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Two-dimensional generalized finite integral transform method for new analytic bending solutions of orthotropic rectangular thin foundation plates

Jinghui Zhang[a], Chao Zhou[b], Salamat Ullah[a], Yang Zhong[a], Rui Li[b,c,*]

[a] *Faculty of Infrastructure Engineering, Dalian University of Technology, Dalian 116024, China*
[b] *State Key Laboratory of Structural Analysis for Industrial Equipment, Department of Engineering Mechanics, and International Research Center for Computational Mechanics, Dalian University of Technology, Dalian 116024, China*
[c] *State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China*

Abstract: In this paper, a two-dimensional generalized finite integral transform method is developed for new analytic bending solutions of orthotropic rectangular thin foundation plates. The vibrating beam functions are adopted as the integral kernels to construct the integral transform pairs. By imposing the transform to the governing equation, utilizing some inherent properties of the beam functions, the title problem is converted to that of solving a system of linear algebraic equations, by which the new analytic solutions are elegantly obtained in a straightforward way. Numerical examples validate the present method as well as the solutions yielded by satisfactory agreement with the literature and finite element analysis.

Keywords: Generalized finite integral transform method, orthotropic thin plate, elastic foundation, analytic solution.

1. Introduction

Rectangular plates are widely used as key structural elements in various engineering fields such as civil, mechanical, marine and aerospace engineering. The mechanical behavior of such structures is of permanent interest for both scientists and engineers since theoretical analysis and practical design are both indispensable for the safety of structures.

Many previous studies have dealt with plate problems with different combinations of boundary conditions, load patterns and material properties by using various approximate or numerical methods. Besides the classical methods such as the finite difference method [1], finite element method (FEM) [2] and boundary element method [3], which are still popular in handling plate problems, some recently developed effective approaches have shown important progresses in the field, including the meshless method [4], isogeometric collocation method [5], boundary particle method [6], finite volume method [7], virtual element method [8], discrete singular convolution method [9], simple hp cloud method [10], finite-layer method [11], etc. In comparison with the numerical methods, analytic methods are sparse, which is attributed to the difficulty in seeking analytic solutions to the complex boundary value problems (BVPs) of higher-order partial differential equations (PDEs) that describe the plate problems. Besides the well-known semi-inverse superposition method [12] that was applied for some simple plate problems, few new analytic methods have been found in the literature, including the symplectic approach [13-16], Fourier-type finite integral transform method [17, 18], etc. It is notable that the one-dimensional generalized finite integral transform method has been applied in the fields of thermodynamics and fluid mechanics [19, 20], by which solving PDEs reduces to solving ordinary differential equations where special mathematical techniques are still required.

This paper presents a first endeavor to extend the one-dimensional generalized finite integral transform to two-dimensional transform for new analytic bending solutions of orthotropic rectangular thin foundation plates, with focus on typical clamped plates that were difficult to solve by the other analytic methods. Taking vibrating beam functions as the integral kernels and conducting the double integral transform, solving the governing PDEs reduces to solving a system of linear algebraic equations, by which the problems are solved in a straightforward way. Compared with the Fourier-type finite integral transform methods, the present method has the advantage of faster

* Corresponding author.
E-mail address: ruili@dlut.edu.cn (R. Li)

convergence with much fewer series terms taken. The validity of the present method is confirmed by satisfactory agreement of the obtained solutions with those available in the literature and by the FEM.

2. Two-dimensional generalized finite integral transform solutions for orthotropic rectangular thin foundation plates

We consider a clamped orthotropic rectangular thin plate resting on an elastic Winkler-type foundation occupying the domain $0 \le x \le a$ and $0 \le y \le b$ in the *xoy* coordinate system, as shown in Fig. 1. The governing bending equation of the plate as well associated boundary conditions are

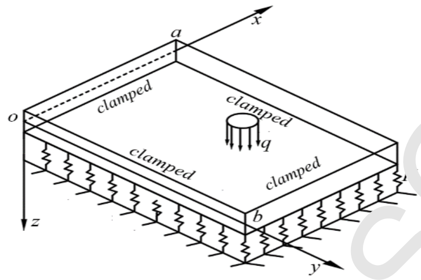


Fig. 1. Schematic illustration of a clamped orthotropic rectangular thin foundation plate.

$$D_x \frac{\partial^4 W(x,y)}{\partial x^4} + 2H \frac{\partial^4 W(x,y)}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 W(x,y)}{\partial y^4} = q(x,y) - KW(x,y) \tag{1}$$

$$W(x,y)\big|_{x=0,a} = \frac{\partial W(x,y)}{\partial x}\bigg|_{x=0,a} = 0, W(x,y)\big|_{y=0,b} = \frac{\partial W(x,y)}{\partial y}\bigg|_{y=0,b} = 0 \tag{2}$$

where D_x and D_y are flexural rigidities in the *x* and *y* directions, respectively; $H = D_1 + 2D_{xy}$ is the effective torsional rigidity in terms of the torsional rigidity D_{xy}, in which $D_1 = \nu_y D_x = \nu_x D_y$, with ν_x and ν_y being the Poisson's ratios; $W(x,y)$ is the deflection, $q(x,y)$ the load, and *K* the Winkler foundation modulus.

The following two-dimensional generalized finite integral transform pair is defined:

$$W_{mn} = \int_0^a \int_0^b W(x,y) X_m(x) Y_n(y) \mathrm{d}x\mathrm{d}y \quad (\text{transform}) \tag{3}$$

$$W(x,y) = \frac{1}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} X_m(x) Y_n(y) \quad (\text{inversion}) \tag{4}$$

where $X_m(x)$ and $Y_n(y)$ are the vibrating beam functions [20]:

$$\begin{aligned} X_m(x) &= \mathrm{ch}(\alpha_m x) - \cos(\alpha_m x) - c_m\left[\mathrm{sh}(\alpha_m x) - \sin(\alpha_m x)\right] \\ Y_n(y) &= \mathrm{ch}(\beta_n y) - \cos(\beta_n y) - c_n\left[\mathrm{sh}(\beta_n y) - \sin(\beta_n y)\right] \end{aligned} \tag{5}$$

in which α_m and β_n are the roots of the transcendental beam frequency equations $\mathrm{ch}(\alpha_m a)\cos(\alpha_m a) = 1$ and $\mathrm{ch}(\beta_n b)\cos(\beta_n b) = 1$, respectively; c_m and c_n are determined by

$$c_m = \frac{\mathrm{ch}(\alpha_m a) - \cos(\alpha_m a)}{\mathrm{sh}(\alpha_m a) - \sin(\alpha_m a)}, c_n = \frac{\mathrm{ch}(\beta_n b) - \cos(\beta_n b)}{\mathrm{ch}(\beta_n b) - \sin(\beta_n b)} \tag{6}$$

The integral kernels here satisfy the following relationships, boundary conditions, and orthogonality:

$$\frac{\mathrm{d}^4 X_m(x)}{\mathrm{d}x^4} = \alpha_m^4 X_m(x), \quad \frac{\mathrm{d}^4 Y_n(y)}{\mathrm{d}y^4} = \beta_n^4 Y_n(y) \tag{7}$$

$$X_m(x)\big|_{x=0,a} = \frac{\mathrm{d}X_m(x)}{\mathrm{d}x}\bigg|_{x=0,a} = 0, \quad Y_n(y)\big|_{y=0,b} = \frac{\mathrm{d}Y_n(y)}{\mathrm{d}y}\bigg|_{y=0,b} = 0 \tag{8}$$

$$\int_0^a X_m(x) X_i(x) \mathrm{d}x = \begin{cases} 0, & m \ne i \\ a, & m = i \end{cases}, \quad \int_0^b Y_n(y) Y_i(y) \mathrm{d}y = \begin{cases} 0, & n \ne i \\ b, & n = i \end{cases} \tag{9}$$

Applying the generalized integral transform as shown in Eq. (3) to each term of Eq. (1), putting the boundary conditions in Eq. (2), the following simplified relationships are derived in sequence:

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial x^4}X_m(x)Y_n(y)\mathrm{d}x\mathrm{d}y$$
$$=\int_0^b\left[\frac{\partial^3 W}{\partial x^3}X_m(x)-\frac{\partial^2 W}{\partial x^2}\frac{\mathrm{d}X_m(x)}{\mathrm{d}x}+\frac{\partial W}{\partial x}\frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}-W\frac{\mathrm{d}^3X_m(x)}{\mathrm{d}x^3}\right]\Bigg|_{x=0}^{x=a}Y_n(y)\mathrm{d}y \tag{10}$$
$$+\int_0^a\int_0^b W(x,y)\frac{\mathrm{d}^4X_m(x)}{\mathrm{d}x^4}Y_n(y)\mathrm{d}x\mathrm{d}y=\alpha_m^4W_{mn}$$

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial y^4}X_m(x)Y_n(y)\mathrm{d}x\mathrm{d}y$$
$$=\int_0^a\left[\frac{\partial^3 W}{\partial y^3}Y_n(y)-\frac{\partial^2 W}{\partial y^2}\frac{\mathrm{d}Y_n(y)}{\mathrm{d}y}+\frac{\partial W}{\partial y}\frac{\mathrm{d}^2Y_n(y)}{\mathrm{d}y^2}-W\frac{\mathrm{d}^3Y_n(y)}{\mathrm{d}y^3}\right]\Bigg|_{y=0}^{y=b}X_m(x)\mathrm{d}x \tag{11}$$
$$+\int_0^a\int_0^b W(x,y)X_m(x)\frac{\mathrm{d}^4Y_n(y)}{\mathrm{d}y^4}\mathrm{d}x\mathrm{d}y=\beta_n^4W_{mn}$$

$$\int_0^a\int_0^b \frac{\partial^4 W}{\partial x^2\partial y^2}X_m(x)Y_n(y)\mathrm{d}x\mathrm{d}y=\int_0^b\left[\frac{\partial^3 W}{\partial x\partial y^2}X_m(x)-\frac{\partial^2 W}{\partial y^2}\frac{\mathrm{d}X_m(x)}{\mathrm{d}x}\right]\Bigg|_{x=0}^{x=a}Y_n(y)\mathrm{d}y$$
$$+\int_0^a\left[\frac{\partial W}{\partial y}Y_n(y)-W\frac{\mathrm{d}Y_n(y)}{\mathrm{d}y}\right]\Bigg|_{y=0}^{y=b}\frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}\mathrm{d}x+\int_0^a\int_0^b W(x,y)\frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2Y_n(y)}{\mathrm{d}y^2}\mathrm{d}x\mathrm{d}y \tag{12}$$
$$=\int_0^a\int_0^b W(x,y)\frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2Y_n(y)}{\mathrm{d}y^2}\mathrm{d}x\mathrm{d}y$$

Substitution of the inversion in Eq. (4) into Eq. (12) leads to

$$\int_0^a\int_0^b W(x,y)\frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}\frac{\mathrm{d}^2Y_n(y)}{\mathrm{d}y^2}\mathrm{d}x\mathrm{d}y=\frac{1}{ab}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}W_{rs}I_{mr}J_{ns} \tag{13}$$

where $I_{mr}=\int_0^a X_r(x)\frac{\mathrm{d}^2X_m(x)}{\mathrm{d}x^2}\mathrm{d}x$ and $J_{ns}=\int_0^b Y_s(y)\frac{\mathrm{d}^2Y_n(y)}{\mathrm{d}y^2}\mathrm{d}y$, the values of which are

$$I_{mr}=\begin{cases}c_m\alpha_m\left[2-c_m(\alpha_m a)\right], & m=r\\ \dfrac{4(\alpha_m a)^2(\alpha_r a)^2\left[c_r(\alpha_r a)-c_m(\alpha_m a)\right]}{a\left[(\alpha_r a)^4-(\alpha_m a)^4\right]}\left[1+(-1)^{m+r}\right], & m\neq r\end{cases} \tag{14}$$

$$J_{ns}=\begin{cases}c_n\beta_n\left[2-c_n(\beta_n b)\right], & n=s\\ \dfrac{4(\beta_n b)^2(\beta_s b)^2\left[c_s(\beta_s b)-c_n(\beta_n b)\right]}{b\left[(\beta_s b)^4-(\beta_n b)^4\right]}\left[1+(-1)^{n+s}\right], & n\neq s\end{cases} \tag{15}$$

Define q_{mn} as the transform of the load function $q(x,y)$,

$$q_{mn}=\int_0^a\int_0^b q(x,y)X_m(x)Y_n(y)\mathrm{d}x\mathrm{d}y \tag{16}$$

The integral transform of Eq. (1) finally gives

$$A_{mn}W_{mn}+\frac{2H}{ab}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}W_{rs}I_{mr}J_{ns}=q_{mn} \tag{17}$$

where $A_{mn}=D_x\alpha_m^4+D_y\beta_n^4+K$.

Equation (17) constitutes a system of infinite linear equations, where *m*, *n*, *r*, and *s* are any positive integers, with their upper limit taken as *t* in practical calculation for convenience. Therefore, the matrix form of Eq. (17) is

$$\begin{bmatrix} M_{11}^{11} & \cdots & M_{11}^{1t} & \cdots & M_{11}^{t1} & \cdots & M_{11}^{tt} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ M_{1t}^{11} & \cdots & M_{1t}^{1t} & \cdots & M_{1t}^{t1} & \cdots & M_{1t}^{tt} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ M_{t1}^{11} & \cdots & M_{t1}^{1t} & \cdots & M_{t1}^{t1} & \cdots & M_{t1}^{tt} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ M_{tt}^{11} & \cdots & M_{tt}^{1t} & \cdots & M_{tt}^{t1} & \cdots & M_{tt}^{tt} \end{bmatrix} \begin{bmatrix} W_{11} \\ \vdots \\ W_{1t} \\ \vdots \\ W_{t1} \\ \vdots \\ W_{tt} \end{bmatrix} = \begin{bmatrix} q_{11} \\ \vdots \\ q_{1t} \\ \vdots \\ q_{t1} \\ \vdots \\ q_{tt} \end{bmatrix} \tag{18}$$

where

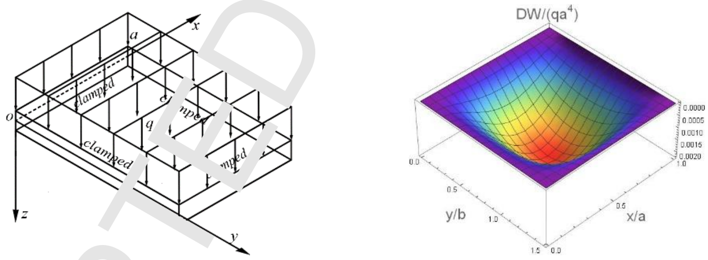
$$M_{mn}^{rs} = \begin{cases} A_{mn} + \dfrac{2HI_{mr}J_{ns}}{ab}, & m = r \text{ and } n = s \\ \dfrac{2HI_{mr}J_{ns}}{ab}, & \text{otherwise} \end{cases} \tag{19}$$

Solving Eq. (18) for W_{mn}, the analytic solutions of plate deflections are obtained by Eq. (4). The other quantities, e.g., the bending moments M_x and M_y, can be readily obtained by proper combinations of the derivatives of governing deflection solutions. For example, $M_x = -\left(D_x \partial^2 W/\partial x^2 + D_1 \partial^2 W/\partial y^2\right)$ and $M_y = -\left(D_y \partial^2 W/\partial y^2 + D_1 \partial^2 W/\partial x^2\right)$.

3. Comprehensive numerical examples

To validate the present method and the obtained analytic solutions, we conduct comprehensive examinations on the plates under three different loading/support conditions.

(1) The first example is on uniformly loaded isotropic rectangular plates without foundation [Fig. 2(a)]. Satisfactory convergence and accuracy are observed from the numerical results listed in Table 1, including both deflections and bending moments, by convergence of the present solutions to the last significant digit of four with only 26 series terms in one direction (i.e., t=26) as well as good agreement between the present solutions and those from the literature [12] and FEM by the commercial software ABAQUS, in which the thickness-to-width ratio of the plates is uniformly set to be 10^{-4} while the 4-node thin shell element S4R and the uniform mesh size a/400 are taken here and hereafter. The non-dimensional 3D deflection of such a plate is plotted in Fig. 2(b).



(a) (b)

Fig. 2. (a) Schematic and (b) 3D plot of a uniformly loaded isotropic plate.

Table 1 Deflections and bending moments of uniformly loaded isotropic plates.

b/a	t	$DW/(qa^4)$ $(x=a/2, y=b/2)$			$M_x/(qa^2)$ $(x=a/2, y=b/2)$			$M_x/(qa^2)$ $(x=0, y=b/2)$		
		Present	Ref. [12]	FEM	Present	Ref. [12]	FEM	Present	Ref. [12]	FEM
1.0	5	0.001267			0.02362			−0.04839		
	10	0.001266			0.02309			−0.05011		
	25	0.001265			0.02344			−0.05125		
	26	0.001265	0.00127	0.001265	0.02344	0.0231	0.02291	−0.05125	−0.0513	−0.05079
1.3	5	0.001915			0.03335			−0.06630		
	10	0.001912			0.03293			−0.06768		
	25	0.001912			0.03296			−0.06855		
	26	0.001912	0.00191	0.001912	0.03296	0.0327	0.03273	−0.06855	−0.0687	−0.06809
1.5	5	0.002202			0.03760			−0.07365		
	10	0.002196			0.03671			−0.07532		
	25	0.002197			0.03697			−0.07552		
	26	0.002197	0.00220	0.002197	0.03697	0.0368	0.03677	−0.07552	−0.0757	−0.07502
1.7	5	0.002391	0.00238	0.002382	0.04017	0.0392	0.03927	−0.07823	−0.0799	−0.07919

	10	0.002383			0.03950			−0.07906		
	25	0.002382			0.03945			−0.07975		
	26	0.002382			0.03945			−0.07975		
2.0	5	0.002548			0.04221			−0.08183		
	10	0.002533			0.04108			−0.08262		
	25	0.002533			0.04109			−0.08275		
	26	0.002533	0.00254	0.002533	0.04109	0.0412	0.04115	−0.08275	−0.0[illegible]29	−0.08222

(2) The second example is on uniformly loaded orthotropic rectangular plates resting on an elastic foundation, where $D_y = 4D_x$, $D_{xy} = 0.85D_x$, $\nu_x = 0.075$, $\nu_y = 0.3$, and $[illegible]/D_x = 100$. t=24 is taken to yield the present convergent solutions. Due to lack of comparable analytic solutions, the present results are only compared with those by FEM, as shown in Table 2, where good agreement for both transverse deflections and bending moments is found.

Table 2. Deflections and bending moments of uniformly loaded orthotropic foundation plates.

$\frac{b}{a}$	$D_xW/(qa^4)$ $(x=a/2, y=b/2)$		$M_x/(qa^2)$ $(x=a/2, y=b/2)$		$M_x/(qa^2)$ $(x=0, y=b/2)$	
	Present	FEM	Present	FEM	Present	FEM
1.0	0.0005047	0.0005047	0.007809	0.007[illegible]44	−0.02749	−0.02734
1.3	0.0009896	0.0009896	0.01560	0.01[illegible]	−0.04272	−0.04246
1.5	0.001298	0.001298	0.02053	0.[illegible]057	−0.05143	−0.05110
1.7	0.001555	0.001555	0.02464	0.0246[illegible]	−0.05826	−0.05787
2.0	0.001835	0.001835	0.02906	[illegible].02909	−0.06508	−0.06467

(3) The final example is on orthotropic rectangular foundation plates under central concentrated loading with intensity P [Fig. 3(a)], which share the same plate and foundation properties with Example 2. t=24 is taken to yield the present convergent solutions. From Table 3, it is seen again that the present solutions agree well with those by FEM. Figure 3(b) plots the non-dimensional 3D deflection of such a plate.

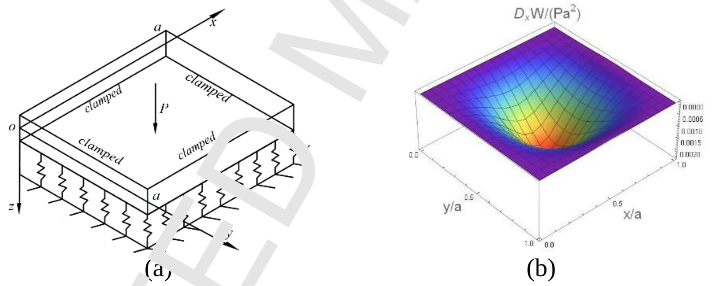


Fig. 3. (a) Schematic and (b) 3D plot of an orthotropic foundation plate under central concentrated loading.

Table 3. Deflections and bending moments of orthotropic foundation plates under central concentrated loading.

$\frac{b}{a}$	$D_xW/([illegible])$ $(x=a/2, y=b/2)$		M_x/P $(x=0, y=b/2)$	
	Present	FEM	Present	FEM
1.0	0.0023[illegible]6	0.002392	−0.04107	−0.04208
1.3	0.003[illegible]25	0.003433	−0.07158	−0.07231
1.5	0.00[illegible]	0.003911	−0.08487	−0.08545
1.7	0.0[illegible]4209	0.004217	−0.09291	−0.09343
2.0	[illegible].00444[illegible]	0.004452	−0.09878	−0.09913

On a workstation with Intel Xeon Processor E5-2697 v4 (x2) (45M Cache, 2.30 GHz), the computational times of the present method in the software Wolfram Mathematica 10.0 *versus* the FEM in ABAQUS 6.13 are 96.30 s *versus* 120 s for example 1, 69.03 s *versus* 117 s for example 2, and 69.3[illegible] s *versus* 119 s for example 3 when b/a=1, for example; the condition numbers of the matrix in Eq. (18) are 174850 for example 1 and 115246 for examples 2 and 3, without warnings of ill-conditioned matrix in calculations. All above examples confirm the validity and accuracy of the two-dimensional generalized finite integral transform method for analyzing the bending problems of orthotropic rectangular thin foundation plates.

4. Conclusions

This work presents a two-dimensional generalized finite integral transform method for new analytic bending solutions of orthotropic rectangular thin foundation plates. The primary advantage of the method is its simplicity and generality in handling a class of complex BVPs of higher-order PDEs as represented by the plate problems; it provides an easy-to-implement tool for exploring more analytic solutions of similar intractable problems.

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