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Energy estimates and local well-posedness of 3D interfacial hydroelastic waves between two incompressible fluids

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Abstract

In the current paper, we are concerned with waves propagating through the deformation of a thin elastic sheet between two incompressible and inviscid fluids, which are usually called hydroelastic waves in the literature to model deformable sheets interacting with surrounding fluids. The main purpose of the present study is to solve a basic question on the theoretical side, i.e. the local well-posedness issue. The problem is formulated by the full Euler equations (without the assumption of irrotationality) for fluids, combined with the Plotnikov-Toland model for the elastic sheet. Based on geometric considerations, we derive energy estimates and prove the local existence and uniqueness of solution for this system in n ($\geqslant 2$) dimensions even if velocity fields are rotational.

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Keywords: Energy estimates; Well-posedness; Hydroelastic wave; Rotational

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1. Introduction

1.1. Mathematical formulation

This paper is devoted to theoretical studies of a free boundary problem. We first obtain local-in-time energy estimates, and then prove the local existence and uniqueness of hydroelastic waves. We adopt the elastic model proposed by Plotnikov & Toland in [34], but do not assume the irrotationality of the flow.

We focus on interfacial waves between two incompressible and inviscid fluids that occupy domains Ω_t^+ and Ω_t^- in $\mathbb{R}^n$ ($n \geqslant 2$) at time t. Assume that $\mathbb{R}^n = \Omega_t^+ \cup \Omega_t^- \cup S_t$ where $S_t = \partial\Omega_t^{\pm}$, and let the unknown functions $p_\pm$ and $v_\pm$, and constants $\rho_\pm > 0$ denote the pressure, velocity vector field, and density, respectively. On the interface S_t, we let $N_\pm(t,x)$ denote the unit normal vectors to $\Omega_t^{\pm}$ pointing outward (thus $N_+ + N_- = 0$), $H(t,x) \in (T_x S_t)^\perp$ denote the mean curvature vector, and $\kappa_\pm = H \cdot N_\pm$. Thus the motions of fluids away from the interface is governed by the Euler equations

$$\begin{cases} v_t + \nabla_v v = -\nabla p, & x \in \mathbb{R}^n \setminus S_t\,, \\ \nabla \cdot v = 0, & x \in \mathbb{R}^n \setminus S_t\,. \end{cases} \tag{1.1}$$

The boundary conditions for the evolution of the interface and the pressure jump are given by

$$\begin{cases} \mathbf{D}_t = \partial_t + v \cdot \nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbb{R}^{n+1}\,, \\ p_+(t,x) - p_-(t,x) = \kappa_{E,+}(t,x), \quad x \in S_t\,. \end{cases} \tag{1.2}$$

Here $\kappa_{E,\pm}$ are defined as

$$\kappa_{E,\pm}(t,x) = -\Delta_{S_t}\kappa_\pm(t,x) + \left[-\frac{1}{2}\kappa_\pm^3(t,x) + 2\kappa_\pm(t,x)\sigma(t,x)\right] \triangleq \kappa_{E,\pm}^h + \kappa_{E,\pm}^l \tag{1.3}$$

with the Guass curvature $\sigma(t,x)$ at the interface S_t, where Δ_{S_t} is the Beltrami-Laplace on S_t. It is obvious that $\kappa_+ = -\kappa_-$, hence $\kappa_{E+} = -\kappa_{E-}$. In the following, we also write κ_+ as κ, and κ_{E+} as κ_E.

Throughout this paper, we neglect the effect of gravity as it only contributes to lower-order terms in energy estimates (the interested reader is referred to the discussion of Section 6 in [37] for more details). For the sake of convenience, S_t is assumed to be compact, and the same results also hold if we assume it is asymptotically flat.

1.2. Known results

The free boundary problem for the Euler equations, i.e. free-surface water waves, is a classical problem in fluid dynamics, and has been studied extensively. The first basic question is the local well-posedness, which has been proved to be highly non-trivial due to the complicated nature of the equations. Early results on local well-posedness of pure gravity waves can only deal with small perturbations of a flat surface (see Nalimov [32], Shinbrot [38], Yosihara [46,47] and Craig [12]). In recent years, a breakthrough for handling the local well-posedness with general initial data is due to Wu [42,43] for irrotational flows. Later on, Christodoulou & Lindblad [10] and

Lindblad [28] considered the problem with vorticity, Beyer & Gunther [8] took into account the effects of surface tension, and Lannes [27] treated the problem with variable bottom topography. Furthermore, Ambrose & Masmoudi [5,6], Coutand & Shkoller [14], and Shatah & Zeng [35–37] extended these results to the two-fluid system where surface tension is necessary to control the Kelvin-Helmholtz instability. In terms of the local regularity problem, Christianson et al. [11] and Alazard et al. [1] proved recently a nonlinear smoothing effect arising directly from the dispersive property of free-surface water waves.

Another important problem in this research field is the global existence of solutions and relatively fewer results have been obtained. All the results in this aspect were obtained for irrotational flows. The first progress was made by Wu [44], who proved the almost global existence of pure gravity waves in two dimensions. Later on, Germain et al. [16] and Wu [45] independently proved the global well-posedness of gravity waves in three dimensions. Wu's two-dimensional almost global result was also improved to the global result by the independent work of Ionescu & Pusateri [23] and Alazard & Delort [2,3]. Recently, Ifrim & Tataru provided shorter and simpler proofs for two-dimensional gravity waves, first for the almost global well-posedness result in joint work with Hunter [19], and then for the global result in [20]. When the surface tension effect is much stronger than gravity, Germain et al. [17] proved the global existence for three-dimensional pure capillary wave problem. And later on, the two-dimensional capillary wave problem was also shown to have global solutions by Ifrim & Tataru [21]. A similar result was obtained by Ionescu & Pusateri [22], based on a different argument. When gravity and surface tension are equally important, Deng et al. [13] proved the global regularity for capillary-gravity waves in three dimensions. On the other hand, some mathematicians were concerned with large initial data problems which lead to the finite-time breakdown (see, for example, the papers on 'splash' singularities [9,15]). The interested reader is referred to [24] for more results on the local and global existence theories for the initial value problem of water waves.

In the present paper, we are interested in hydroelastic waves which describe interactions between elastic sheet and hydrodynamics. This problem is important in biology, medicine and ocean engineering (see [4,25,30,31,33,39,40] and references therein). Korobkin et al. [26] summarizes recent work on analyses, numerical simulations and applications of hydroelastic waves. The mathematical description of the problem is similar to the classic water-wave problem, but with the restoring forces due to gravity or surface tension replaced by the flexural elasticity. A nonlinear model describing the deformation of a thin and heavy elastic sheet was recently proposed by Toland [41] and Plotnikov & Toland [34] for two and three dimensions respectively. Their derivation is based on the Cosserat theory of shells satisfying Kirchhoff's hypothesis. This new model has a clear elastic potential energy which is equivalent to the Willmore functional. For hydroelastic waves, most of the theoretical results till now were obtained in two-dimensional flows. Of note is the work of Groves et al. [18] who proved the existence of solitary waves in relatively shallow fluids, Ambrose & Siegel [7] who proved the local well-posedness in potential flows, and Liu & Ambrose [29] who proved the local well-posedness when the inertial effect is also taken into account. In this paper, we consider a more general case, that is, the local well-posedness of hydroelastic waves with vorticity in dimension n ($\geqslant 2$).

1.3. Main results

We now state in precise the main results of this paper.

Theorem 1.1. *Given initial surface* $S_0 \in H^{\frac{5}{2}k+2}$ *and initial velocity* $v_0 \in H^{\frac{5}{2}k}(\Omega_0)$ *with* $\frac{5}{2}k > \frac{n}{2} + 1$*, then the problem* (1.1)-(1.2) *has a solution in the space*

$$S_t \in C^0\left([0,T], H^{\frac{5}{2}k+2}\right) \quad \textit{and} \quad v \in C^0\left([0,T], H^{\frac{5}{2}k}(\Omega_t)\right)$$

for some small time interval $[0,T]$*, and the energy estimate* (3.30) *holds. If* $k > 2$*, then the problem is locally well-posed, that is, the solution is unique and depends continuously on the initial data.*

1.4. Difficulties

As mentioned above, the local well-posedness theory for water waves has been extensively investigated by various groups using different formulations. For our problem, the flexural elasticity κ_E is very complicated and involves a lot of geometry. This observation motivates us to use the geometric point of view due to Shatah & Zeng [35–37] to formulate the problem. In contrast to pure capillary waves, we need to handle the flexural elasticity κ_E in the hydroelastic wave problem instead of the mean curvature κ. Since κ_E introduces a higher-order term in comparison with κ, the geometric calculations related to κ_E are more complicated, and the following difficulties distinguish our problem from pure capillary waves.

- For the capillary wave problem, Shatah and Zeng reduced the Euler equations with a free boundary to the evolution equation of the mean curvature κ. For the hydroelastic wave problem, if we perform the same transformation, then $S_t \in \Lambda\left(S_0, \frac{5}{2}k + \frac{1}{2}, \delta\right)$ is required for the local well-posedness. However, from the regularity of Lagrangian coordinate maps, we can only obtain $S_t \in \Lambda\left(S_0, \frac{5}{2}k - \frac{1}{2}, \delta\right)$ at most. We have to reduce the Euler equations with free boundary to the equation of $\kappa_E^h = -\Delta_{S_t}\kappa$. Since the geometric formulas involved the higher-order term κ_E^h is much more complicated, it will be more difficult to distinguish these formulas' leading-order terms and lower-order terms.
- In [36], it is enough to reduce the system to the following form

$$\partial_t^2\kappa - \Delta_{S_t}\mathcal{N}\kappa = \text{the lower-order terms},$$

  where $\mathcal{N}$ is the Dirichlet-Neumann operator. For our problem, the corresponding equation reads

$$\partial_t^2\kappa_E^h + \Delta_{S_t}^2\mathcal{N}\kappa_E^h = \text{the lower-order terms}.$$

  However, in the process of deriving energy estimates, we find that the integral arising from the lower-order terms contains a derivative out of control. We will overcome this difficulty by constructing an auxiliary 'Energy' to cancel these terms.

The rest of the paper is organized as follows. In Section 2 we introduce notations used throughout the paper. In Section 3 we focus on a priori estimates. In Section 4, we provide the proof of the local well-posedness. Some detailed calculations associated with geometric properties of the Dirichlet-Neumann operators $\mathcal{N}$ and $\mathcal{N}_\pm$ that can be found in [35,36], are omitted.

2. Notations

- A^*: adjoint operator of an operator
- q_+: the quantity defined on Ω_t^+
- q_-: the quantity defined on Ω_t^-
- $\perp$ and $\top$: the normal and the tangential components of the relevant quantities
- $\Delta_\pm^{-1}$: the inverse Laplacian with zero Dirichlet data in $\Omega_t^\pm$
- $\mathcal{H}_\pm$: the harmonic extension of functions defined on S_t into $\Omega_t^\pm$
- $\mathcal{N}_\pm$: the Dirichlet-Neumann operators in the domain $\Omega_t^\pm$
- $\mathcal{N} := \frac{1}{\rho^+}\mathcal{N}_+ + \frac{1}{\rho^-}\mathcal{N}_-$, where $\rho_\pm$ are the densities in $\Omega_t^\pm$
- $\mathcal{N}^{-1}$: the inverse of the operator $\mathcal{N}$
- $\mathcal{D}$: the covariant differentiation on S_t
- $\mathbf{D}_t = \partial_t + v \cdot \nabla$: the material derivative along the particle path.
- $\Pi_\pm$: the second fundamental form of S_t associated with $N_\pm$
- $\Lambda\left(S_0, \frac{5k}{2} - \frac{1}{2}, \delta\right)$: the collection of all hypersurfaces $\tilde{S}$ such that a diffeomorphism $F : S_0 \to \tilde{S}$ exists with $|F - id_{S_0}|_{H^{\frac{5}{2}k-\frac{1}{2}}(S_0)} < \delta$, where S_0 is a given hypersurface.

3. A priori estimates

3.1. Preliminary results

3.1.1. Material derivative $\mathbf{D}_t$

In this section, we recall some expressions involved the material derivative $\mathbf{D}_t = \partial_t + v \cdot \nabla$. In [35], the authors obtained the following formulas:

$$\mathbf{D}_t N_\pm = -((Dv_\pm)^*(N_\pm))^\top . \tag{3.1}$$

Henceforth, the script $\pm$ stands for $+$ or $-$ corresponding to the quantities in $\Omega_t^\pm$ respectively. For example, identity (3.1) implies

$$\mathbf{D}_t N_+ = -((Dv_+)^*(N_+))^\top , \quad \mathbf{D}_t N_- = -((Dv_-)^*(N_-))^\top .$$

On the hypersurface S_t, we have

$$\mathbf{D}_t dS = (v_\pm^\perp \kappa_\pm + \mathcal{D} \cdot v_\pm^\top) dS . \tag{3.2}$$

For $\mathbf{D}_t \kappa_\pm$, we have

$$\begin{aligned} \mathbf{D}_t \kappa_\pm &= -\Delta_{S_t} v_\pm \cdot N_\pm - 2\Pi_\pm \cdot ((D^\top|_{TS_t}) v_\pm) \\ &= -\Delta_{S_t} v_\pm^\perp - v_\pm^\perp |\Pi_\pm|^2 + (\mathcal{D} \cdot \Pi_\pm)(v_\pm^\top) , \end{aligned} \tag{3.3}$$

and for any smooth function $f(t,x)$, $x \in S_t$, we have

$$[\mathbf{D}_t, \mathcal{H}_\pm] f = \Delta_\pm^{-1}(2Dv_\pm \cdot D^2 f_{\mathcal{H}_\pm} + \nabla f_{\mathcal{H}_\pm} \cdot \Delta v_\pm) , \tag{3.4}$$

$$[\mathbf{D}_t, \Delta_\pm^{-1}] f = \Delta_\pm^{-1}(2Dv_\pm \cdot D^2 \Delta_\pm^{-1} f + \Delta v_\pm \cdot \nabla \Delta_\pm^{-1} f) , \tag{3.5}$$

$$[\mathbf{D}_t, \mathcal{N}_\pm]f = \nabla_{N_\pm}\Delta_\pm^{-1}(2Dv_\pm \cdot D^2 f_{\mathcal{H}_\pm} + \nabla f_{\mathcal{H}_\pm} \cdot \Delta v_\pm) - \nabla f_{\mathcal{H}_\pm} \cdot \nabla_{N_\pm} v_\pm - \nabla_{\nabla^\top f} v_\pm \cdot N_\pm \,, \tag{3.6}$$

$$[\mathbf{D}_t, \Delta_{S_t}]f = -2\mathcal{D}^2 f \cdot ((D^\top|_{TS_t})v_\pm) - \nabla^\top f \cdot \Delta_{S_t} v_\pm + \kappa_\pm \nabla_{\nabla^\top f} v_\pm \cdot N_\pm \,. \tag{3.7}$$

Note that,

$$\mathbf{D}_t \kappa^h_{E\pm} = \mathbf{D}_t(-\Delta_{S_t}\kappa_\pm) = -\Delta_{S_t}\mathbf{D}_t\kappa_\pm - [\mathbf{D}_t, \Delta_{S_t}]\kappa_\pm \,,$$

from (3.3) and (3.7), we have

$$\begin{aligned}
\mathbf{D}_t \kappa^h_{E\pm} =& - \Delta_{S_t}\mathbf{D}_t\kappa_\pm - [\mathbf{D}_t, \Delta_{S_t}]\kappa_\pm \\
=&\Delta_{S_t}\left(\Delta_{S_t} v_\pm^\perp + v^\perp|\Pi_\pm|^2 - \nabla_{v_\pm^\top}\kappa_\pm\right) + 2\mathcal{D}^2\kappa_\pm \cdot ((D^\top|_{TS_t})v_\pm) \\
&+ \nabla^\top \kappa_\pm \cdot \Delta_{S_t} v_\pm - \kappa_\pm \nabla_{\nabla^\top \kappa_\pm} v_\pm \cdot N_\pm \\
=&\Delta^2_{S_t} v_\pm^\perp + \Delta_{S_t}(v_\pm^\perp|\Pi_\pm|^2) - \Delta_{S_t}\left(\nabla_{v_\pm^\top}\kappa_\pm\right) + 2\mathcal{D}^2\kappa_\pm \cdot ((D^\top|_{TS_t})v_\pm) \\
&+ \nabla^\top \kappa_\pm \cdot \Delta_{S_t} v_\pm - \kappa_\pm \nabla_{\nabla^\top \kappa_\pm} v_\pm \cdot N_\pm \\
:=&\Delta^2_{S_t} v_\pm^\perp + r \,.
\end{aligned} \tag{3.8}$$

3.1.2. The expression of pressure

In this part, we explain how to express the pressure in terms of the velocity. The most of the calculations is the same as [36]. For the sake of completeness, we give details of calculations.

Taking the dot product of Euler equations (1.1) with $N_\pm$,

$$-N_\pm \cdot \nabla p_\pm = \rho_\pm \mathbf{D}_{t\pm}(v_\pm \cdot N_\pm) - \rho_\pm v_\pm \cdot \mathbf{D}_{t\pm} N_\pm \,.$$

By using the fact that $v_+^\perp + v_-^\perp = 0$, we obtain

$$\frac{1}{\rho_+}\nabla_{N_+} p_+ + \frac{1}{\rho_-}\nabla_{N_-} p_- = v_+ \cdot \mathbf{D}_{t+} N_+ + v_- \cdot \mathbf{D}_{t-} N_- - \nabla_{v_+^\top - v_-^\top} v_+^\perp \,.$$

From (3.1), we have

$$\frac{1}{\rho_+}\nabla_{N_+} p_+ + \frac{1}{\rho_-}\nabla_{N_-} p_- = \Pi_+(v_+^\top, v_+^\top) + \Pi_-(v_-^\top, v_-^\top) - 2\nabla_{v_+^\top - v_-^\top} v_+^\perp \,.$$

Since $p_\pm = \mathcal{H}_\pm(p_\pm|_{S_t}) + \Delta_\pm^{-1}\Delta p_\pm$ in $\Omega_t^\pm$, on S_t we have

$$\begin{aligned}
&\frac{1}{\rho_+}\nabla_{N_+} p_+ + \frac{1}{\rho_-}\nabla_{N_-} p_- \\
&= - \frac{1}{\rho_+}\nabla_{N_+}\Delta_+^{-1}\Delta p_+ - \frac{1}{\rho_-}\nabla_{N_-}\Delta_-^{-1}\Delta p_- \\
&\quad + \Pi_+(v_+^\top, v_+^\top) + \Pi_-(v_-^\top, v_-^\top) - 2\nabla_{v_+^\top - v_-^\top} v_+^\perp \,.
\end{aligned}$$

The boundary condition $p_+ - p_- = \kappa_{E,+}$ on S_t implies that

$$p_\pm = \mathcal{N}^{-1}\Big[-\frac{1}{\rho_\mp}\mathcal{N}_\mp \kappa_{E,\mp} - \frac{1}{\rho_+}\nabla_{N_+}\Delta_+^{-1}\Delta p_+ - \frac{1}{\rho_-}\nabla_{N_-}\Delta_-^{-1}\Delta p_- + \Pi_+(v_+^\top, v_+^\top) + \Pi_-(v_-^\top, v_-^\top) - 2\nabla_{v_+^\top - v_-^\top} v_+^\perp \Big].$$

Finally, since $\nabla \cdot v = 0$ in $\mathbb{R}^n \setminus S_t$, we have

$$-\Delta p = \rho \nabla \cdot (\nabla_v v) = \rho \operatorname{tr}(Dv)^2 \tag{3.9}$$

for $x \in \mathbb{R}^n \setminus S_t$. Therefore,

$$p_\pm|_{S_t} = \mathcal{N}^{-1}\Big[-\frac{1}{\rho_\mp}\mathcal{N}_\mp \kappa_{E,\mp} - \frac{1}{\rho_+}\nabla_{N_+}\Delta_+^{-1}\operatorname{tr}(Dv)^2 - \frac{1}{\rho_-}\nabla_{N_-}\Delta_-^{-1}\operatorname{tr}(Dv)^2 + \Pi_+(v_+^\top, v_+^\top) + \Pi_-(v_-^\top, v_-^\top) - 2\nabla_{v_+^\top - v_-^\top} v_+^\perp \Big]. \tag{3.10}$$

Note that the quantity inside brackets has zero mean on S_t and thus p is well-defined.

3.2. Lagrangian formulation

We will obtain a priori estimates by the energy method in this section. The main difficulty is to find an appropriate energy, and the following analyses will give a clue on how to construct the 'proper' energy. We first establish the geometric formulation of the problem (1.1)-(1.2) and obtain the linearization of the problem, which explains the motivation for the specific expression of energy. The calculations are essentially same as [36], and the major difference is that the surface tension κ should be replaced by the flexural elasticity κ_E.

Multiplying the Euler equations by $v = v_+ 1_{\Omega_t^+} + v_- 1_{\Omega_t^-}$ and integrating over $\mathbb{R}^n \setminus S_t$ give the conserved energy E_0:

$$E_0 = E_0(S_t, v) = \int_{\mathbb{R}^n \setminus S_t} \frac{\rho |v|^2}{2} dx + \int_{S_t} \frac{1}{2}\kappa^2 dS, \tag{3.11}$$

where $\rho = \rho_+ 1_{\Omega_t^+} + \rho_- 1_{\Omega_t^-}$, and in the following, we write q means $q = q_+ 1_{\Omega_t^+} + q_- 1_{\Omega_t^-}$ for any quantity q defined on $\mathbb{R}^n \setminus S_t$.

For $y \in \Omega_0^\pm$, let $u = u_+ 1_{\Omega_t^+} + u_- 1_{\Omega_t^-}$ be the Lagrangian coordinate map solving

$$\frac{dx}{dt} = v(t, x), \quad x(0) = y, \tag{3.12}$$

then we have $v = u_t \circ u^{-1}$, and for any vector field w on $x \in \mathbb{R}^n \setminus S_t$, $\mathbf{D}_t w = (w \circ u)_t \circ u^{-1}$. Therefore, in Lagrangian coordinates, the Euler equations take the form

$$\rho u_{tt} = -(\nabla p) \circ u \quad \text{and} \quad u(0) = id_{\mathbb{R}^n \setminus S_0}, \tag{3.13}$$

where the pressure p is given by (3.9) and (3.10).

Now, let $\Phi_\pm$ satisfying (i) $\Phi_\pm : \bar{\Omega}_0^\pm \to \Phi_\pm(\bar{\Omega}_0^\pm)$, a volume-preserving homeomorphism, and (ii) $\partial\Phi_\pm(\Omega_0^\pm) = \Phi_\pm(\partial\Omega_0^\pm = S_0)$, and set $\Gamma = \{\Phi = \Phi_+ 1_{\Omega_0^+} + \Phi_- 1_{\Omega_0^-}\}$. Then, as a manifold, the tangent space of Γ is given by divergence-free vector fields with matching normal component in Eulerian coordinates:

$$T_\Phi\Gamma = \{\bar{w} : \mathbb{R}^n \setminus S_0 \to \mathbb{R}^n | \nabla \cdot w = 0 \text{ and } w_+^\perp + w_-^\perp|_{\Phi(S_0)} = 0, \text{ where } w = (\bar{w} \circ \Phi^{-1})\}\,.$$

Denoting $S(\Phi; \kappa^2) = \int_{\Phi(S_0)} \frac{1}{2}\kappa^2 dS$, then the energy E_0 in Lagrangian coordinates can be rewritten as

$$E_0 = E_0(u, u_t) = \frac{1}{2} \int\limits_{\mathbb{R}^n \setminus S_0} \rho |u_t|^2 dy + S(u; \kappa^2)\,, \quad (u, u_t) \in T\Gamma\,, \tag{3.14}$$

where the volume-preserving property of u is used. This conservation of energy suggests: (1) $T\Gamma$ is endowed with the $L^2(\rho dy)$ metric, and (2) the free boundary problem of the Euler equations has a Lagrangian action

$$I(u) = \frac{1}{2} \iint\limits_{\mathbb{R}^n \setminus S_0} \rho |u_t|^2 dx dt - \int S(u; \kappa^2) dt\,, \quad u(t, \cdot) \in \Gamma\,.$$

Let $\bar{\mathscr{D}}$ denote the covariant derivative associated with the metric on Γ, and a critical path $u(t, \cdot)$ of $I(u)$ satisfies

$$\bar{\mathscr{D}}_t u_t + S'(u; \kappa^2) = 0\,. \tag{3.15}$$

Next, we will verify that the Lagrangian coordinate map $u(t, \cdot)$ of a solution of (1.1) and (1.2) is indeed a critical path of $I(u)$.

We first recall Hodge decomposition. For any vector field X defined on $\Phi(\mathbb{R}^n \setminus S_0)$, we have Hodge decomposition: $X = w - \nabla\psi$ with $\psi = \psi_+ 1_{\Omega_0^+} + \psi_- 1_{\Omega_0^-}$, where $\bar{w} = w \circ \Phi \in T_\Phi\Gamma$ and $\nabla\psi \circ \Phi \in (T_\Phi\Gamma)^\perp$. Thanks to [36], we have

$$(T_\Phi\Gamma)^\perp = \{-(\nabla\psi) \circ \Phi | \rho_+\psi_+ = \rho_-\psi_- \text{ on } \Phi(S_0)\}$$

and

$$\begin{cases} -\Delta\psi = \nabla \cdot X\,, \\ \psi_\pm|_{\Phi(S_0)} = \frac{1}{\rho_\pm}\psi^S \\ \qquad = -\frac{1}{\rho_\pm}\mathcal{N}^{-1}(X_+^\perp + X_-^\perp - \nabla_{N_+}\Delta_+^{-1}\Delta\nabla \cdot X - \nabla_{N_-}\psi_-\Delta_-^{-1}\Delta\nabla \cdot X)\,, \end{cases} \tag{3.16}$$

where $\psi^S \triangleq \rho_+\psi_+ = \rho_-\psi_-$.

For a given path $u(t, \cdot) \in \Gamma$, let $\bar{v} = u_t$, $S_t = u(t, S_0)$, and $\bar{w}(t, \cdot) \in T_{u(t)}\Gamma$. Then the covariant derivative $\bar{\mathscr{D}}_t\bar{w}$, $\Pi_{u(t)}(\bar{w}, \bar{v})$, and the second fundamental form respectively satisfy

$$\bar{w}_t = \bar{\mathscr{D}}_t\bar{w} + \Pi_{u(t)}(\bar{w}, \bar{v})\,, \quad \bar{\mathscr{D}}_t\bar{w} \in T_{u(t)}\Gamma\,, \quad \Pi_{u(t)}(\bar{w}, \bar{v}) \in (T_{u(t)}\Gamma)^\perp\,.$$

Let $v = u_t \circ u^{-1} = \bar{v} \circ u^{-1}$ and $w = \bar{w} \circ u^{-1}$. From the above Hodge decomposition, for $X = \mathbf{D}_t w$, there exists $p_{w,v} = p^+_{w,v} 1_{\Omega_t^+} + p^-_{w,v} 1_{\Omega_t^-} : \mathbb{R}^n \setminus u(t, S_0) :\to \mathbb{R}$ determined by (3.16) such that

$$\rho_+ p^+_{w,v} = \rho_- p^-_{w,v} \text{ on } S_t , \quad \Pi_{u(t)}(\bar{w}, \bar{v}) = -(\nabla p_{w,v}) \circ u \in (T_{u(t)}\Gamma)^\perp .$$

Hence, in Eulerian coordinates, the expression of the covariant derivative is

$$\mathscr{D}_t w = (\bar{\mathscr{D}}_t \bar{w}) \circ u^{-1} = \mathbf{D}_t w + \nabla p_{w,v} ,$$

where $p_{w,v}$ is given by

$$\begin{cases} -\Delta p_{w,v} = tr(DvDw) , \\ p^\pm_{w,v}|_{S_t} = \frac{1}{\rho_\pm} p^S_{w,v} = \\ \qquad -\frac{1}{\rho_\pm} \mathcal{N}^{-1} [\nabla_{v_+^\top - v_-^\top} w_+^\perp + \nabla_{w_+^\top - w_-^\top} v_+^\perp - \Pi_+(v_+^\top, w_+^\top) - \Pi_-(v_-^\top, w_-^\top) \\ \qquad\qquad -\nabla_{N_+} \Delta_+^{-1} tr(DvDw) - \nabla_{N_-} \Delta_-^{-1} tr(DvDw)] . \end{cases} \tag{3.17}$$

From the divergence decomposition formula,

$$0 = \nabla \cdot v_\pm = \mathcal{D} \cdot v_\pm^\top + \kappa_\pm v_\pm^\perp + N_\pm \cdot \nabla_{N_\pm} v_\pm \quad \text{on } S_t , \tag{3.18}$$

where $\mathcal{D}$ is the covariant derivative on S_t. Hence, we have

$$\nabla_{w_\pm} v_\pm \cdot N_\pm = \nabla_{w_\pm^\top} v_\pm \cdot N_\pm - \kappa_\pm w_\pm^\perp v_\pm^\top - \mathcal{D} \cdot (w_\pm^\perp v_\pm^\top) + \nabla_{v_\pm^\top} w_\pm^\perp .$$

Thus, we can also write $p^S_{w,v}$ as follows:

$$\begin{aligned} p^S_{w,v} = -\mathcal{N}^{-1} \{ & \nabla_{w_+} v_+ \cdot N_+ + \nabla_{w_-} v_- \cdot N_- + \mathcal{D} \cdot (w_+^\perp (v_+^\top - v_-^\top)) \\ & - \nabla_{N_+} \Delta_+^{-1} tr(DvDw) - \nabla_{N_-} \Delta_-^{-1} tr(DvDw) \} . \end{aligned}$$

Moreover, for any smooth function f defined on S_t, from the divergence theorem we have

$$\int_{S_t} -f \nabla_{N_\pm} \Delta_\pm^{-1} tr(Dv_\pm Dw_\pm) dS = - \int_{\Omega_t^\pm} \nabla f_{\mathcal{H}_\pm} \cdot \nabla \Delta_\pm^{-1} tr(Dv_\pm Dw_\pm) + f_{\mathcal{H}_\pm} tr(Dv_\pm Dw_\pm) dx .$$

Again, by the divergence theorem, the first term vanishes, and the second term can be rewritten as

$$\begin{aligned} & \int_{S_t} -f \nabla_{N_\pm} \Delta_\pm^{-1} tr(Dv_\pm Dw_\pm) dS = \\ & \qquad \int_{\Omega_t^\pm} -f \nabla_{w_\pm} v_\pm \cdot N_\pm + w_\pm^\perp \nabla f_{\mathcal{H}_\pm} \cdot v_\pm dS - \int_{\Omega_t^\pm} D^2 f_{\mathcal{H}_\pm}(v_\pm, w_\pm) dx . \end{aligned} \tag{3.19}$$

Thus, using the decomposition $\nabla f_{\mathcal{H}_\pm} = \nabla^\top f + (\mathcal{N}_\pm f)N_\pm$ and letting $f = -\mathcal{N}^{-1}g$, we have

$$\int_{S_t} g p^S_{w,v} dS = \int_{S_t} -w_+^\perp v_+^\perp (\mathcal{N}_+ + \mathcal{N}_-)\mathcal{N}^{-1} g dS + \int_{\mathbb{R}^n\setminus S_t} D^2(\mathcal{H}_\pm(\mathcal{N}^{-1}g))(v,w)dx\,. \tag{3.20}$$

Now, we compute $S'(u;\kappa^2)$. By a variation of the mean curvature formula, see [34], for any $\bar{w} \in T_\Phi\Gamma$, we have

$$\langle S'(u;\kappa^2), \bar{w}\rangle_{L^2(\mathbb{R}^n\setminus S_0,\rho dy)} = \int_{S_t} \kappa_{E+} w_+^\perp dS = \int_{S_t} \kappa_{E-} w_-^\perp dS\,. \tag{3.21}$$

It follows from Lemma 3.1 in [36] that

$$S'(u;\kappa^2) = \nabla p_{\kappa_E} = \nabla p_{\kappa_E^h} + \nabla p_{\kappa_E^l}\,, \tag{3.22}$$

where $p_{\kappa_E} = p_{\kappa_{E+}} 1_{\Omega_t^+} + p_{\kappa_{E-}} 1_{\Omega_t^-}$ and

$$p_{\kappa^h_{E\pm}} = \frac{1}{\rho_+\rho_-}\mathcal{H}_\pm \mathcal{N}^{-1}\mathcal{N}_\mp \kappa^h_{E,\pm}\,, \quad p_{\kappa^l_{E\pm}} = \frac{1}{\rho_+\rho_-}\mathcal{H}_\pm \mathcal{N}^{-1}\mathcal{N}_\mp \kappa^l_{E,\pm}\,.$$

From the above calculations, we have $\rho(p_{v,v} + p_{\kappa_E}) = p$. Therefore, we have obtained the equivalence between equation (3.15) for critical paths of I and the Euler equation (1.1) with the free boundary condition (1.2). In particular, the Euler equations can also be written as

$$\mathbf{D}_t v + \nabla p_{v,v} + \nabla p_{\kappa_E} = 0\,. \tag{3.23}$$

3.2.1. Linearization

Now we start with the linearization of the problem. From (3.15), the linearized equation takes the form

$$\bar{\mathscr{D}}_t^2 \bar{w} + \bar{\mathscr{R}}(u_t,\bar{w})u_t + \bar{\mathscr{D}}^2 S(u;\kappa^2)(\bar{w}) = 0\,, \quad \bar{w}(t,\cdot) \in T_{u(t,\cdot)}\Gamma\,, \tag{3.24}$$

where $\bar{\mathscr{R}}$ is the curvature tensor of the infinite-dimensional manifold Γ.

Next, we compute the leading-order terms of $\bar{\mathscr{R}}(u_t,\bar{w})u_t$ and $\bar{\mathscr{D}}^2 S(u;\kappa^2)(\bar{w})$. The leading-order term of $\bar{\mathscr{R}}(u_t,\bar{w})u_t$ had been obtained by Shatah and Zeng [36]. Let

$$\begin{cases} \mathscr{R}_0(v)(w) = \nabla f_+ 1_{\Omega_t^+} + \nabla f_- 1_{\Omega_t^-}\,, \\ f_\pm = \frac{1}{\rho_+\rho_-}\mathcal{H}_\pm \mathcal{N}^{-1}\mathcal{N}_\mp \nabla_{v_+^\top - v_-^\top}\mathcal{N}^{-1}\mathcal{D}\cdot(w_\pm^\perp(v_+^\top - v_-^\top))\,, \end{cases}$$

they proved that

$$\bar{\mathscr{R}}(u)(\bar{v},\bar{w})\bar{v} = \bar{\mathscr{R}}_0(\bar{v}) + \text{at most first-order differential operators}\,. \tag{3.25}$$

We now compute the leading-order term of $\bar{\mathscr{D}}^2 S(u;\kappa^2)(\bar{w})$. Differentiating (3.21) yields

$$\bar{\mathscr{D}}^2 S(u;\kappa^2)(\bar{w},\bar{w}) = \frac{d}{dt}\int_{S_t} \kappa_{E\pm} w_\pm \cdot N_\pm dS.$$

Substituting the expressions of $\mathbf{D}_t N$, $\mathbf{D}_t dS$ and $\mathbf{D}_t \kappa_{E\pm}$ gives

$$\begin{aligned}\bar{\mathscr{D}}^2 S(u;\kappa^2)(\bar{w},\bar{w}) = &\int_{S_t} \kappa_{E\pm} w_\pm^\perp (\kappa_\pm w_\pm^\perp + \mathcal{D}\cdot w_\pm^\top) + \kappa_{E\pm}\mathbf{D}_t w_\pm \cdot N_\pm \\ &+ \kappa_{E\pm} w_\pm \cdot \mathbf{D}_t N_\pm + w_\pm^\perp \mathbf{D}_t \kappa_{E\pm} dS \\ = &\int_{S_t} \kappa_{E\pm} w_\pm^\perp (\kappa_\pm w_\pm^\perp + \mathcal{D}\cdot w_\pm^\top) - \kappa_{E\pm}\nabla_{N\pm} p_{w,w}^\pm \\ &- \kappa_{E\pm}\nabla_{w_\pm^\top} w_\pm \cdot N + w_\pm^\perp (\Delta_{S_t}^2 w_\pm^\perp + r + \mathbf{D}_t \kappa_{E\pm}^l) dS\,,\end{aligned}$$

where r is given in (3.8). Noting that $\kappa_{E\pm}^l = -\frac{1}{2}\kappa_\pm^3(t,x) + 2\kappa_\pm(t,x)\sigma(t,x)$, hence $\kappa_{E\pm}^l$ have the same regularity with κ. Therefore, by using the divergence theorem, we have

$$\left|\bar{\mathscr{D}}^2(\bar{w},\bar{w}) - \int_{S_t} |\Delta_{S_t} w_+^\perp|^2 dS\right| \le \left|\int_{S_t} \kappa_{E+}(\nabla_{N_+} p_{w,w}^+ + \nabla_{w_+^\top} w_+ \cdot N_+ + \nabla_{w_+^\top} w_+^\perp) dS\right| + C|w|_{H^1(S_t)}^2\,,$$

where the constant $C > 0$ depends on the geometry of S_t. Recalling (3.18), (3.19), and upon noting

$$\begin{aligned}\nabla_{N_+} p_{w,w}^+ =& \mathcal{N}_+(p_{w,w}^+|_{S_t} + \nabla_{N_+}\Delta_+^{-1}\Delta p_{w,w}^+) \\ =& \frac{1}{\rho_+}\mathcal{N}_+ p_{w,w}^S - \nabla_{N_+}\Delta_+^{-1} tr(Dw)^2\,,\end{aligned}$$

we have

$$\left|\bar{\mathscr{D}}^2(\bar{w},\bar{w}) - \int_{S_t} |\Delta_{S_t} w_+^\perp|^2 dS\right| \le \left|\int_{S_t} p_{w,w}^S \frac{1}{\rho_+}\mathcal{N}_+ \kappa_{E+} dS\right| + C\left(|w|_{H^1(S_t)}^2 + |w|_{L^2(\mathbb{R}^n\setminus S_t)}^2\right),$$

where the constant $C > 0$ depends on the geometry of S_t. Thus, by (3.20)

$$\left|\bar{\mathscr{D}}^2(\bar{w},\bar{w}) - \int_{S_t} |\Delta_{S_t} w_+^\perp|^2 dS\right| \le C\left(|w|_{H^1(S_t)}^2 + |w|_{L^2(\mathbb{R}^n\setminus S_t)}^2\right).$$

Let

$$\mathscr{A}(u)(w) = \nabla f_+ 1_{\Omega_t^+} + \nabla f_- 1_{\Omega_t^-}, \tag{3.26}$$

where

$$f_\pm = \frac{1}{\rho_+\rho_-}\mathcal{H}_\pm \mathcal{N}^{-1}\mathcal{N}_\mp(\Delta_{S_t}^2) w_\pm^\perp .$$

Clearly $\bar{\mathscr{A}}(u)$ satisfies

$$\bar{\mathscr{A}}(\bar{w}, \bar{w}) = \int_{S_t} |\Delta_{S_t} w_+^\perp|^2 dS . \tag{3.27}$$

Then, it follows that

$$\bar{\mathscr{D}}^2 S(u;\kappa^2) = \bar{\mathscr{A}}(u) + \text{at most third-order differential operators.} \tag{3.28}$$

Finally, from (3.25) and (3.28), we have that (3.24) can be written as

$$\bar{\mathscr{D}}_t^2 \bar{w} + \bar{\mathscr{A}}(u)\bar{w} + \bar{\mathscr{R}}_0(\bar{u}_t)\bar{w} = \text{the lower-order terms} . \tag{3.29}$$

3.3. Local energy estimates

In this section, we derive the local estimates. We will show that the solutions of (1.1) with the boundary condition (1.2) are locally bounded in

$$v(t,\cdot) \in H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_t) \quad \text{and } S_t \in H^{\frac{5}{2}k+2},$$

where k is an integer satisfying $\frac{5}{2}k > \frac{n}{2} + 1$ (equivalently, $\frac{5}{2}k \geqslant \frac{n}{2} + \frac{3}{2}$).

3.3.1. Definition of energies

We start with the choice of energy. It is well known that the basic energy is given by

$$E_0 = \int_{\mathbb{R}^n \setminus S_t} \frac{\rho|v|^2}{2} dx + \int_{S_t} \frac{1}{2}\kappa^2 dS.$$

Equation (3.29) motivates us to define the following high-order energies.

Definition 3.1. Given domains Ω_t^+ and Ω_t^- separated by the interface S_t, where Ω_t^+ is compact and $\Omega_t^\pm \in H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_t)$. The velocity fields $v_\pm$ in respective domain satisfy $v_+^\perp + v_-^\perp|_{S_t} = 0$ and $\nabla \cdot v_\pm = 0$. We define the energy $E(S_t, v)$, often written as E for short, by

$$E = \int_{\mathbb{R}^n \setminus S_t} \frac{1}{2}|\mathscr{A}^{\frac{k}{2}} v|^2 + \frac{1}{2}|\mathscr{A}^{\frac{k}{2}-\frac{1}{2}} \nabla p_{\kappa_E^h}|^2 dx + |\omega|^2_{H^{\frac{5}{2}k-1}(\mathbb{R}^n \setminus S_t)},$$

where $\omega = Dv - (Dv)^*$ is the vorticity of the vector v.

It is remarked that $\nabla p_{\kappa_E^h}$ behaves like $\mathbf{D}_t v$ due to Equation (3.23), and the vorticity term in E is to control the tangential component of the velocity field. Next, we fix $0 < \delta \ll 1$ and let $\Lambda_0 \triangleq \Lambda\left(S_0, \frac{5}{2}k - \frac{1}{2}, \delta\right)$. Then, it is easy to obtain

$$|\nabla p_{\kappa_E^h}|_{H^{s-\frac{1}{2}}(\mathbb{R}^n \setminus S_t)} \leqslant C|\kappa_E^h|_{H^s(S_t)}, \quad s \in \left[\frac{1}{2}, \frac{5}{2}k - \frac{1}{2}\right],$$

and

$$|\mathscr{A}|_{L(H^s(\mathbb{R}^n\setminus S_t), H^{s-5}(\mathbb{R}^n\setminus S_t))} \leqslant C, \quad s \in \left[6 - \frac{5}{2}k, \frac{5}{2}k - 1\right],$$

where C is uniform in $S_t \in \Lambda_0$.

Theorem 3.2. *For fixed $\delta > 0$ sufficiently small, there exists $L > 0$ such that, if a solution of the system (1.1) with the boundary condition (1.2) is given by $S_t \in H^{\frac{5}{2}k+2}$ and $v(t, \cdot) \in C_t^0(H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_t))$, then there exists $t^* > 0$, depending only on $|v(0, \cdot)|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)}$, L, and the set Λ_0 such that, for all $t \in [0, t^*]$,*

$$\begin{aligned} &S_t \in \Lambda_0 \quad \text{and} \quad |\kappa_E^h|_{H^{\frac{5}{2}k-2}(\partial\Omega_t)} \leqslant L, \\ &E(S_t, v(t, \cdot)) \leqslant 2E(S_0, v(0, \cdot)) + C_1 + \int_0^t P(E_0, E(S_{t'}, v(t', \cdot)))dt', \end{aligned} \tag{3.30}$$

where $P(\cdot)$ is a polynomial of positive coefficients determined only by the set Λ_0, and C_1 is a constant determined only by $|v(0, \cdot)|_{H^{\frac{5}{2}k-\frac{3}{2}}(\mathbb{R}^n\setminus S_0)}$ and the set Λ_0.

To prove Theorem 3.2, we first establish the following proposition.

Proposition 3.3. *For $S_t \in \Lambda_0$ with $S_t \in H^{\frac{5}{2}k+2}$, we have*

$$\begin{aligned} |\kappa_E^h|^2_{H^{\frac{5}{2}k-2}(S_t)} &\leqslant C_0(1 + E), \\ |v|^2_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)} &\leqslant C_0(E + E_0)^m, \end{aligned}$$

where the integer $m > 0$ depends only on k and n, and the constant $C_0 > 0$ depends only on the set Λ_0.

It follows from the identity $\kappa_E^h(t, x) = -\Delta_{S_t}\kappa(t, x)$ that $|\kappa|_{L^2(S_t)} + |\kappa_E^h|_{H^{s-2}(S_t)}$ is equivalent to $|\kappa|_{H^s(S_t)}$ for any $s \geqslant 2$ when S_t is smooth. Hence we have the following lemmas, which are needed for proving Proposition 3.3, the detailed proofs of these lemmas can be found in [36].

Lemma 3.4. *For $S_t \in \Lambda_0$, if $\kappa \in H^s(S_t)$ and $s \in \left[\frac{5}{2}k - \frac{5}{2}, \frac{5}{2}k - 1\right]$, then we have*

$$|\Pi|_{H^s(S_t)} + |N|_{H^{s+1}(S_t)} \leqslant C(1 + |\kappa|_{H^s(S_t)}) \leqslant C\left(1 + |\kappa_E^h|_{H^{s-2}(S_t)}\right)$$

for some $C > 0$ uniform in $S_t \in \Lambda_0$.

Lemma 3.5. *Assuming $\kappa_E^h \in H^{\frac{5}{2}k-\frac{7}{2}}(S_t)$, $g \in H^{\frac{5}{2}k-1}(\Omega_t^{\pm}) \cap (\dot{H}_0^1(\Omega_t^{\pm}))^*$, and $q = -\Delta_{\pm}^{-1} g$, the following inequality holds*

$$|\nabla_{N_{\mathcal{H}_\pm}} q|_{H^{\frac{5}{2}k}(\Omega_t^{\pm})} \leqslant C\left(1 + |\kappa_E^h|_{H^{\frac{5}{2}k-\frac{7}{2}}(S_t)}\right)\left(|g|_{H^{\frac{5}{2}k-1}(\Omega_t^{\pm})} + |g|_{(\dot{H}_0^1(\Omega_t^{\pm}))^*}\right)$$

for some $C > 0$ uniform in $S_t \in \Lambda_0$.

Here $(\dot{H}_0^1(\Omega_t^{\pm}))^*$ is the dual of $\dot{H}_0^1(\Omega_t^{\pm})$, and $\dot{H}_0^1(\Omega_t^{\pm})$ is the completion of $C_0^\infty(\Omega_t^{\pm})$ under the metric $|\cdot|_{\dot{H}^1(\Omega_t^{\pm})}$.

Lemma 3.6. *If $\kappa_E^h \in H^{\frac{5}{2}k-\frac{7}{2}}(S_t)$ with $S_t \in \Lambda_0$, then for any $s' \in \left[\frac{1}{2} - \frac{5}{2}k, \frac{5}{2}k - \frac{1}{2}\right]$,*

$$|(\Delta_{S_t})^{\frac{1}{2}} - \mathcal{N}_\pm|_{L(H^{s'}(S_t))} \leqslant C\left(1 + |\kappa_E^h|_{H^{\frac{5}{2}k-\frac{7}{2}}(S_t)}\right).$$

By using these lemmas, we can prove Proposition 3.3.

Proof of Proposition 3.3. First, it is easy to get that

$$|\mathscr{A}^{\frac{k}{2}} v|_{L^2(\mathbb{R}^n \setminus S_t)}^2 = \int_{S_t} v_+^{\perp} (\Delta_{S_t}^2 \bar{\mathcal{N}})^{k-1} \Delta_{S_t}^2 v_+^{\perp} dS \tag{3.31}$$

and

$$|\mathscr{A}^{\frac{k}{2}-\frac{1}{2}} \nabla p_{\kappa_E^h}|_{L^2(\mathbb{R}^n \setminus S_t)}^2 = \int_{S_t} \kappa_{E+}^h \bar{\mathcal{N}} (\Delta_{S_t}^2 \bar{\mathcal{N}})^{k-1} (-\Delta_{S_t}) \kappa_{E+}^h dS, \tag{3.32}$$

where

$$\bar{\mathcal{N}} = \left(\frac{1}{\rho_+}\mathcal{N}_+\right)\mathcal{N}^{-1}\left(\frac{1}{\rho_-}\mathcal{N}_-\right) = \left[\left(\frac{1}{\rho_+}\mathcal{N}_+\right)^{-1} + \left(\frac{1}{\rho_-}\mathcal{N}_-\right)^{-1}\right]^{-1}. \tag{3.33}$$

By Lemma 3.6, $\mathcal{N}_\pm$ behaves like $(-\Delta_{S_t})^{\frac{1}{2}}$, hence we have the estimates of $|\kappa_E^h|_{H^{\frac{5}{2}k-2}(S_t)}$ and $|v_+^{\perp}|_{H^{\frac{5}{2}k-\frac{1}{2}}(S_t)}$. Finally, considering the equation

$$\Delta v^i = \partial_j \omega_j^i,$$

we can obtain the estimate of $|v|_{H^{\frac{5}{2}k-\frac{1}{2}}(S_t)}$. The details can be found in Proposition 4.3 of [36], and the main difference is that Lemma 4.5 in [36] should be replaced by our Lemma 3.4. □

3.3.2. Proof of Theorem 3.2

As in [36], we divide the proof of Theorem 3.2 into three steps.

Step 1. To estimate κ_E^h, and make sure $S_t \in \Lambda_0$. First of all, due to the assumption, the equation $u_t(t,y) = v(t, u(t,y))$ is well-posed. Since $\frac{5}{2}k > \frac{n}{2}+1$, for $s \in \left[0, \frac{5}{2}k\right]$ and $f \in H^s(\Omega_t^+)$,

$$|f \circ u(t,\cdot)|_{H^s(\Omega_0^+)} \leqslant C|f|_{H^s(\Omega_t^+)}|u(t,\cdot)|^s_{H^{\frac{5}{2}k}(\Omega_0^+)},$$

where $C > 0$ depends on n and k, and hence there is a constant $C_1 > 0$ depending on n and k such that

$$|u(t,\cdot) - I|_{H^{\frac{5}{2}k}(\Omega_0^+)} \leqslant C_1 \int_0^t |v(t',\cdot)|_{H^{\frac{5}{2}k}(\Omega_t^+)} |u(t',\cdot)|^{\frac{5}{2}k}_{H^{\frac{5}{2}k}(\Omega_t^+)} dt'. \tag{3.34}$$

Set

$$t_0 = \sup\{t \,|\, |v(t',\cdot)|_{H^{\frac{5}{2}k}(\Omega_t^+)} < \mu, \quad \forall\, t' \in [0,t]\},$$

where $\mu > 0$ is a positive large number to be specified later. Then for all $t \in [0, t_0]$,

$$|u(t,\cdot) - I|_{H^{\frac{5}{2}k}(\Omega_0^+)} \leqslant \mu \int_0^t |u(t',\cdot)|^{\frac{5}{2}k}_{H^{\frac{5}{2}k}(\Omega_0^+)} dt'.$$

Thus there exist $t_1 > 0$ and $C_2 > 0$ depending only on μ such that, for all $0 < t \leqslant \min\{t_0, t_1\}$,

$$|u(t,\cdot) - I|_{H^{\frac{5}{2}k}(\Omega_0^+)} \leqslant C_2 t\,.$$

This implies that for all $0 \leqslant t \leqslant \min\{t_0, t_1\}$,

$$|\kappa^h_{E,+}(t,\cdot)|_{H^{5k-\frac{9}{2}}(S_t)} \leqslant |\kappa^h_{E,+}(0,\cdot)|_{H^{5k-\frac{9}{2}}(S_0)} + C_3 t\,,$$

where $C_3 > 0$ is determined only by μ and the set Λ_0. Finally, there exists $t_2 > 0$ determined by μ and the set Λ_0 such that $S_t \in \Lambda_0$ for $0 \leqslant t \leqslant \min\{t_0, t_2\}$.

Step2. To obtain the estimate of vorticity. Firstly,

$$\begin{aligned}\mathbf{D}_t\omega =& D\mathbf{D}_t v - (D\mathbf{D}_t v)^* + ((Dv)^*)^2 - (Dv)^2 \\ =& ((Dv)^*)^2 - (Dv)^2 \\ =& -(Dv)^*\omega - \omega Dv\,.\end{aligned}$$

Thus we can obtain

$$\frac{d}{dt}\int\limits_{\mathbb{R}^n\setminus S_t}|D^{\frac{5}{2}k-1}\omega|^2dx=\int\limits_{\mathbb{R}^n\setminus S_t}\mathbf{D}_t|D^{\frac{5}{2}k-1}\omega|^2dx$$
$$\leqslant C|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)}|\omega|^2_{H^{\frac{5}{2}k-1}(\mathbb{R}^n\setminus S_t)},$$

where the constant $C>0$ is determined by the set Λ_0.

Step 3. To establish the energy estimate. As [35], the following two estimates hold, for any function f defined on S_t, there is a constant $C>0$ depending on Λ_0 such that

$$|[\mathbf{D}_t,\Delta_{S_t}]|_{L(H^{s_1}(S_t),H^{s_1-2}(S_t))}\leqslant C|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)},\quad s_1\in\left(\frac{7}{2}-\frac{5}{2}k,\frac{5}{2}k-\frac{1}{2}\right],\tag{3.35}$$

and

$$|[\mathbf{D}_t,\mathcal{N}_\pm]|_{L(H^{s_2}(S_t),H^{s_2-1}(S_t))}\leqslant C|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)},\quad s_2\in\left[\frac{1}{2},\frac{5}{2}k-\frac{1}{2}\right].\tag{3.36}$$

Hence,

$$|[\mathbf{D}_t,\mathcal{N}^{-1}]|_{L(H^{s_1}(S_t),H^{s_1-1}(S_t))}\leqslant C|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)},\quad s_1\in\left[-\frac{1}{2},\frac{5}{2}k-\frac{3}{2}\right],\tag{3.37}$$

and

$$|[\mathbf{D}_t,\bar{\mathcal{N}}]|_{L(H^{s_2}(S_t),H^{s_2-1}(S_t))}\leqslant C|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)},\quad s_2\in\left[\frac{1}{2},\frac{5}{2}k-\frac{1}{2}\right].\tag{3.38}$$

In the following, we let $Q=Q\left(|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)},|\kappa_E^h|_{H^{\frac{5}{2}k-2}(\mathbb{R}^n\setminus S_t)}\right)$ denote a generic positive polynomial function of $|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)}$ and $|\kappa_E^h|_{H^{\frac{5}{2}k-2}(\mathbb{R}^n\setminus S_t)}$ with coefficients depending only on the set Λ_0.

The main difficulty is to close the energy estimate. To this end, we first define the following two 'Energies':

$$E_{Aux}=\frac{1}{2}\int\limits_{S_t}\kappa_{E+}^h\bar{\mathcal{N}}(\Delta_{S_t}^2\bar{\mathcal{N}})^{k-2}\Delta_{S_t}\bar{\mathcal{N}}(\kappa_{E+}^h|\Pi|^2)dS,$$

and

$$E_{ex}=\frac{\rho_+}{2(\rho_++\rho_-)}\int\limits_{S_t}\nabla_{v_+^\top}\kappa_+\cdot(\Delta_{S_t}^2\bar{\mathcal{N}})^{k-1}\nabla_{v_+^\top}\kappa_+dS$$
$$+\frac{\rho_-}{2(\rho_++\rho_-)}\int\limits_{S_t}\nabla_{v_+^\top}\kappa_+\cdot(\Delta_{S_t}^2\bar{\mathcal{N}})^{k-1}\nabla_{v_+^\top}\kappa_+dS.$$

In the subsequent analyses, we will prove the following estimates.

$$\left|\frac{d}{dt}\left(\frac{1}{2}|\mathscr{A}^{\frac{k}{2}-\frac{1}{2}}\nabla p_{\kappa_E^h}|^2_{L^2(\mathbb{R}^n\setminus S_t)}-E_{Aux}\right)-\int_{S_t}\kappa^h_{E+}\bar{\mathcal{N}}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\Delta^2_{S_t}v_+^\perp dS\right|\leqslant Q \tag{I}$$

and

$$\left|\frac{d}{dt}\left(\frac{1}{2}|\mathscr{A}^{\frac{k}{2}}v|^2_{L^2(\mathbb{R}^n\setminus S_t)}-E_{ex}\right)+\int_{S_t}v_+^\perp(\Delta^2_{S_t}\bar{\mathcal{N}})^k\kappa^h_{E+}dS\right|\leqslant Q\,. \tag{II}$$

Proof of Estimate (I). From (3.32), (3.35) and (3.38), we can obtain that

$$\left|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}-\frac{1}{2}}\nabla p_{\kappa_E^h}|^2_{L^2(\mathbb{R}^n\setminus S_t)}-\int_{S_t}\kappa^h_{E+}\bar{\mathcal{N}}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\mathbf{D}_{t+}\kappa^h_{E+}dS\right|\leqslant Q \tag{3.39}$$

and

$$\left|\frac{d}{dt}E_{Aux}-\int_{S_t}\kappa^h_{E+}\bar{\mathcal{N}}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-2}\Delta_{S_t}\bar{\mathcal{N}}(\mathbf{D}_t\kappa^h_{E+}|\Pi|^2)dS\right|\leqslant Q\,. \tag{3.40}$$

Recall (3.8),

$$\begin{aligned}\mathbf{D}_t\kappa^h_{E\pm}=&\Delta^2_{S_t}v_\pm^\perp+\Delta_{S_t}(v_\pm^\perp|\Pi_\pm|^2)-\Delta_{S_t}\left(\nabla_{v_\pm^\top}\kappa_\pm\right)+2\mathcal{D}^2\kappa_\pm\cdot((D^\top|_{TS_t})v_\pm)\\&+\nabla^\top\kappa_\pm\cdot\Delta_{S_t}v_\pm-\kappa_\pm\nabla_{\nabla^\top\kappa_\pm}v_\pm\cdot N_\pm\,.\end{aligned}$$

Note that

$$\begin{aligned}&-\Delta_{S_t}\left(\nabla_{v_\pm^\top}\kappa_\pm\right)+\nabla^\top\kappa_\pm\cdot\Delta_{S_t}v_\pm\\=&-\Delta_{S_t}\left(\nabla_{v_\pm^\top}\kappa_\pm\right)+\nabla^\top\kappa_\pm\cdot\Delta_{S_t}v_\pm^\top+\nabla^\top\kappa_\pm\cdot\Delta_{S_t}(v_\pm^\perp N_\pm)\\=&[\nabla^\top\kappa_\pm\cdot,\Delta_{S_t}]v_\pm^\top+\nabla^\top\kappa_\pm\cdot\Delta_{S_t}(v_\pm^\perp N_\pm)-\Delta_{S_t}(v_\pm^\perp\nabla^\top\kappa_\pm\cdot N_\pm)\\=&[\nabla^\top\kappa_\pm\cdot,\Delta_{S_t}]v_\pm^\top+[\nabla^\top\kappa_\pm\cdot,\Delta_{S_t}](v_\pm^\perp N_\pm)\,.\end{aligned}$$

Hence, from the definition of Q, we have the leading-order terms of $\mathbf{D}_t\kappa^h_{E\pm}$ are $\Delta^2_{S_t}v_\pm^\perp+\Delta_{S_t}(v_\pm^\perp|\Pi_\pm|^2)$. From this fact, substituting $\mathbf{D}_t\kappa^h_{E\pm}$ into (3.39) and (3.40), and using (3.35) and (3.38), we can get

$$\left|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}-\frac{1}{2}}\nabla p_{\kappa_E^h}|^2_{L^2(\mathbb{R}^n\setminus S_t)}-\int_{S_t}\kappa^h_{E+}\bar{\mathcal{N}}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\left(\Delta^2_{S_t}v_\pm^\perp+|\Pi|^2\Delta_{S_t}v_\pm^\perp\right)dS\right|\leqslant Q$$

and

$$\left| \frac{d}{dt} E_{Aux} - \int_{S_t} \kappa^h_{E+} \bar{\mathcal{N}} (\Delta^2_{S_t} \bar{\mathcal{N}})^{k-1} (|\Pi|^2 \Delta_{S_t} v^\perp_\pm) dS \right| \leqslant Q \,.$$

The above two equations give the estimate (I).

Proof of Estimate (II). By (3.35) and (3.38), we have

$$\left| \frac{1}{2} \frac{d}{dt} |\mathscr{A}^{\frac{k}{2}} v|^2_{L^2(\mathbb{R}^n \setminus S_t)} - \int_{S_t} v^\perp_+ (\Delta^2_{S_t} \bar{\mathcal{N}})^{k-1} \Delta^2_{S_t} \mathbf{D}_{t+} v^\perp_+ dS \right| \leqslant Q \,,$$

where

$$\mathbf{D}_{t+} v^\perp_+ = -\frac{1}{\rho_+} \mathcal{N}_+ p^S_{v,v} - \nabla_{N+} \Delta^{-1}_+ tr(Dv)^2 - \bar{\mathcal{N}} \kappa_{E+} - \nabla_{v^\top_+} v^\perp_+ + \Pi_+(v^\top_+, v^\top_+)$$

due to (3.23). It follows from Lemma 3.5 that

$$|\nabla_{N_\pm} \Delta^{-1}_\pm tr(Dv)^2|_{H^{\frac{5}{2}k-\frac{1}{2}}(S_t)} \leqslant Q \,.$$

Therefore, one can obtain

$$\begin{aligned} &\left| \frac{1}{2} \frac{d}{dt} |\mathscr{A}^{\frac{k}{2}} v|^2_{L^2(\mathbb{R}^n \setminus S_t)} - \int_{S_t} v^\perp_+ (\Delta^2_{S_t} \bar{\mathcal{N}})^{k-1} \Delta^2_{S_t} \right. \\ &\qquad \left. \times \left(-\frac{1}{\rho_+} \mathcal{N}_+ p^S_{v,v} - \bar{\mathcal{N}} \kappa^h_{E+} - \nabla_{v^\top_+} v^\perp_+ + \Pi_+(v^\top_+, v^\top_+) \right) dS \right| \leqslant Q \,. \end{aligned}$$

On the other hand,

$$\begin{aligned} -\frac{1}{\rho_+} \mathcal{N}_+ p^S_{v,v} =& \frac{1}{\rho_+} \mathcal{N}_+ \bar{\mathcal{N}}^{-1} \Big[2 \nabla_{v^\top_+ - v^\top_-} v^\perp_+ - \Pi_+(v^\top_+, v^\top_+) \\ & - \Pi_-(v^\top_-, v^\top_-) - \nabla_{N_+} \Delta^{-1}_+ tr(Dv)^2 - \nabla_{N_-} \Delta^{-1}_- tr(Dv)^2 \Big] . \end{aligned}$$

From Lemma 3.6, we get

$$\left| \mathcal{N}_+ \bar{\mathcal{N}}^{-1} - \frac{\rho_+ \rho_-}{\rho_+ + \rho_-} \right|_{L(H^{\frac{5}{2}k-\frac{3}{2}}(S_t), H^{\frac{5}{2}k-\frac{1}{2}}(S_t))} \leqslant Q \,.$$

Hence,

$$\Big| -\frac{1}{\rho_+} \mathcal{N}_+ p^S_{v,v} - \frac{\rho_-}{\rho_+ + \rho_-} (2 \nabla_{v^\top_+ - v^\top_-} v^\perp_+ - \Pi_+(v^\top_+, v^\top_+) - \Pi_-(v^\top_-, v^\top_-)) \Big|_{H^{\frac{5}{2}k-\frac{1}{2}}(S_t)} \leqslant Q \,.$$

It follows that

$$\Big|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}}v|^2_{L^2(\mathbb{R}^n\setminus S_t)} - \int_{S_t} v_+^{\perp}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\Delta^2_{S_t} \times \Big\{\frac{\rho_+}{\rho_+ + \rho_-}\Pi_+(v_+^{\top}, v_+^{\top}) - \frac{\rho_-}{\rho_+ + \rho_-}\Pi_+(v_-^{\top}, v_-^{\top}) - \bar{\mathcal{N}}\kappa_{E+} + \nabla v_+^{\perp}\cdot\Big(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}v_+^{\top} - \frac{2\rho_-}{\rho_- + \rho_+}v_-^{\top}\Big)\Big\}dS\Big| \leqslant Q\,.$$

Commuting $\nabla_{v_\pm^{\perp}}$ yields

$$\left|\int_{S_t} v_+^{\perp}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\Delta^2_{S_t}\nabla_{v_+^{\perp}}\cdot\Big(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}v_+^{\top} - \frac{2\rho_-}{\rho_- + \rho_+}v_-^{\top}\Big)dS\right| \leqslant Q\,.$$

Therefore,

$$\Big|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}}v|^2_{L^2(\mathbb{R}^n\setminus S_t)} - \int_{S_t} v_+^{\perp}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\Delta^2_{S_t} \times \Big\{\frac{\rho_+}{\rho_+ + \rho_-}\Pi_+(v_+^{\top}, v_+^{\top}) - \frac{\rho_-}{\rho_+ + \rho_-}\Pi_-(v_-^{\top}, v_-^{\top}) - \bar{\mathcal{N}}\kappa^h_{E+}\Big\}dS\Big| \leqslant Q\,.$$

By noticing

$$|\Delta_{S_t}(\Pi_\pm(v_\pm^{\top}, v_\pm^{\top})) - \mathcal{D}^2\kappa_\pm(v_\pm^{\top}, v_\pm^{\top})|_{H^{\frac{5}{2}k-\frac{5}{2}}(S_t)} \leqslant Q$$

and

$$\mathcal{D}^2_{\kappa_\pm}(v_\pm^{\top}, v_\pm^{\top}) = \nabla_{v_\pm^{\top}}\nabla_{v_\pm^{\top}}\kappa_\pm - \mathcal{D}_{v_\pm^{\top}}v_\pm^{\top}\cdot\nabla\kappa_\pm\,,$$

we can obtain that

$$\Big|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}}v|^2_{L^2(\mathbb{R}^n\setminus S_t)} - \int_{S_t} v_+^{\perp}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\Delta_{S_t} \times \Big\{-\frac{\rho_+}{\rho_+ + \rho_-}\nabla_{v_+^{\top}}\nabla_{v_+^{\top}}\kappa_+ + \frac{\rho_-}{\rho_+ + \rho_-}\nabla_{v_-^{\top}}\nabla_{v_-^{\top}}\kappa_- + \Delta_{S_t}\bar{\mathcal{N}}\kappa^h_{E+}\Big\}dS\Big| \leqslant Q\,.$$

Commuting $\nabla_{v_\pm^{\top}}$ gives

$$\Big|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}}v|^2_{L^2(\mathbb{R}^n\setminus S_t)} - \frac{\rho_+}{\rho_+ + \rho_-}\int_{S_t} \nabla_{v_+^\top}(\Delta_{S_t} v_+^\perp)\cdot(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\nabla_{v_+^\top}\kappa_+ dS$$
$$- \frac{\rho_-}{\rho_+ + \rho_-}\int_{S_t} \nabla_{v_-^\top}(\Delta_{S_t} v_+^\perp)\cdot(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\nabla_{v_-^\top}\kappa_+ dS + \int_{S_t} v_+^\perp(\Delta^2_{S_t}\bar{\mathcal{N}})^k\kappa^h_{E+}dS\Big| \leqslant Q\,.$$

Finally, from

$$|-\Delta_{S_t} v_+^\perp - \mathbf{D}_{t+}\kappa_+|_{H^{\frac{5}{2}k-2}(S_t)} \leqslant Q\,,$$

we have

$$\Big|\frac{1}{2}\frac{d}{dt}|\mathscr{A}^{\frac{k}{2}}v|^2_{L^2(\mathbb{R}^n\setminus S_t)} - \frac{\rho_+}{\rho_+ + \rho_-}\int_{S_t} \nabla_{v_+^\top}\mathbf{D}_{t+}\kappa_+\cdot(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\nabla_{v_+^\top}\kappa_+ dS$$
$$- \frac{\rho_-}{\rho_+ + \rho_-}\int_{S_t} \nabla_{v_-^\top}\mathbf{D}_{t+}\kappa_+\cdot(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-1}\nabla_{v_-^\top}\kappa_+ dS + \int_{S_t} v_+^\perp(\Delta_{S_t}\bar{\mathcal{N}})^k\kappa^h_{E+}dS\Big| \leqslant Q\,.$$

Thus, using (3.35) and (3.36), we can obtain the estimate (II).

Combining the estimates (I) and (II) gives

$$E(t)-E(0)-(E_{Aux}(t)-E_{Aux}(0))-(E_{ex}(t)-E_{ex}(0)) \leqslant \int_0^t Q(|v|_{H^{5k}(\mathbb{R}^n\setminus S_{t'})}, |\kappa|_{H^{5k-2}(S_{t'})})dt'\,.$$

It is easy to obtain

$$|E_{ex}| \leqslant C|v|^2_{H^{\frac{5}{2}k-\frac{5}{8}}(\mathbb{R}^n\setminus S_t)}|\kappa^h_E|^2_{H^{\frac{5}{2}k-\frac{7}{2}}(S_t)}\,.$$

Next, a calculation similar to (3.34) gives

$$|E_{ex}| \leqslant \frac{1}{4}E + C_1\left(1+|v|^m_{H^{\frac{5}{2}k-\frac{3}{2}}(\mathbb{R}^n\setminus S_t)}\right) \leqslant \frac{1}{4}E + C_1 + \int_0^t Qdt'\,,$$

where C_1 is determined by $|v(0,\cdot)|_{H^{\frac{5}{2}k-\frac{3}{2}}(\mathbb{R}^n\setminus S_t)}$ and the set Λ_0. On the other hand, it is easy to obtain

$$|E_{Aux}| = |\frac{1}{2}\int_{S_t} \kappa^h_{E+}\bar{\mathcal{N}}(\Delta^2_{S_t}\bar{\mathcal{N}})^{k-2}\Delta_{S_t}\bar{\mathcal{N}}(\kappa^h_{E+}|\Pi|^2)dS| \leqslant \frac{1}{4}E + C_1\,.$$

Therefore

$$E(S_t, v(t,\cdot)) \leqslant 2E(S_0, v(0,\cdot)) + C_1 + \int_0^t Qdt'\,.$$

Finally, using Proposition 3.3 and choosing μ large enough in comparison with the initial data, we finish the proof of Theorem 3.2.

4. Local well-posedness

After obtaining the estimates for the system (1.1) with the boundary condition (1.2), the proof of the local well-posedness is relatively standard. We will adopt the argument similar to [37], and focus on the main different points.

4.1. Preliminary results

Let k be an integer such that $\frac{5}{2}k > \frac{n}{2} + 1$ and $S_* \subset \mathbb{R}^n$ be a compact reference hypersurface of the Sobolev class $H^{\frac{5}{2}k+1}$ that separates $\mathbb{R}^n$ into domains Ω_*^+ and Ω_*^- with $S_* = \partial\Omega_*^+ = \partial\Omega_*^-$. Denote by $N_{*\pm}$ the outward unit normal vector, $\Pi_{*\pm}$ the second fundamental form, and $\kappa_{*\pm}$ the mean curvature of S_*. In addition, we denote $\kappa^h_{E,*\pm} \triangleq -\Delta_{S_*}\kappa_{*\pm}$.

Fix a unit vector field $\nu \in H^{\frac{5}{2}k+2}(S_*, \mathbb{S}^{n-1})$ such that its normal component $\nu_+^\perp \triangleq \nu \cdot N_{*+} \geqslant \frac{8}{9}$. From the implicit function theorem, there exists a δ_0 determined by S_* and ν such that

$$\phi : S_* \times [-\delta_0, \delta_0] \to \mathbb{R}^n \quad as \quad \phi(p, d) = p + d\nu(p)$$

is an $H^{\frac{5}{2}k+2}$ diffeomorphism from its domain to a neighborhood of S_*.

This coordinate system associates each hypersurface S close to S_* in the C^1 topology with a unique function $d_S : S_* \to \mathbb{R}$ such that

$$\Phi_S(p) \triangleq p + d_S(p)\nu(p)\,.$$

For $\delta > 0$ and $s \in \left(\frac{n+1}{2}, \frac{5}{2}k + 2\right]$, we let $\Lambda(S_*, s, \delta)$ be the collection of all hypersurfaces S such that its associated $d_S : S_* \to \mathbb{R}$ satisfies $|d_S|_{H^s(S_*)} < \delta$.

Given a surface $S \in \Lambda(S_*, s, \delta)$ separating two regions $\Omega^\pm$, we construct harmonic coordinates on $\Omega^\pm$ in the following manner. Consider the map $\Phi_S(p) = p + d_s(p)\nu(p)$ from $S_* \to S$ and let

$$\mathscr{X}_S^\pm = \mathcal{H}_{*\pm}(\Phi_S - id_{S_*}) + id : \Omega_* \to \Omega^\pm\,,$$

where $\mathcal{H}_{*\pm}$ is a harmonic extension operator on the domains $\Omega_*^\pm$. It is clear that

$$|D\mathscr{X}_S^\pm - I|_{H^{s-\frac{1}{2}}(\Omega_*^\pm)} \leqslant C|d_S|_{H^s(S_*)}$$

with $C > 0$ uniform in $S \in \Lambda(S_*, s, \delta)$ and thus $\mathscr{X}_S^\pm$ is a diffeomorphism from $\Omega_*^\pm \to \Omega^\pm$ if $\delta \ll 1$. The maps $\mathscr{X}_S^\pm$ are used as coordinates on $\Omega_S^\pm$, and we write

$$\mathscr{X}_S = \mathscr{X}_S^+ 1_{\Omega^+} + \mathscr{X}_S^- 1_{\Omega^-} : \mathbb{R}^n \setminus S \to \mathbb{R}^n\,.$$

For any $S \in \Lambda(S_*, s, \delta)$, let

$$\mathcal{K}(d_S)(p) = \kappa^h_{E,a}(p) \triangleq \kappa^h_{E,+}(\Phi_S(p)) + a^2 d_S(p)\,,$$

where $p \in S_*$ and a is a large constant depending on S_*, and we also set

$$\Lambda_* \triangleq \Lambda\left(S_*, \frac{5}{2}k - \frac{1}{2}, \delta\right).$$

Then, by the implicit function theorem, we can obtain the following lemma (see also Lemma 2.2 of [37]):

Lemma 4.1. *There exist $C, \delta, \delta_1, a_0 > 0$ determined only by S_* such that, for any $a \geqslant a_0$, $\mathcal{K}$ is a diffeomorphism from Λ_*, an open subset in $H^{\frac{5}{2}k-\frac{1}{2}}(S_*)$, to $H^{\frac{5}{2}k-\frac{9}{2}}(S_*)$. Let*

$$B_{\delta_1} \triangleq \left\{\kappa^h_{E,a} : |\kappa^h_{E,a} - \kappa^h_{E,*}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)} < \delta_1\right\},$$

where $\kappa^h_{E,*} = -\Delta_{\partial\Omega_t}\kappa_{*\pm}$, and it follows that

$$|\mathcal{K}^{-1}|_{C^3(B_{\delta_1}, H^{\frac{5}{2}k-\frac{1}{2}}(S_*))} \leqslant C\,.$$

Moreover, if $\kappa^h_{E,a} \in B_{\delta_1} \cap H^{s-4}(S_*)$ for $s \in \left[\frac{5}{2}k - \frac{1}{2}, \frac{5}{2}k + 2\right]$, then $d_S, \Phi_S \in H^s(S_*)$, and for any $\max\{s' - 4, -s\} \leqslant s'' \leqslant s' \leqslant s$,

$$|D\mathcal{K}^{-1}|_{L(H^{s''}(S_*), H^{s'}(S_*))} \leqslant C a^{s'-s''-4}\left(1 + |\kappa^h_{E,a}|_{H^{s-4}(S_*)}\right).$$

4.2. Reduction of the problem

In this section, we decompose the problem into a coupled system of the evolutions of the interface represented by $\kappa^h_{E,a}$ and the rotational part of the velocity field.

4.2.1. Velocity fields and boundary motion

Let $v = v_+ 1_{\Omega_t^+} + v_- 1_{\Omega_t^-} : \mathbb{R}^n \setminus S_t \to \mathbb{R}^n$ be a velocity field of $\Omega_t^\pm$. Define the operator $\mathcal{L}$ as follows: for $g : \Omega \to \mathbb{R}$ and $f : \partial\Omega \to \mathbb{R}$, let $\mathcal{L}(g, f) = \nabla h$ with

$$\nabla_N h|_{\partial\Omega} = f \quad \text{and} \quad \Delta h = g + \left(\int_{\partial\Omega} f dS - \int_{\Omega} g dx\right)\gamma\,,$$

where $\gamma = (\int_\Omega dx)^{-1}$ if Ω is bounded; $supp(\gamma) \subset\subset \Omega$ and $\int_{\mathbb{R}^2} \gamma dx = 1$ if $\Omega \subset \mathbb{R}^2$ is unbounded; $\gamma \equiv 0$ if $\Omega \subset \mathbb{R}^n$ is unbounded and $n > 2$. For the vector field v, by means of the Hodge decomposition we can decompose v as the rotational part v_r and the irrotational part v_{ir}. From Appendix A in [37], the irrotational part v_{ir} takes the form of

$$v_{ir\pm} = \mathcal{L}_\pm\left(0, (\partial_t d_{S_t} \nu) \circ \Phi_{S_t}^{-1} \cdot N_\pm\right). \tag{4.1}$$

Thus, the rotational part can be expressed as

$$v_{r\pm} = v_\pm - v_{ir\pm} = v_\pm - \mathcal{L}_\pm\left(0, (\partial_t d_{S_t}\nu)\circ\Phi_{S_t}^{-1}\cdot N_\pm\right) := P(S_t, v_\pm)\,.$$

We will use the harmonic coordinates $\mathscr{X}_{S_t}^{\pm}$ to pull v_r back to $\Omega_\pm^*$, i.e.

$$v_r = \left(D\mathscr{X}_{S_t}^{\pm}(v_{r*\pm})\right)^{-1}\left(P(S_t, v_\pm)\circ\mathscr{X}_{S_t}^{\pm}\right).$$

Here, we have $v_{r^*\pm}(x) \in T_x S_*$.

Now, given $\kappa_{E,a}^h$, $\partial_t\kappa_{E,a}^h$ and $v_{r^*\pm}(x) \in T_x S_*$, from the above analysis, we can decompose the velocity field as

$$v_\pm = v_{ir\pm} + v_{r\pm} = \mathcal{L}_\pm\left(0, (\partial_t d_{S_t}\nu)\circ\Phi_{S_t}^{-1}\cdot N_\pm\right) + \left(D\mathscr{X}_{S_t}^{\pm}(v_{r*\pm})\right)\circ\left(\mathscr{X}_{S_t}^{\pm}\right)^{-1}, \tag{4.2}$$

where $d_{S_t} = \mathcal{K}^{-1}(\kappa_E^h)$ and $\mathcal{L}_\pm$ are defined on $\Omega_t^\pm$ respectively. In addition, we have the following estimate

$$|v|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_t)} \leqslant Q\left(|\kappa_{E,a}^h|_{H^{\frac{5}{2}k-\frac{7}{2}}(S_*)}\right)\left(|v_{r*}|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_*)} + |\partial_t\kappa_{E,a}^h|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)}\right).$$

4.2.2. Velocity fields on the interface

Given $\kappa_{E,a}^h(t,\cdot) : S_* \to \mathbb{R}$ and $v_{r*} : \mathbb{R}^n \setminus S_t \to \mathbb{R}^n$ with $v_{r*\pm}|_{S_*} \in TS_*$, let

$$v_M = \frac{\rho_+ v_+ + \rho_- v_-}{\rho_+ + \rho_-}, \quad \mathbf{D}_{\bar t} = \partial_t + \nabla_{v_M} = \frac{\rho_+}{\rho_+ + \rho_-}\mathbf{D}_{t+} + \frac{\rho_-}{\rho_+ + \rho_-}\mathbf{D}_{t-}\,.$$

The velocity field v_M defines a flow map $U(t,\cdot) : S_0 \to S_t$ by

$$U(t,\cdot) = id_{S_0}, \quad \partial_t U(t,\cdot) = v_M(t, U(t,\cdot)),$$

and

$$U_*(t,\cdot) = \Phi_{S_t}^{-1}\circ U(t,\cdot)\circ\Phi_{S_0}\,.$$

The velocity field induced by this family of transformations $U_*(t,\cdot)$ on S_* is given by

$$v_{M*} = \partial_t U_*\circ U^{-1} = D\Phi_{S_t}^{-1}(v_M\circ\Phi_{S_t} - (\partial_t d_{S_t})\nu) = D\Phi_{S_t}^{-1}(v_M^\top\circ\Phi_{S_t} - (\partial_t d_{S_t})\nu^\top)\,. \tag{4.3}$$

The material differentiation associated with v_{M*} is defined by

$$\mathbf{D}_{t*} = \partial_t + \nabla_{v_{M*}}\,,$$

and moreover,

$$(\mathbf{D}_{\bar t} f)\circ\Phi_{S_t} = \mathbf{D}_{t*}(f\circ\Phi_{S_t})$$

for any function $f(t,\cdot)$ defined on S_t.

Now, we assume that $\kappa^h_{E,a}$ and v_{r*} depend on some parameter β. Then from (3.10) and Lemma 3.1 of [37], we also have

$$\partial_\beta v_{M*} = B_1(\kappa^h_{E,a})\partial_{t\beta}\kappa^h_{E,a} + \frac{\rho_+\partial_\beta v_{r*+} + \rho_-\partial_\beta v_{r*-}}{\rho_+ + \rho_-} + R_1(\kappa^h_{E,a}, \partial_t\kappa^h_{E,a}),$$

where the linear operators $B_1(\kappa^h_{E,a})$ and $R_1(\kappa^h_{E,a}, \partial_t\kappa^h_{E,a})$ satisfy the following lemma:

Lemma 4.2. *Assume $a \geqslant a_0$, where a_0 is given in Lemma 4.1. For $\kappa^h_{E,a} \in B_{\delta_1}$, $\max\{-\frac{9}{2}, s'-4\} \leqslant s'' \leqslant s' \leqslant \frac{5}{2}k - \frac{3}{2}$ and $-\frac{9}{2} \leqslant s \leqslant \frac{5}{2}k - \frac{11}{2}$, we have*

$$|B_1(\kappa_a)|_{L(H^{s''}(S_*,\mathbb{R}),H^{s'}(S_*,TS_*))} \leqslant Ca^{s'-s''-4},$$
$$|DB_1(\kappa_a)|_{L(H^{\frac{5}{2}k-\frac{9}{2}}(S_*),L(H^s(S_*),H^{s+4}(S_*)))} \leqslant C,$$

where C depends only on S_* and δ. If $\kappa^h_{E,a} \in B_{\delta_1} \cap H^{\frac{5}{2}k-2}(S_*)$ and $\partial_t\kappa^h_{E,a}$, $\partial_\beta\kappa^h_{E,a} \in H^{\frac{5}{2}k-\frac{9}{2}}(S_*)$, then for any $\max\{-\frac{9}{2}, s'-4\} \leqslant s'' \leqslant s' \leqslant \frac{5}{2}k$,

$$|B_1(\kappa^h_{E,a})|_{L(H^{s''}(S_*,\mathbb{R}),H^{s'}(S_*,TS_*))} \leqslant Ca^{s'-s''-4}Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-2}}\right),$$
$$|R_1(\kappa^h_{E,a}, \partial_t\kappa^h_{E,a})|_{L(H^{\frac{5}{2}k-\frac{9}{2}}(S_*),H^{\frac{5}{2}k-\frac{3}{2}}(S_*))} \leqslant Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-2}}, |\partial_t\kappa^h_{E,a}|_{H^{\frac{5}{2}k-\frac{9}{2}}}\right).$$

Moreover, for $s \in \left[\frac{5}{2}k - \frac{9}{2}, \frac{5}{2}k - 2\right]$,

$$|DR_1|_{L(H^s(S_*)\times H^{s-\frac{5}{2}}(S_*),L(H^{s-\frac{5}{2}}(S_*)\times H^{s+\frac{1}{2}}(S_*)))} \leqslant Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-2}(S_*)}, |\partial_t\kappa^h_{E,a}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)}\right).$$

4.2.3. Evolution of $\kappa^h_{E,a}$

Let (S_t, v) be a solution to (1.1)-(1.2) for $t \in [0,T]$ with $S_t \in \Lambda_* \triangleq \Lambda(S_*, \frac{5}{2}k - \frac{1}{2}, \delta)$ and $S_t \in H^{\frac{5}{2}k+2}$ and $v(t,\cdot) \in H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_t)$. The interface S_t is determined by $\kappa^h_{E,a}$ whose leading order term is $\kappa^h_{E,+}$. We first consider the evolution of $\kappa^h_{E,+}$ in the direction of weighted mean velocity. Set

$$v_M = \frac{\rho_+}{\rho_+ + \rho_-}v_+ + \frac{\rho_-}{\rho_+ + \rho_-}v_-,$$
$$\mathbf{D}_{\bar{t}} \triangleq \partial_t + \nabla_{v_M} = \frac{\rho_+}{\rho_+ + \rho_-}\mathbf{D}_{t+} + \frac{\rho_-}{\rho_+ + \rho_-}\mathbf{D}_{t-} \quad \text{on} \quad S_t.$$

For any $S_t \in H^{\frac{5}{2}k+2}$ and any tangential vector field $\mathcal{X}$ on S_t, define the operators $\mathscr{A}$ and $\mathscr{R}_0$ by

$$\mathscr{A}(S_t) = \Delta_S^2\bar{\mathcal{N}}, \quad \mathscr{R}_0(S_t, \mathcal{X}) = \nabla_{\mathcal{X}}\mathcal{N}^{-1}\mathcal{D}\cdot(\mathcal{X}\bar{\mathcal{N}}(\cdot)),$$

where $\mathcal{N} = \frac{1}{\rho_+}\mathcal{N}_+ + \frac{1}{\rho_-}\mathcal{N}_-$ and

$$\bar{\mathcal{N}} = \left(\frac{1}{\rho_+}\mathcal{N}_+\right)\mathcal{N}^{-1}\left(\frac{1}{\rho_-}\mathcal{N}_-\right) = \left[\left(\frac{1}{\rho_+}\mathcal{N}_+\right)^{-1} + \left(\frac{1}{\rho_-}\mathcal{N}_-\right)^{-1}\right]^{-1}.$$

From the procedure of the proof of Theorem 3.2, we can obtain the following lemma (the interested reader is referred to the proof of Lemma 3.2 in [37] for details).

Lemma 4.3. *There exists $\delta > 0$, which depends only on S_*, such that for any solution (S_t, v) of (1.1)-(1.2) for $t \in [0, T]$ with $S_t \in \Lambda_*$, $S_t \in H^{\frac{5}{2}k+2}$, and $v(t,\cdot) \in H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_t)$, $\kappa^h_{E,+}$ satisfies*

$$\mathbf{D}_{\bar{t}}^2\kappa^h_{E,+} + \mathscr{A}(S_t)\kappa^h_{E,+} + \mathscr{R}_0(S_t, (v_+ - v_-)|_{S_t})\kappa^h_{E,+} \\ + (|\Pi_+|^2 - \nabla^\top\kappa_+ \cdot \nabla^\top\mathcal{N}_+^{-1})\Delta_{S_t}\bar{\mathcal{N}}\kappa^h_{E,+} = R_0(S_t, v),$$

where $R_0(S_t, v) : S_t \to \mathbb{R}$ satisfies

$$|R_0(S_t, v)|_{H^{\frac{5}{2}k-\frac{5}{2}}(S_t)} \leqslant Q = Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k}(S_*)}, |v|_{H^{\frac{5}{2}k-2}(\mathbb{R}^n\setminus S_t)}\right).$$

Remark 4.1. Comparing with the case of pure capillary waves (see [37]), we have extra terms $(|\Pi_+|^2 - \nabla^\top\kappa_+ \cdot \nabla^\top\mathcal{N}_+^{-1})\Delta_{S_t}\bar{\mathcal{N}}\kappa^h_{E,+}$.

Now, given a surface $S \in \Lambda_*$ and $\mathcal{X}_* \in TS_*$, let

$$\mathscr{A}_M(\kappa^h_{E,a})f = [\mathscr{A}(S)(f \circ \Phi_S^{-1})] \circ \Phi_S,$$
$$\mathscr{R}_M(\kappa^h_{E,a}, \mathcal{X}_*) = [\mathscr{R}_0(S, \mathcal{X})(f \circ \Phi_S^{-1})] \circ \Phi_S, \quad \mathcal{X} = D\Phi_S(\mathcal{X}_*),$$
$$\mathscr{C}_M(\kappa^h_{E,a})f = \left[\left((|\Pi_+|^2 - \nabla^\top\kappa_+ \cdot \nabla^\top\mathcal{N}_+^{-1})\Delta_S\bar{\mathcal{N}}\right)(f \circ \Phi_S^{-1})\right] \circ \Phi_S$$

be operators acting on the function f defined on S_*. Therefore, similar to Lemma 3.3 of [37], we have

Lemma 4.4. *There exist $C, \delta_1 > 0$ determined only by S_* such that for $\kappa^h_{E,a} \in B_{\delta_1}$, $\mathcal{X}_* \in H^{\frac{5}{2}k-\frac{1}{2}}(S_*)$, the following inequalities hold*

$$|\mathscr{A}_M(\kappa^h_{E,a})|_{L(H^s(S_*),H^{s-5}(S_*))} \leqslant C,$$
$$|\mathscr{R}_M(\kappa_a, \mathcal{X}_*)|_{L(H^s(S_*),H^{s-2}(S_*))} \leqslant C|\mathcal{X}_*|^2_{H^{\frac{5}{2}k-\frac{1}{2}}(S_*)},$$
$$|D\mathscr{A}_M|_{L(H^{\frac{5}{2}k-\frac{9}{2}}(S_*),L(H^{s_1}(S_*),H^{s_1-5}))} \leqslant C,$$

where $s \in \left[\frac{7}{2} - \frac{5}{2}k, \frac{5}{2}k - \frac{1}{2}\right]$ and $s_1 \in \left[\frac{9}{2} - \frac{5}{2}k, \frac{3}{2}k - \frac{1}{2}\right]$. Moreover, if $\kappa^h_{E,a} \in B_{\delta_1} \cap H^{\frac{5}{2}k-\frac{5}{2}}(S_*)$, then for $s \in \left[4 - \frac{5}{2}k, \frac{5}{2}k - \frac{1}{2}\right]$,

$$
\begin{aligned}
&|D\mathscr{R}_M|_{L(H^{\frac{5}{2}k-\frac{5}{2}}(S_*)\times H^{\frac{5}{2}k-2}(S_*),L(H^s(S_*),H^{s-\frac{5}{2}}(S_*)))} \\
&\leqslant Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-\frac{5}{2}}(S_*)}, |\mathcal{X}_*|_{H^{\frac{5}{2}k-\frac{1}{2}}(S_*)}\right).
\end{aligned}
$$

By (4.3) and a similar procedure for deriving the evolution equation of κ_a in [37], we can obtain the governing equation for $\kappa^h_{E,a}$. We state the results as follows. If (S_t, v) is a solution to (1.1)-(1.2), then

$$
\begin{aligned}
&\left(\partial_{tt} + 2\nabla_{v_{M*}}\partial_t + \nabla_{v_{M*}}\nabla_{v_{M*}} + \mathscr{A}_M + \mathscr{R}_M(\kappa_a, (D\Phi_{S_t})^{-1}(v_+ - v_-)) + \mathscr{C}_M\right)\kappa^h_{E,a} \\
&= B_2(\kappa^h_{E,a})(\partial_t v_{r*+}|_{S_*}) + R_2(\kappa^h_{E,a}, \partial_t\kappa^h_{E,a}, v_{r*}),
\end{aligned}
$$

and if $\kappa^h_{E,a} \in H^{\frac{5}{2}k-2}(S_*)$, $\partial_t\kappa^h_{E,a} \in H^{\frac{5}{2}k-\frac{9}{2}}(S_*)$ and $v_{r*} \in H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_*)$, then we have for $\varepsilon > 0$ and $3 - \frac{5}{2}k < s \leqslant \frac{5}{2}k - 3$ with $s \geqslant -\frac{9}{2}$,

$$
\begin{aligned}
|B_2(\kappa^h_{E,a})|_{L(H^{s+\varepsilon}(S_*),H^s(S_*))} &\leqslant Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-2}(S_*)}\right), \\
|DB_2|_{L(H^{\frac{5}{2}k-\frac{9}{2}}(S_*),L(H^{\frac{5}{2}k-\frac{3}{2}},H^{\frac{5}{2}k-\frac{11}{2}}(S_*)))} &\leqslant Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-2}(S_*)}\right), \\
|R_2|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)}, |DR_2|_{L(H^{\frac{5}{2}k-\frac{9}{2}}(S_*)\times H^{\frac{5}{2}k-7}(S_*)\times H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n\setminus S_*),H^{\frac{5}{2}k-5}(S_*))}& \\
\leqslant a^4 Q\left(|\kappa^h_{E,a}|_{H^{\frac{5}{2}k-2}(S_*)}, |\partial_t\kappa^h_{E,a}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)}, |v_{r*}|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_*)}\right)&.
\end{aligned}
$$

4.3. The linear problem along the interface

Let $S_* \in H^{\frac{5}{2}k+2}$, $\frac{5}{2}k > \frac{n}{2} + 1$, be a reference hypersurface and $1 \gg \delta > 0$ be fixed such that

$$
\Lambda_* \triangleq \Lambda\left(S_*, \frac{5}{2}k - \frac{1}{2}, \delta\right).
$$

Assume $S_t \in \Lambda_*$ is a family of hypersurfaces parameterized by $t \in [0, T]$ and $v_{M*}, \mathcal{X}_* : S_* \to TS_*$ are tangential vector fields on S_* such that

$$
\kappa^h_{E,a} \in C^0([0,T], H^{\frac{5}{2}k-2}(S_*)) \cap C^1([0,T], B_{\delta_1} \subset H^{\frac{5}{2}k-\frac{9}{2}}(S_*)), \tag{H1}
$$

$$
v_{M*}, \mathcal{X}_* \in C^0([0,T], H^{\frac{5}{2}k-\frac{1}{2}}(S_*)) \cap C^1([0,T], B_{\delta_1} \subset H^{\frac{5}{2}k-3}(S_*)). \tag{H2}
$$

We consider the linear initial value problem

$$
\begin{cases}
\left(\partial_{tt} + 2\nabla_{v_{M*}}\partial_t + \nabla_{v_{M*}}\nabla_{v_{M*}} + \mathscr{A}_M + \mathscr{R}_M(\kappa_a, \mathcal{X}_*) + \mathscr{C}_M\right) f = g \\
f(0,\cdot) = f_0, \quad \partial_t f(0,\cdot) = f_1
\end{cases} \tag{4.4}
$$

for given functions $f_0, f_1, g(t,\cdot) : S_* \to \mathbb{R}$. Then by similar proofs of Proposition 4.1 and Lemma 4.3 in [37], we have

Proposition 4.5. *For $s \in \left[2 - \frac{5}{2}k, \frac{5}{2}k - 2\right]$ and $g \in C^0([0, T], H^s(S_*))$, Equation (4.4) is well-posed in $H^{s+\frac{5}{2}}(S_*) \times H^s(S_*)$.*

Lemma 4.6. *There exists C_0 determined by the set Λ_* such that, for any integer $l \in [1-k, k-2]$ and $t \in \left[0, \frac{1}{Q}\right]$, we have*

$$|f|^2_{H^{\frac{5}{2}l+3}(S_*)} + |\partial_t f|^2_{H^{\frac{5}{2}l+\frac{1}{2}}(S_*)} \leqslant C_0 e^{Q_0 Q t}\Big[|f_0|^2_{H^{\frac{5}{2}l+3}(S_*)} + |f_1|^2_{H^{\frac{5}{2}l+\frac{1}{2}}(S_*)} + Q_0 |f_0|^2_{H^{\frac{5}{2}l+\frac{1}{2}}(S_*)} + |g|^2_{L^2([0,T], H^{\frac{5}{2}l+\frac{1}{2}}(S_*))}\Big],$$

where Q is a polynomial of norms of $\kappa^h_{E,a}$, $\partial_t \kappa^h_{E,a}$, v_{M_*}, $\partial_t v_{M_*}$, $\mathcal{X}_*$ and $\partial_t \mathcal{X}_*$, given in assumptions $(H1-2)$ with coefficients depending only on S_* and δ, and Q_0 is a polynomial of $|\mathcal{X}_*(0, \cdot)|_{H^{\frac{5}{2}k-2}(S_*)}$ and $|v_{M*}(0, \cdot)|_{H^{\frac{5}{2}k-2}(S_*)}$.

4.4. Proof of the local well-posedness

We are ready to prove the local well-posedness. We first define a set Σ as follows.

Definition 4.7. For given constants $T, L, L_0, L_1, L_r, L_\kappa$, define the set Σ as the collection of $(\kappa^h_{E,a}, v_{r*})$, which satisfies

$$\begin{aligned}
|\kappa^h_{E,a}(0,\cdot) - \kappa^h_{E,*+}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)} &\leqslant \delta_1\,,\\
|\partial_t \kappa^h_{E,a}(0,\cdot)|_{H^{\frac{5}{2}k-7}(S_*)}, |v_{r*}(0,\cdot)|_{H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n\setminus S_*)} &\leqslant L\,,\\
|\kappa^h_{E,a}|_{C^0([0,T], H^{\frac{5}{2}k-2}(S_*))} &\leqslant L_0\,,\\
|\partial_t \kappa^h_{E,a}|_{C^0([0,T], H^{\frac{5}{2}k-\frac{9}{2}}(S_*))}, |v_{r*}|_{C^0([0,T], H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_*))} &\leqslant L_1\,,\\
|\partial_t v_{r*}|_{C^0([0,T], H^{\frac{5}{2}k-1}(\mathbb{R}^n\setminus S_*))} &\leqslant L_r\,,\\
|\partial_{tt} \kappa^h_{E,a}|_{C^0([0,T], H^{\frac{5}{2}k-7}(S_*))} &\leqslant a^4 L_\kappa\,.
\end{aligned}$$

For $0 < \varepsilon \ll \delta$ and $A > 0$, consider a collection of initial data

$$I(\varepsilon, A) = \Big\{(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,a})_I, w_{Ir*}) \in H^{\frac{5}{2}k-2}(S_*) \times H^{\frac{5}{2}k-\frac{9}{2}}(S_*) \times H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_*) : |\kappa^h_{E,aI} - \kappa^h_{E,*+}|_{H^{\frac{5}{2}k-2}(S_*)} < \varepsilon, |(\partial_t \kappa^h_{E,a})_I|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)}, |w_{Ir*}|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_*)} < A\Big\}.$$

The iteration map $\mathscr{T}$. Fix the initial data $\left(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,a})_I, w_{Ir*}\right) \in \mathcal{I}(\varepsilon, A)$. Given $(\kappa^h_{E,a}, v_{r*}) \in \Sigma$, let $\tilde{\kappa}^h_{E,a}$ be a solution to the nonhomogeneous linear equation

$$\begin{cases}(\partial_{tt}+2\nabla_{v_{M*}}\partial_t+\nabla_{v_{M*}}\nabla_{v_{M*}}+\mathscr{A}_M+\mathscr{R}_M(\kappa_a,J_*)+\mathscr{C}_M)\tilde{\kappa}^h_{E,a}\\ =R_2(\kappa^h_{E,a},\partial_t\kappa^h_{E,a},v_{r*})+B_2(\kappa^h_{E,a})(\partial_t v_{r*}|_{S_*}),\\ \tilde{\kappa}^h_{E,a}(0,\cdot)=\kappa^h_{aI}, \quad \partial_t\tilde{\kappa}^h_{E,a}(0,\cdot)=(\partial_t\kappa^h_{E,a})_I,\end{cases} \tag{4.5}$$

where v_M is the weighted mean velocity on S_t, v_{M*} defined in (4.3), and

$$\mathcal{X}_*=(D\Phi_{S_t})^{-1}(v_+-v_-).$$

Since $(\kappa^h_a, v_{r*})\in\Sigma$, then

$$|v_{M*}(0,\cdot)|_{H^{\frac{5}{2}k-3}(S_*)}, |J_*(0,\cdot)|_{H^{\frac{5}{2}k-3}(S_*)}\leqslant Q(L)$$

and

$$|\partial_t v_{M*}|_{C^0([0,T],H^{\frac{5}{2}k-3}(S_*))}, |\partial_t\mathcal{X}_*|_{C^0([0,T],H^{\frac{5}{2}k-3}(S_*))}\leqslant Q(L_0,L_1,a^4L_\kappa,L_r).$$

Using estimates of (4.6), we have

$$|\tilde{\kappa}^h_a|^2_{C^0([0,T],H^{\frac{5}{2}k-2}(S_*))}+|\partial_t\tilde{\kappa}^h_a|^2_{C^0([0,T],H^{\frac{5}{2}k-\frac{9}{2}}(S_*))} \leqslant C_0e^{Q(L_0,L_1,a^2L_\kappa,L_r)T}(Q(L)+a^4Q(L_0,L_1,L_r)T).$$

By choosing L_0 and L_1 large in comparison with L and T sufficiently small, we have

$$\begin{aligned}|\tilde{\kappa}^h_{E,a}|^2_{C^0([0,T],H^{\frac{5}{2}k-2}(S_*))}&\leqslant Q(L)\leqslant L_0,\\ |\partial_t\tilde{\kappa}^h_{E,a}|^2_{C^0([0,T],H^{\frac{5}{2}k-\frac{9}{2}}(S_*))}&\leqslant Q(L)\leqslant L_1.\end{aligned}$$

This implies

$$|\partial_{tt}\tilde{\kappa}^h_{E,a}|^2_{C^0([0,T],H^{\frac{5}{2}k-7}(S_*))}\leqslant a^4Q(L_0,L_1,L_r)\leqslant a^2L_\kappa,$$

where L_κ large compared to L_0, L_1 and L_r. Let $F_{0\pm}=u\circ\mathscr{X}^{\pm}_{S_0}\circ(\mathscr{X}^{\pm}_{S_t})^{-1}$. Using F_0 we define $\tilde{v}_{r*}$ and $\hat{v}_r$ by

$$\begin{aligned}\hat{v}_r&=w-w_{ir},\\ \tilde{v}_{r*\pm}&=(D\mathscr{X}^{\pm}_{S_t})^{-1}\{\hat{v}_r\circ\mathscr{X}^{\pm}_{S_t}\},\end{aligned}$$

where $w_\pm=(DF^{-1}_{0\pm})^*(v_{I\pm}\circ F^{-1}_{0\pm}):\Omega^{\pm}_t\to\mathbb{R}^n$ and w_{ir} is defined in (4.1). Then it follows that

$$|\tilde{v}_{r*}|_{H^{\frac{5}{2}k}(\mathbb{R}^n\setminus S_*)}\leqslant Q(L_0)\leqslant L_1,$$

for an appropriate choice of L_1, and

$$|\partial_t \tilde{v}_{r*}|_{H^{\frac{5}{2}k}(\mathbb{R}^n \setminus S_*)} \leqslant Q(L_0, L_1) \leqslant L_r \,,$$

for an appropriate choice of L_r (see [37] for details).

Thus, we have the following lemma

Lemma 4.8. *Assume $\frac{5}{2}k > \frac{n}{2} + 1$. For any $0 < \varepsilon \ll \delta$ and $A > 0$, there exist $L, L_0, L_1, L_r, L_\kappa$ such that for sufficiently small $T > 0$, we have*

$$\mathscr{T}((\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,aI}), w_{Ir*}), (\kappa^h_{E,a}, v_{r*})) \triangleq (\tilde{\kappa}^h_{E,a}, \tilde{v}_{r*}) \in \Sigma \,,$$

where $\left(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,aI}), w_{Ir*}\right) \in I(\varepsilon, A)$ and $(\kappa^h_{E,a}, v_{r*}) \in \Sigma$.

Contraction mapping. We define the norm $|\cdot|_{\Sigma,\lambda}$ as

$$\begin{aligned} |(\kappa^h_{E,a}, v_{r*})|_{\Sigma,\lambda} = {} & |\kappa^h_{E,a}|_{C^0([0,T], H^{\frac{5}{2}k-\frac{9}{2}}(S_*))} + |\partial_t \kappa^h_{E,a}|_{C^0([0,T], H^{\frac{5}{2}k-7}(S_*))} \\ & + |v_{r*}|_{C^0([0,T], H^{\frac{5}{2}k-\frac{5}{2}}(S_*))} + \lambda |\partial_t v_{r*}|_{C^0([0,T], H^{\frac{5}{2}k-\frac{9}{2}}(\mathbb{R}^n \setminus S_*))} \,, \end{aligned}$$

where $\lambda \in [0, 1]$ to be determined later while estimating $\partial_t v_{r*}$.

Assume $\frac{5}{2}k > 5$ so that we can take traces of functions in $H^{\frac{5}{2}k-\frac{9}{2}}(\mathbb{R}^n \setminus S_*)$. For a parameter τ, consider a family $\left(\kappa^h_{E,a}(\tau), v_{r*}(\tau)\right) \in \Sigma$ with initial data $(\kappa^h_{E,aI}, v_I(\tau))$, and let $(\tilde{\kappa}^h_{E,a}, \tilde{v}_{r*}(\tau)) = \mathscr{T}(\kappa^h_{E,a}(\tau), v_{r*}(\tau))$. Differentiating (4.5) with respect to τ yields

$$\begin{aligned} & (\partial_{tt} + 2\nabla_{v_{M*}} \partial_t + \nabla_{v_{M*}} \nabla_{v_{M*}} + \mathscr{A}_M + \mathscr{R}_M(\kappa_a, J_*) + \mathscr{C}_M) \partial_\tau \tilde{\kappa}^h_{E,a} \\ & = - \{2\nabla_{\partial_\tau v_{M*}} \partial_t + \nabla_{\partial_\tau v_{M*}} \nabla_{v_{M*}} + \nabla_{v_{M*}} \nabla_{\partial_\tau v_{M*}} \\ & \quad + D(\mathscr{A}_M + \mathscr{C}_M)(\partial_\tau \kappa^h_{E,a}) + D\mathscr{R}_M(\partial_\tau \kappa_a, \partial_{tau} J_*)\} \\ & \quad + DR_2(\kappa^h_{E,a}, \partial_t \kappa^h_{E,a}, v_{r*})(\partial_\tau \kappa^h_{E,a}, \partial_t \partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*}) \\ & \quad + B_2(\kappa^h_{E,a}) \partial_{t\tau} v_{r*} + DB_2(\kappa^h_{E,a})(\partial_\tau \kappa^h_{E,a}) \partial_t v_{r*} \,, \end{aligned} \tag{4.6}$$

and at $t = 0$

$$\partial_\tau \tilde{\kappa}^h_{E,a}(0, \cdot) = \kappa^h_{aI}, \quad \partial_t \partial_\tau \tilde{\kappa}_{E,a}(0, \cdot) = (\partial_t \kappa^h_{E,a})_I \,.$$

From (4.2), we have

$$|\partial_\tau v_{M*}|_{C^0([0,T], H^{\frac{5}{2}k-3}(S_*))}, |\partial_\tau J_*|_{C^0([0,T], H^{\frac{5}{2}k-3}(S_*))} \leqslant Q(L_0, L_1) |(\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,0} \,.$$

Hence, by (4.6) and (4.4), we can obtain

$$\begin{aligned}
&|\partial_\tau \tilde{\kappa}^h_{E,a}|^2_{C^0([0,T],H^{\frac{5}{2}k-\frac{9}{2}}(S_*))} + |\partial_t \partial_\tau \tilde{\kappa}^h_{E,a}|^2_{C^0([0,T],H^{\frac{5}{2}k-7}(S_*))} \\
&\quad \leqslant T Q(L_0, L_1, L_r)| \left((\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,0} + |\partial_{t\tau} v_{r*}|_{C^0([0,T],H^{\frac{5}{2}k-\frac{9}{2}}(S_*))} \right) \\
&\quad + Q(L) \left(|\partial_\tau \kappa^h_{E,aI}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)} + |\partial_\tau ((\partial_t \kappa^h_{E,aI})_I)|_{H^{\frac{5}{2}k-7}(S_*)} \right).
\end{aligned} \tag{4.7}$$

From [37], we have

$$\begin{aligned}
&\partial_\tau \tilde{v}_{r*\pm} = (D\mathscr{X}^\pm_{S_t})^{-1} \left(-(D\partial_\tau \mathscr{X}^\pm_{S_t}) \tilde{v}_{r*} + ((\partial_\tau + \nabla_X)\hat{v}_r) \circ \mathscr{X}^\pm_{S_t} \right), \\
&D\mathscr{X}^\pm_{S_t}(\partial_{t\tau} \tilde{v}_{r*}) + D\partial_t \mathscr{X}^\pm_{S_t}(\partial_\tau \tilde{v}_{r*}) + D\partial_{t\tau} \mathscr{X}^\pm_{S_t}(\tilde{v}_{r*}) = ((\partial_\tau + \nabla_X)(\partial_t + \nabla_Z)\hat{v}_r) \circ \mathscr{X}^\pm_{S_t},
\end{aligned}$$

where $X = \partial_\tau \mathscr{X}^\pm_{S_t} \circ (\mathscr{X}^\pm_{S_t})^{-1}$ and $Z = \partial_t \mathscr{X}^\pm_{S_t} \circ (\mathscr{X}^\pm_{S_t})^{-1}$. By the similar procedure shown in [37], we can obtain the estimates for $\partial_\tau \tilde{v}_{r*\pm}$ and $\partial_{t\tau} \tilde{v}_{r*\pm}$

$$\begin{aligned}
|\partial_\tau \tilde{v}_{r*\pm}|_{H^{\frac{5}{2}k-\frac{5}{2}}(S_*)} \leqslant& (t + a^{-\frac{3}{2}}) Q(L_0, L_1) |(\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,0} \\
&+ C_0 |\partial_\tau ((\mathscr{X}^\pm_{S_I})^* v_I)|_{H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n \setminus S_*)},
\end{aligned} \tag{4.8}$$

where $C_0 > 0$ depends on S_* and

$$|\partial_{t\tau} \tilde{v}_{r*\pm}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)} \leqslant Q(L_0, L_1) \Big(|(\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,0} + |\partial_\tau ((\mathscr{X}^\pm_{S_I})^* v_I)|_{H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n \setminus S_*)} \Big). \tag{4.9}$$

It is noted that our case is easier to obtain due to the higher regularity of d_{S_t}. Next, if choosing T, a and λ_0 such that

$$\lambda_0 Q(L_0, L_1) \leqslant \frac{1}{6}, \quad (T + a^{-\frac{3}{2}}) Q(L_0, L_1) \leqslant \frac{1}{6}, \quad T Q(L_0, L_1) \leqslant \frac{1}{6} \lambda_0, \tag{4.10}$$

then from the estimates (4.7)-(4.9) and the definition of Σ, we can obtain the following lemma.

Lemma 4.9. *Assume $\frac{5}{2}k > \max\{5, \frac{n}{2} + 1\}$. For any $0 < \varepsilon \ll \delta$ and $A > 0$, there exist $L, L_0, L_1, L_r, L_\kappa$, if T, a and λ_0 satisfy (4.10), then*

$$\begin{aligned}
|\mathscr{T}(\kappa^h_{E,a}(\tau), v_{r*}(\tau))|_{\Sigma,\lambda_0} \leqslant& \frac{1}{2} |(\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,\lambda_0} + Q(L_0, L_1) \Big[|\partial_\tau \kappa^h_{E,aI}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)} \\
&+ |\partial_\tau ((\partial_t \kappa^h_{E,a})_I)|_{H^{\frac{5}{2}k-7}(S_*)} + |\partial_\tau w_{Ir*}|_{H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n \setminus S_*)} \Big].
\end{aligned}$$

By virtue of this lemma, fixing the initial data $\left(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,a})_I, w_{Ir*} \right) \in I(\varepsilon, A)$, for any $k \in \mathbb{N}$, we have

$$
\begin{aligned}
|\mathscr{T}^k(\kappa^h_{E,a}(\tau), v_{r*}(\tau))|_{\Sigma,\lambda_0} \leqslant & (\sum_{i=1}^{k} \frac{1}{2^i})|(\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,\lambda_0} \\
& + (\sum_{i=0}^{k-1} \frac{1}{2^i}) Q(L_0, L_1)\Big[|\partial_\tau \kappa^h_{E,aI}|_{H^{\frac{5}{2}k-\frac{9}{2}}(S_*)} \\
& + |\partial_\tau ((\partial_t \kappa^h_{E,a})_I)|_{H^{\frac{5}{2}k-7}(S_*)} + |\partial_\tau w_{Ir*}|_{H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n\setminus S_*)}\Big],
\end{aligned}
$$

and for any k_1, $k_2 \in \mathbb{N}$ and $k_1 < k_2$, we have

$$
|\mathscr{T}^{k_1}(\kappa^h_{E,a}(\tau), v_{r*}(\tau)) - \mathscr{T}^{k_2}(\kappa^h_{E,a}(\tau), v_{r*}(\tau))|_{\Sigma,\lambda_0} \leqslant \left(\sum_{i=k_1}^{k_2} \frac{1}{2^i}\right) |(\partial_\tau \kappa^h_{E,a}, \partial_\tau v_{r*})|_{\Sigma,\lambda_0}.
$$

Hence the iteration sequence $\mathscr{T}^k(\kappa^h_{E,a}(\tau), v_{r*}(\tau))$ is a Cauchy sequence in the complete metric space Σ with the norm $|\cdot|_{\Sigma,\lambda_0}$. Let $(\bar{\kappa}^h_{E,a}(\tau), \bar{v}_{r*}(\tau))$ be a limit of $\mathscr{T}^k(\kappa^h_{E,a}(\tau), v_{r*}(\tau))$ in Σ, and then it follows that $(\bar{\kappa}^h_{E,a}(\tau), \bar{v}_{r*}(\tau))$ is the fixed point of the operator $\mathscr{T}$. Hence, we have

Proposition 4.10. *Assume $\frac{5}{2}k > \max\{5, \frac{n}{2} + 1\}$. For any $0 < \varepsilon \ll \delta$ and $A > 0$, there exist $L, L_0, L_1, L_r, L_\kappa$, if T, a and λ_0 satisfying (4.10), then there exists $\mathcal{F} : \mathcal{I}(\varepsilon, A) \to \Sigma$ satisfying*

$$
\mathscr{T}\left((\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,aI})_I, w_{Ir*}), \mathcal{F}(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,aI})_I, w_{Ir*})\right) = \mathcal{F}\left(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,aI})_I, w_{Ir*}\right)
$$

$$
|D\mathcal{F}|_{L\left(H^{\frac{5}{2}k-\frac{9}{2}}(S_*)\times H^{\frac{5}{2}k-7}(S_*)\times H^{\frac{5}{2}k-\frac{5}{2}}(\mathbb{R}^n\setminus S_*), |\cdot|_{\Sigma,\lambda_0}\right)} \leqslant Q(L_0, L_1).
$$

Let $\left(\kappa^h_{E,aI}, (\partial_t \kappa^h_{E,a})_I, w_{Ir*}\right) \in I(\varepsilon, A)$ and $(\kappa^h_{E,a}, v_{r*})$ be its fixed point. From $\kappa^h_{E,a}$ and v_{r*}, we can define a family of interfaces S_t and velocity field $v(t, \cdot)$ by (4.2) whose initial values coincide with the interface S_I and velocity field v_I defined by $\left(\kappa^h_{E,a}, (\partial_t \kappa^h_{E,a})_I, w_{Ir*}\right)$. Finally, by the similar calculations of [37], we can check that (S_t, v) is the solution to (1.1)-(1.2) with initial data (S_I, v_I). Thus, we complete the proof of the local well-posedness.

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