*\mathbf{u})(t)-\mathbf{u}(t)\right)dt.\end{aligned} \tag{11}$$

By Lemma 4, this is bounded by

$$C(t_0)\|\mathbf{u}\|_{p,q}\|\nabla\mathbf{u}\|_{r,r}\|k_\varepsilon*\mathbf{u}-\mathbf{u}\|_{2,\infty}^{\theta}\|k_\varepsilon*\mathbf{u}-\mathbf{u}\|_{p,q}^{\frac{q}{2}-1}.$$

This goes to zero because of usual properties of mollifiers. Thus (11) goes to zero. Now, we prove $\int_0^{t_0}\left(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),\mathbf{u}(t)\right)=0$. From Lemma 4, we have that the function F defined by

$$F(\boldsymbol{\psi},\boldsymbol{\chi}) = \int_0^{t_0}\left(\mathbf{u}(t)\cdot\nabla\boldsymbol{\psi}(t),\boldsymbol{\chi}(t)\right)dt$$

is continuous on $L^2(0,t_0;\dot{H}_0^{1,r}(\Omega))\times L^q(0,t_0;L^p(\Omega))$. On the other hand, integration by parts shows that $F(\boldsymbol{\psi},\boldsymbol{\psi})=0$ if $\boldsymbol{\psi}$ is smooth. Let $\{\mathbf{u}_m\}$ be a sequence from $C_0^\infty([0,T);C_{0,\sigma}^\infty(\Omega))$ converging to $\mathbf{u}$ in the appropriate spaces. Then we find

$$0=F(\mathbf{u}_m,\mathbf{u}_m)\to F(\mathbf{u},\mathbf{u}).$$

All of this shows that

$$\int_0^{t_0}\int_0^{t_0} k_\varepsilon(t-\tau)\left(\mathbf{u}(t)\cdot\nabla\mathbf{u}(t),\mathbf{u}(\tau)\right)d\tau dt \to 0$$

as $\varepsilon\to 0$. Now, let $\varepsilon\to 0$ in (9), we have

$$\frac{1}{2}\|\mathbf{u}(t_0)\|_2^2+\int_0^t\|D(\mathbf{u})(t)\|_r^r d\tau = \|\mathbf{u}_0\|_2^2,$$

which is (3) for $t=t_0$. Since t_0 is arbitrary, we have finished the proof of Theorem 1.

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