Regularity criteria for 3D shear thinning fluids via two velocity components

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ABSTRACT

Bae & Choe in a 1997 paper established a regularity criteria for the incompressible Navier–Stokes equations in the whole space $\mathbb{R}^3$ based on two velocity components. The goal of this note is to extend their result to the non-Newtonian fluids with shear thinning viscosity.

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1. Introduction

In this paper, we are concerned with the non-Newtonian incompressible fluids which is governed by the following system

$$\begin{cases} \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \operatorname{div} \boldsymbol{\sigma} + \nabla \pi = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \mathbb{R}^3 \times (0, T), \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), & \text{in } \mathbb{R}^3, \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ denotes the unknown velocity of the fluids, π denotes the pressure, and

$$\boldsymbol{\sigma} = |D(\mathbf{u})|^{p-2} D(\mathbf{u}), \quad D(\mathbf{u}) = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right)$$

denotes the shear stress. Since we are concerned with the study of sufficient condition for regularity of weak solutions to the above system, we start by giving the well known definitions of weak and strong solutions to (1).

Definition 1 (*[1,2]*). Let $\mathbf{u}_0 \in L^2_\sigma(\mathbb{R}^3)$. We call $\mathbf{u} \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^p(0, T; W^{1,p}_{0,\sigma}(\mathbb{R}^3))$ a weak solution of (1), if

$$\int_{\mathbb{R}^3 \times (0,T)} -\mathbf{u} \cdot \boldsymbol{\varphi}_t \, dxdt - \int_{\mathbb{R}^3 \times (0,T)} \mathbf{u} \otimes \mathbf{u} : \nabla \boldsymbol{\varphi} \, dxdt$$

$$+ \int_{\mathbb{R}^3 \times (0,T)} |D(\mathbf{u})|^{p-2} D(\mathbf{u}) : D(\boldsymbol{\varphi}) dxdt = \int_{\mathbb{R}^3} \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0) \, dx$$

for all $\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3 \times [0, T))$ satisfying $\operatorname{div} \boldsymbol{\varphi} = 0$ and, furthermore, if the following energy inequality

$$\frac{1}{2} \|\mathbf{u}(t)\|_2^2 + \int_0^t \int_{\mathbb{R}^3} |D(\mathbf{u})|^p dxdt \le \frac{1}{2} \|\mathbf{u}_0\|_2^2, \tag{2}$$

holds for almost all $t \in (0, T)$.

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We use the following notations and definitions: $C_0^\infty(\mathbb{R}^3 \times [0, T))$ and $C_0^\infty(\mathbb{R}^3)$ denote two spaces of smooth functions with compact support, and $C_{0,\sigma}^\infty(\mathbb{R}^3) = \{\boldsymbol{\varphi} \in C_0^\infty(\mathbb{R}^3) | \nabla \cdot \boldsymbol{\varphi} = 0\}$. The norm in $L^p(\mathbb{R}^3)$ will be denoted by $\| \cdot \|_p$, and

$$\| \cdot \|_{W^{k,p}} := \sum_{0 \le |\alpha| \le k} \|\partial^\alpha(\cdot)\|_p, \quad k \ge 1 \,.$$

$L_\sigma^p(\mathbb{R}^3)$ denotes the closure of $C_{0,\sigma}^\infty(\mathbb{R}^3)$ in the norm $\| \cdot \|_p$, and $W_{0,\sigma}^{1,p}(\mathbb{R}^3)$ the closure of $C_{0,\sigma}^\infty(\mathbb{R}^3)$ in the norm $\| \cdot \|_{W^{1,p}}$.

Definition 2 ([3]). Let $\mathbf{u}_0 \in W_{0,\sigma}^{1,2}(\mathbb{R}^3)$. We say that a weak solution $\mathbf{u}$ to (1) is strong if

$$\mathbf{u} \in L^p(0, T; W^{2,p}(\mathbb{R}^3)) \cap L^\infty(0, T; W_{0,\sigma}^{1,2}(\mathbb{R}^3)) \,.$$

The bibliography concerning the system (1) is too large to be dealt here. So, the aim of our references is merely to help the reading of the notes. In particular, we essentially restrict us to the evolution problem, in the case $p \le 2$.

For the mathematical study of Eq. (1) one must recall the pioneering study done by O.A. Ladyzhenskaya in reference [4] where, in particular, existence of strong global solutions with periodic boundary conditions is proved for $p \ge \frac{11}{5}$. The existence of weak solutions of (1) is shown in [4] with the periodic boundary condition, and in [5] in the whole space for $p > \frac{9}{5}$. In [2], J. Wolf showed the existence of weak solutions with Dirichlet boundary condition for $p > \frac{8}{5}$. For $\frac{7}{5} < p \le 2$, i.e. in the shear thinning case, the short time existence results of strong solutions with the periodic boundary condition or in the whole space, are established in [6,7]. Moreover, when $\frac{8}{5} < p \le 2$, Bae et al. [3] obtained the following Prodi–Serrin type regularity criteria.

Lemma 1 (Theorem 2.3, [3]). *Let $\frac{8}{5} < p \le 2$. Suppose that $\mathbf{u}$ is a weak solution to (1) in $[0, T)$. Assume further that*

$$\mathbf{u} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{5p - 6}{2\beta} \le \frac{5p - 8}{2}, \quad \alpha > \frac{6}{5p - 8} \,.$$

Then $\mathbf{u}$ is a strong solution to (1) in $[0, T)$.

For a related shear thickening fluids result, $2 < p < \frac{11}{5}$, see reference [1].

When $p = 2$, the system (1) reduces to the classical Navier–Stokes equations. In this case the global existence of weak solutions goes back to Leray [8] and Hopf [9]. It is one of the most challenging open mathematical problems to prove, or disprove, that weak solutions are necessarily strong under reasonable but general assumptions. In this context, a remarkable and classical sufficient condition for uniqueness and regularity is the so-called Prodi–Serrin condition, namely

$$\mathbf{u} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \frac{2}{\beta} + \frac{3}{\alpha} \le 1, \quad \alpha > 3 \,. \tag{3}$$

Weak solutions satisfying the condition (3) are known to be strong and unique, see [10–13]. In [14], see also [15], the authors proved, in the whole space case, that a solution is strong if merely two components of the velocity satisfy the above condition. Very recently, this result was extended to the half-space in [16], and cylindrical boundaries in [17], both cases under slip boundary conditions. Below we extend the result proved in [3] to the system (1) with $p < 2$, i.e., we will improve Lemma 1. Our main result reads as follows.

Theorem 1. *Let $\mathbf{u}$ be a weak solution to the system (1) with $\frac{8}{5} < p \le 2$. Furthermore, let $\bar{\mathbf{u}} = (u_1, u_2, 0)$, and assume that*

$$\bar{\mathbf{u}} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3)), \quad \frac{3}{\alpha} + \frac{5p - 6}{2\beta} \le \frac{5p - 8}{2}, \quad \alpha > \frac{6}{5p - 8} \,. \tag{4}$$

then $\mathbf{u}$ is a strong solution.

Remark 1. Following [3], and from the following proof, we may show that the above result still holds when $\bar{\mathbf{u}} \in L^\infty(0, T; L^{\frac{6}{5p-8}}(\mathbb{R}^3))$ and $\|\bar{\mathbf{u}}\|_{L^\infty(0,T;L^{\frac{6}{5p-8}})}$ is sufficiently small.

Remark 2. Note that if $p = 2$ we reobtain the result shown in reference [14].

Remark 3. The above choice $\bar{u} = (u_1, u_2, 0)$ is equivalent to say that $\bar{u}$ is a (two dimensional) component of the velocity parallel to a fixed, arbitrary plane.

2. Preliminary results

Lemma 2 ([18,19], Korn inequality). *Let $\mathbf{u} \in W^{1,p}(\mathbb{R}^3)$, for $1 < p < \infty$. Then*

$$\|\nabla \mathbf{u}\|_p \le C \|D(\mathbf{u})\|_p \,.$$

Hereinafter, we define C as being a generic constant which value may vary line by line.

Lemma 3 (*[18,20], generalized Korn inequality*)**.** *Let* $\mathbf{u} \in W^{2,p}(\mathbb{R}^3)$, *for* $1 < p < \infty$. *Then*

$$\|\nabla^2 \mathbf{u}\|_p \leq C \|\nabla D(\mathbf{u})\|_p \,.$$

By virtue of Lemmas 2 and 3, and noting that

$$\sum_{i,j=1}^{3} \nabla(|D|^{p-2} D_{ij}) \cdot \nabla D_{ij} \geq (p-1) \int |D|^{p-2} |\nabla D|^2 \, dx \,,$$

a straightforward calculation implies the following lemma, the details can be found in Lemma 2.1 of [3].

Lemma 4. *Let* $1 < p < 2$. *Suppose* $\mathbf{u} \in W^{2,s}(\mathbb{R}^3)$ *for* $1 < s < 2$, *then*

$$\int_{\mathbb{R}^3} |\nabla^2 \mathbf{u}|^s \, dx \leq C \left(\int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D|^{p-2} D_{ij}) \cdot \nabla D_{ij} \, dx \right)^{\frac{s}{2}} \left(\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^{\frac{(2-p)s}{2-s}} \, dx \right)^{\frac{2-s}{2}}$$

$$\leq C \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D|^{p-2} D_{ij}) \cdot \nabla D_{ij} \, dx + C \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^{\frac{(2-p)s}{2-s}} \, dx \,.$$

Hereinafter, for convenience, we set $D = D(\mathbf{u})$ *and* $D_{ij} = \frac{\partial_i u_j + \partial_j u_i}{2}$.

3. Proof of Theorem 1

Proof. Applying the inner product $\nabla \times (\nabla \times \mathbf{u})$ to (1), we have

$$\int_{\mathbb{R}^3} \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \mathrm{div} \left(|D|^{p-2} D \right) \right) \cdot (\nabla \times (\nabla \times \mathbf{u})) \, dx = 0 \,.$$

After integrating by parts, we easily obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 \, dx + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \nabla \left(|D|^{p-2} D_{ij} \right) \cdot \nabla D_{ij} \, dx = \int_{\mathbb{R}^3} \boldsymbol{\omega} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega} \, dx$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. We have used the following facts:

$$-\Delta \mathbf{u} = \nabla \times (\nabla \times \mathbf{u}), \quad \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2} \nabla |\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}) \,,$$

and

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega} \,.$$

Hence, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 \, dx + \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \nabla \left(|D|^{p-2} D_{ij} \right) \cdot \nabla D_{ij} \, dx \leq \left| \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \omega_j \partial_j u_i \omega_i \, dx \right| \,.$$

Following [14] or [16], we consider separately the three cases $i \neq 3$; $i = 3$ and $j \neq 3$; $i = j = 3$.

Case I: $i \neq 3$. By integration by parts, one has

$$\sum_{j=1}^{3} \int_{\mathbb{R}^3} \omega_j (\partial_j u_i) \omega_i \, dx = -\sum_{j=1}^{3} \int_{\mathbb{R}^3} u_i \omega_j \partial_j \omega_i \, dx \,. \tag{5}$$

Case II: $i = 3$ and $j \neq 3$. By integration by parts, one has

$$\sum_{j=1}^{2} \int_{\mathbb{R}^3} \omega_j (\partial_j u_3) \omega_3 \, dx = \sum_{j=1}^{2} \int_{\mathbb{R}^3} \omega_j \partial_j u_3 (\partial_1 u_2 - \partial_2 u_1) \, dx$$

$$= -\sum_{j=1}^{2} \int_{\mathbb{R}^3} u_2 \left(\partial_1 \omega_j \partial_j u_3 + \omega_j \partial_j \partial_1 u_3 \right) \, dx \tag{6}$$

$$+ \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_1 \left(\partial_2 \omega_j \partial_j u_3 + \omega_j \partial_j \partial_2 u_3 \right) \, dx \,.$$

Case III: $i = j = 3$. Due to div $\mathbf{u} = 0$, one has

$$\int_{\mathbb{R}^3} (\partial_3 u_3)\omega_3^2 \, dx = - \int_{\mathbb{R}^3} (\partial_1 u_1 + \partial_2 u_2) \, \omega_3^2 \, dx$$
$$= 2 \int_{\mathbb{R}^3} u_1 \omega_3 \partial_1 \omega_3 \, dx + 2 \int_{\mathbb{R}^3} u_2 \omega_3 \partial_2 \omega_3 \, dx. \tag{7}$$

Now, from (5)–(7), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 \, dx + \sum_{i,j=1}^{3}\int_{\mathbb{R}^3} \nabla\left(|D|^{p-2}D_{ij}\right) \cdot \nabla D_{ij} \, dx$$
$$\leq C \int_{\mathbb{R}^3} |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}| \, dx. \tag{8}$$

On the other hand, since $\alpha > \frac{6}{5p-8} > \frac{3p}{4p-6} > \frac{p}{p-1}$, we have

$$\int_{\mathbb{R}^3} |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}| dx \leq \|\bar{\mathbf{u}}\|_{\alpha}\|\,|\nabla\mathbf{u}||\nabla^2\mathbf{u}|\,\|_{\frac{\alpha}{\alpha-1}}$$
$$\leq \|\bar{\mathbf{u}}\|_{\alpha}\|\nabla\mathbf{u}\|_{\frac{p\alpha}{p\alpha-p-\alpha}}\|\nabla^2\mathbf{u}\|_p$$
$$\leq \|\bar{\mathbf{u}}\|_{\alpha}\|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}\|\nabla\mathbf{u}\|_{\frac{3p}{3-p}}^{\frac{3(2p+2\alpha-p\alpha)}{(5p-6)\alpha}}\|\nabla^2\mathbf{u}\|_p$$
$$\leq C\|\bar{\mathbf{u}}\|_{\alpha}\|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}\|\nabla^2\mathbf{u}\|_p^{\frac{2p\alpha+6p}{(5p-6)\alpha}}.$$

Hence, for any $0 \leq t_1 < t_2 \leq T$, we have

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}| \, dxdt$$
$$\leq C\left(\int_{t_1}^{t_2}\|\bar{\mathbf{u}}\|_{\alpha}^{\beta}dt\right)^{\frac{1}{\beta}}\left(\int_{t_1}^{t_2}\|\nabla\mathbf{u}\|_2^{\frac{\beta}{\beta-1}\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}\|\nabla^2\mathbf{u}\|_p^{\frac{\beta}{\beta-1}\frac{2p\alpha+6p}{(5p-6)\alpha}} \, dt\right)^{\frac{\beta-1}{\beta}}$$
$$\leq C\left(\int_0^T\|\bar{\mathbf{u}}\|_{\alpha}^{\beta}dt\right)^{\frac{1}{\beta}}\sup_{t\in[t_1,t_2]}\|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}\left(\int_{t_1}^{t_2}\|\nabla^2\mathbf{u}\|_p^{\frac{\beta}{\beta-1}\frac{2p\alpha+6p}{(5p-6)\alpha}} \, dt\right)^{\frac{\beta-1}{\beta}}.$$

Due to (4), we have

$$\frac{\beta}{\beta-1}\frac{2\alpha+6}{(5p-6)\alpha} \leq 1.$$

Therefore

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}| \, dxdt$$
$$\leq C\left(\int_{t_1}^{t_2}\|\bar{\mathbf{u}}\|_{\alpha}^{\beta} \, dt\right)^{\frac{1}{\beta}}\sup_{t\in[t_1,t_2]}\|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}\left(\int_{t_1}^{t_2}\|\nabla^2\mathbf{u}\|_p^p \, dt\right)^{\frac{2\alpha+6}{(5p-6)\alpha}}.$$

In Lemma 4, set $s = p$. By appealing to (2) and Lemma 2, we have

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^3} |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}| \, dxdt$$
$$\leq C\left(\int_{t_1}^{t_2}\|\bar{\mathbf{u}}\|_{\alpha}^{\beta} \, dt\right)^{\frac{1}{\beta}}\sup_{t\in[t_1,t_2]}\|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}\left(\int_{t_1}^{t_2}\|\nabla\mathbf{u}\|_p^p \, dt\right)^{\frac{(2-p)(\alpha+3)}{5p-6}}$$
$$\times \left(\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\nabla(|D|^{p-2}D_{ij}) \cdot \nabla D_{ij} \, dxdt\right)^{\frac{p(\alpha+3)}{5p-6}}$$
$$\leq C\left(\int_{t_1}^{t_2}\|\bar{\mathbf{u}}\|_{\alpha}^{\beta} \, dt\right)^{\frac{1}{\beta}}\sup_{t\in[t_1,t_2]}\|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}$$
$$\times \left(\int_{t_1}^{t_2}\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\nabla(|D|^{p-2}D_{ij}) \cdot \nabla D_{ij} \, dxdt\right)^{\frac{p(\alpha+3)}{(5p-6)\alpha}}. \tag{9}$$

Thus, from (8) and (9), it follows that

$$\|\nabla\mathbf{u}(t_2)\|_2^2 + \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D|^{p-2}D_{ij}) \cdot \nabla D_{ij}\, dxdt$$

$$\leq C \left(\int_{t_1}^{t_2} \|\bar{\mathbf{u}}\|_\alpha^\beta dt\right)^{\frac{1}{\beta}} \sup_{t\in[t_1,t_2]} \|\nabla\mathbf{u}\|_2^{\frac{8p\alpha-12\alpha-6p}{(5p-6)\alpha}}$$

$$\times \left(\int_{t_1}^{t_2} \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D|^{p-2}D_{ij}) \cdot \nabla D_{ij}\, dxdt\right)^{\frac{p(\alpha+3)}{(5p-6)\alpha}} + \|\nabla\mathbf{u}(t_1)\|_2^2.$$

Since $\bar{\mathbf{u}} \in L^\beta(0, T; L^\alpha(\mathbb{R}^3))$ and $\frac{4p\alpha-6\alpha-3p}{(5p-6)\alpha} + \frac{p(\alpha+3)}{(5p-6)\alpha} = 1$, we have

$$\sup_{t\in[0,T]} \|\nabla\mathbf{u}(t)\|_2^2 + \int_0^T \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \nabla(|D|^{p-2}D_{ij}) \cdot \nabla D_{ij}\, dxdt \leq \|\nabla\mathbf{u}_0\|_2^2.$$

See the proof of Theorem 2.3 in [3] for the details. Now from Lemma 4, one has

$$\sup_{t\in[0,T]} \|\nabla\mathbf{u}(t)\|_2^2 + \int_0^T \int_{\mathbb{R}^3} |\nabla^2\mathbf{u}|^p\, dxdt \leq C \int_0^T \int_{\mathbb{R}^3} |\nabla\mathbf{u}|^p\, dxdt + \|\nabla\mathbf{u}_0\|_2^2 \leq C.$$

Thus, the proof of Theorem 1 is complete.

Acknowledgment

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