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WELL-POSEDNESS OF ELECTROHYDRODYNAMIC INTERFACIAL WAVES UNDER TANGENTIAL ELECTRIC FIELD*

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Abstract. We consider the motion of the interface between two inviscid, incompressible, and dielectric fluids with different densities and permittivities, in the presence of a uniform electric field acting in a direction parallel to the undisturbed configuration. The system is assumed to be irrotational except the interface where the discontinuity of the tangential velocity induces vorticity. In this paper, we establish the local existence and uniqueness theory for the initial-value problem in Sobolev spaces for interfacial electrohydrodynamics. As we show, this system is locally well-posed in both two and three dimensions when surface tension is taken into account. More importantly, the tangential electric field provides a significant stabilizing effect for the two-dimensional problem (with a one-dimensional interface) such that we can prove the local-in-time well-posedness for small data even if one neglects the surface tension.

Key words. well-posedness, electrohydrodynamics, tangential electric

AMS subject classifications. 35Q35, 76W05

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1. Introduction. We study wave motions in interfacial electrohydrodynamics (EHD), a research field pioneered by Taylor and Melcher among others in the 1960s (see [20, 21, 22, 31, 32]). An electric field can exert considerably large force at interfaces in a multilayer immiscible fluid system, and it can be stabilizing or destabilizing broadly depending on its orientation with respect to undisturbed interfaces and electric properties of the fluids (see [21, 24]). A tangential electric field, which is parallel to the flat equilibrium, has a stabilizing effect, and relevant problems have been investigated by many scientists in modeling and numerics. A tangential electric field can delay the formation of film rupture (see [33]), remove the Kelvin–Helmholtz instability at all wavelengths in the linear regime even without surface tension (see [12, 13]), and completely suppress the Rayleigh–Taylor instability (see [5, 9]). A normal electric field, as opposed to the tangential electric field, has a destabilizing effect on the interface of two liquids with different permittivities. Extensive coverage of the electrohydrodynamic instability resulting from normal electric fields can be found in [17, 18, 23, 24, 25, 31, 32, 34] and references therein.

Although much effort has been devoted to the linear analysis, multiscale modeling, and direct numerical simulation, until recently there has been very little work done on the well-posedness of the EHD interfacial wave problems. In this paper, we consider a different and more interesting scenario, namely, the local existence and uniqueness of

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interfacial waves between two dielectric fluids under tangential electric fields. It is well known that a two-layer density-stratified system is ill-posed without surface tension due to the Kelvin–Helmholtz instability, but surface tension can control Fourier modes of high wavenumber in the linearization and then make the problem well-posed (see [2]). In the present paper, we prove rigorously that for the two-dimensional problem, surface tension can be replaced by a tangential electric field which also provides a dispersive regularization in local well-posedness, but surface tension remains necessary in the three-dimensional problem.

Let us recall some local well-posedness results of the Cauchy problem for fully nonlinear water-wave equations (without electrical fields). The first breakthrough in this research field was made by Wu [35, 36] who proved that arbitrary irrotational initial data lead to short-time existence in both two and three dimensions. Since then there have been a great number of papers devoted to the short-time problem with increasing generality. Of note is the work of Lannes [14] who treated the case of uneven bottom topography, Beyer and Günther [7] who took the effects of surface tension into consideration, and Christodoulou and Lindblad [8], Lindblad [19], and Coutand and Shkoller [10] who investigated the problem with vorticity. The interested reader is referred to Alazard et al. [1], Ambrose and Masmoudi [3, 4], Schweizer [26], Shatah and Zeng [27], and Zhang and Zhang [38] for more results on this topic. On the other hand, Ambrose [2], Shatah and Zeng [28, 29], and Lannes [16] extended these results to two-fluid systems where surface tension is necessary due to the Kelvin–Helmholtz instability.

1.1. Mathematical formulation. The system is composed of two incompressible and inviscid fluids superimposed on top of each other. Denote by Ω_t^+ and Ω_t^- the domain occupied by the lower and upper fluids, respectively, at time t. We assume the sharp interface between two layers can be parameterized by a function $z = \zeta(t, X)$, where $X = (X_1, \dots, X_d)^\top \in \mathbb{R}^d$ $(d = 1, 2)$ is the horizontal coordinate system and the z-axis points in the opposite direction to the force of gravity with $z = 0$ at the undisturbed interface. The system is bounded below (respectively, above) by a horizontal wall located at $z = -H^+$ (respectively, $z = H^-$). For the sake of convenience, we also denote by Γ_t the interface $\Gamma_t = \{(X, z)^\top, z = \zeta(t, X)\}$ and by $\Gamma^\pm$ the lower and upper boundaries $\Gamma^\pm = \{z = \mp H^\pm\}$. The fluids are assumed to be perfect dielectrics with electric permittivities ϵ^+ and ϵ^- in the corresponding regions $(\epsilon^+ \neq \epsilon^-)$, and a uniform electric field acts parallel to the undisturbed configuration. In the subsequent analyses, we define $[[A^\pm]]$ and $\langle A^\pm \rangle$ as follows:

$$[[A^\pm]] = A^+ - A^- \,, \quad \langle A^\pm \rangle = \frac{A^+ + A^-}{2} \,,$$

where $A^\pm$ can be real numbers, functions, etc.

The flow is assumed to be irrotational everywhere except at the interface; therefore there exist potential functions $\phi^\pm$ such that the velocity fields are $\nabla_{X,z}\phi^\pm$ in the corresponding regions, where $\nabla_{X,z} := (\partial_{X_1}, \dots, \partial_{X_d}, \partial_z)^\top$. If the electric field in each layer is denoted by $E^\pm$, the electrostatic limit of the Maxwell equations yields $\nabla_{X,z} \times E^\pm = 0$, and hence we can introduce voltage potentials $V^\pm$ such that $E^\pm = -\nabla_{X,z} V^\pm$. Under these assumptions, the field equations read

$$\begin{cases} \Delta_{X,z}\phi^\pm = 0 & \text{in } \Omega_t^\pm \,, \\ \Delta_{X,z} V^\pm = 0 & \text{in } \Omega_t^\pm \,, \end{cases}$$

where $\Delta_{X,z} := \partial^2_{X_1} + \cdots + \partial^2_{X_d} + \partial^2_z$. The boundary conditions for the voltage potentials $V^\pm$ at the interface Γ_t are

$$V^+ = V^- , \quad \epsilon^+ \frac{\partial V^+}{\partial \mathbf{n}} = \epsilon^- \frac{\partial V^-}{\partial \mathbf{n}} ,$$

where $\frac{\partial}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla_{X,z}$ and the unit normal vector is

$$\mathbf{n} = \frac{(-\zeta_{X_1}, \ldots, -\zeta_{X_d}, 1)^\top}{\sqrt{1 + |\nabla\zeta|^2}} ,$$

where $\nabla := \nabla_X = (\partial_{X_1}, \ldots, \partial_{X_d})^\top$ is the horizontal gradient. While the hydrodynamic boundary conditions at Γ_t, namely the kinematic boundary conditions are

$$\zeta_t = \phi^\pm_z - \nabla\zeta \cdot \nabla\phi^\pm ,$$

which implies the continuity of the normal velocity at the interface. The Bernoulli equations read

$$\rho^\pm \left(\partial_t \phi^\pm + g\zeta + \frac{1}{2} |\nabla_{X,z} \phi^\pm|^2 \right) = -P^\pm \quad \text{in } \Omega^\pm_t ,$$

and the continuity of normal stresses gives

$$[[P^\pm(t, \cdot)|_{\Gamma_t}]] = \sigma\kappa(\zeta) + [[\mathbf{n} \cdot \Sigma^\pm \cdot \mathbf{n}]]$$

with

$$\Sigma^\pm_{ij} = \epsilon^\pm \left(E^\pm_i E^\pm_j - \frac{1}{2} |\mathbf{E}^\pm|^2 \delta_{ij} \right),$$

where δ_{ij} is the Kronecker delta, $\rho^\pm$ are the densities corresponding to two fluid layers, g accounts for the gravitational acceleration, σ represents the surface tension coefficient between liquids, and $\kappa(\zeta)$ is the mean curvature of the interface,

$$\kappa(\zeta) = -\nabla \cdot \left(\frac{\nabla\zeta}{1 + |\nabla\zeta|^2} \right).$$

A straightforward calculation yields

$$\mathbf{n} \cdot \Sigma^\pm \cdot \mathbf{n} = \epsilon^\pm \left[\left(\frac{\partial V^\pm}{\partial \mathbf{n}} \right)^2 - \frac{1}{2} |\nabla_{X,z} V^\pm|^2 \right].$$

The boundary conditions on the channel walls $\Gamma^\pm$ are a no-penetration condition for hydrodynamics and no-current condition for electrostatics:[1]

$$\frac{\partial V^\pm}{\partial z} = \frac{\partial \phi^\pm}{\partial z} = 0 .$$

Finally, the asymptotic condition in the far field,

$$V^\pm \to E_0 X_1 \quad \text{as } \sqrt{|X|^2 + z^2} \to +\infty,$$

[1]On physical ground, no-current boundary condition for electric field is used to model electrically insulating walls.

completes the mathematical statement of the problem. We now introduce the modified voltage potentials $W^{\pm}$ by defining $W^{\pm} = \frac{V^{\pm}}{E_0} - X_1$. It then follows that $W^{\pm}$ satisfy

$$\begin{cases} \Delta_{X,z} W^{\pm} = 0 & \text{in } \Omega_t^{\pm}\,, \\ W^+ = W^- & \text{on } \Gamma_t\,, \\ \epsilon^+ \frac{\partial W^+}{\partial \mathbf{n}} - \epsilon^- \frac{\partial W^-}{\partial \mathbf{n}} = [[\epsilon^{\pm}]] \frac{\zeta_{X_1}}{\sqrt{1+|\nabla\zeta|^2}} & \text{on } \Gamma_t\,, \\ \frac{\partial W^{\pm}}{\partial z} = 0 & \text{on } \Gamma^{\pm}\,, \end{cases}$$

and

$$\begin{aligned} \mathbf{n}\cdot\Sigma^{\pm}\cdot\mathbf{n} =& \epsilon^{\pm} E_0^2 \Bigg[\left(\frac{\partial W^{\pm}}{\partial \mathbf{n}}\right)^2 - \frac{2\zeta_{X_1}}{\sqrt{1+|\nabla\zeta|^2}} \frac{\partial W^{\pm}}{\partial \mathbf{n}} \\ & + \frac{\zeta_{X_1}^2}{1+|\nabla\zeta|^2} - W^{\pm}_{X_1} - \frac{1}{2}|\nabla_{X,z} W^{\pm}|^2 \Bigg] - \frac{1}{2}\epsilon^{\pm} E_0^2\,. \end{aligned} \tag{1.1}$$

Denoting

$$\psi^{\pm}(X,t) := \phi^{\pm}(X, \zeta(X,t), t)$$

and

$$w(X,t) := W^+(X,\zeta(X,t),t) = W^-(X,\zeta(X,t),t)\,,$$

one can then reduce the system to the Zakharov–Craig–Sulem formulation [11, 37]:

$$\begin{cases} \partial_t \zeta - \mathcal{G}^+[\zeta]\psi^+ = 0\,, \\ \mathcal{G}^+[\zeta]\psi^+ = \mathcal{G}^-[\zeta]\psi^-\,, \quad (\epsilon^+\mathcal{G}^+[\zeta] - \epsilon^-\mathcal{G}^-[\zeta])\, w = [[\epsilon^{\pm}]]\zeta_{X_1}\,, \\ \rho^{\pm}\left(\partial_t\psi^{\pm} + g\zeta + \frac{1}{2}|\nabla\psi^{\pm}|^2 - \frac{(\mathcal{G}^{\pm}[\zeta]\psi^{\pm} + \nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)}\right) = -P^{\pm}|_{\Gamma_t}\,, \\ [[P^{\pm}(t,\cdot)|_{\Gamma_t}]] = \sigma\kappa(\zeta) + [[\mathbf{n}\cdot\Sigma^{\pm}\cdot\mathbf{n}]]\,, \end{cases} \tag{1.2}$$

where $\mathcal{G}^{\pm}[\zeta] = \mathcal{G}^{\pm}[\zeta, H^{\pm}]$ are the Dirichlet–Neumann operators corresponding to two fluid layers (see (2.2) for the precise definition). Next, as in [16], we will reduce the above two-fluid equations to a set of two equations on the surface elevation ζ and of the quantity ψ defined as

$$\psi := \underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^- \quad \text{with} \quad \underline{\rho}^{\pm} = \frac{\rho^{\pm}}{\rho^+ + \rho^-}\,.$$

To this end, we define the operator $\mathcal{G}[\zeta]$ as

$$\mathcal{G}[\zeta] := \mathcal{G}^-[\zeta]\left(\underline{\rho}^+\mathcal{G}^-[\zeta] - \underline{\rho}^-\mathcal{G}^+[\zeta]\right)^{-1}\mathcal{G}^+[\zeta]\,.$$

It then follows that

$$\mathcal{G}[\zeta]\psi = \mathcal{G}^+[\zeta]\psi^+ = \mathcal{G}^-[\zeta]\psi^-, \qquad \psi^{\pm} = \mathcal{G}^{\pm}[\zeta]^{-1}\mathcal{G}[\zeta]\psi.$$

In addition, we define the operator $\tilde{\mathcal{G}}[\zeta]$ as

$$\tilde{\mathcal{G}}[\zeta] := \underline{\epsilon}^+\mathcal{G}^+[\zeta] - \underline{\epsilon}^-\mathcal{G}^-[\zeta] \quad \text{with} \quad \underline{\epsilon}^{\pm} = \frac{\epsilon^{\pm}}{\epsilon^+ + \epsilon^-}\,.$$

Finally, we set

$$\underline{\tilde{Z}}^{\pm} := \frac{\mathcal{G}^{\pm}[\zeta]w + \nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}, \tag{1.3}$$

$$\underline{W}^{\pm} = (\underline{W}_1^{\pm}, \ldots, \underline{W}_d^{\pm})^{\top} := \nabla w - \underline{\tilde{Z}}^{\pm} \nabla \zeta. \tag{1.4}$$

Now, after redefining the velocity potential to absorb constant in (1.1), one can rewrite the system (1.2) in a canonical form (see [6]) as

(1.5)
$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta]\psi = 0\,, \\ \partial_t \psi + g'\zeta + \frac{1}{2}[[\underline{\rho}^{\pm}|\nabla\psi^{\pm}|^2]] - \frac{[[\underline{\rho}^{\pm}(\mathcal{G}^{\pm}[\zeta]\psi^{\pm} + \nabla\zeta\cdot\nabla\psi^{\pm})^2]]}{2(1+|\nabla\zeta|^2)} = -W_e \kappa(\zeta) - E_b \tau(\zeta)\,, \\ w = [[\underline{\epsilon}^{\pm}]]\tilde{\mathcal{G}}[\zeta]^{-1}\partial_{X_1}\zeta\,, \\ \psi^{\pm} = \mathcal{G}^{\pm}[\zeta]^{-1}\mathcal{G}[\zeta]\psi\,, \end{cases}$$

where $g' = (\underline{\rho}^+ - \underline{\rho}^-)g$, $W_e = \frac{\sigma}{\rho^+ + \rho^-}$, $E_b = \frac{E_0^2(\epsilon^+ + \epsilon^-)}{\rho^+ + \rho^-}$, and

$$\tau(\zeta) = \frac{[[\underline{\epsilon}^{\pm}(\mathcal{G}^{\pm}[\zeta]w)^2]]}{1+|\nabla\zeta|^2} - \frac{1}{2}\Big([[\underline{\epsilon}^{\pm}|\underline{W}^{\pm}|^2]] + [[\underline{\epsilon}^{\pm}|\underline{\tilde{Z}}^{\pm}|^2]]\Big) - \frac{[[\underline{\epsilon}^{\pm}]]\zeta_{X_1}^2}{1+|\nabla\zeta|^2} - [[\underline{\epsilon}^{\pm}\underline{W}_1^{\pm}]]\,.$$

1.2. Main results.

THEOREM 1.1. *Let $t_0 \geqslant \frac{3}{2}$, $N \geqslant 5$. Assume that $\sigma > 0$. Let $U^0 = (\zeta^0, \psi^0)^{\top}$ be the initial data satisfying*

$$\exists\, h_{\min}^{\pm} > 0\,, \quad \inf_{X\in\mathbb{R}^d}(H^{\pm} \pm \zeta^0(X)) \geqslant h_{\min}^{\pm}\,,$$

and $\mathcal{E}^N(U^0) < \infty$, where $\mathcal{E}^N(U^0)$ is defined in (5.1)*. Then there exist $T_\sigma > 0$ depending on σ and a unique solution $U = (\zeta,\psi)^{\top} \in E_T^N$ to the system* (1.5) *with initial data U^0, where E_T^N is given by* (3.2)*.*

If one neglects the surface tension, i.e.,

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta]\psi = 0\,, \\ \partial_t \psi + g'\zeta + \frac{1}{2}[[\underline{\rho}^{\pm}|\nabla\psi^{\pm}|^2]] - \frac{[[\underline{\rho}^{\pm}(\mathcal{G}^{\pm}[\zeta]\psi^{\pm} + \nabla\zeta\cdot\nabla\psi^{\pm})^2]]}{2(1+|\nabla\zeta|^2)} = -E_b \tau(\zeta)\,, \\ w = [[\underline{\epsilon}^{\pm}]]\tilde{\mathcal{G}}[\zeta]^{-1}\partial_{X_1}\zeta\,, \\ \psi^{\pm} = \mathcal{G}^{\pm}[\zeta]^{-1}\mathcal{G}[\zeta]\psi\,, \end{cases} \tag{1.6}$$

then one has the following result.

THEOREM 1.2. *Set $d = 1$. Let $t_0 \geqslant \frac{3}{2}$, $N \geqslant 5$, and $\tilde{\mathcal{E}}^N(U^0)$ and $\tilde{E}_T^N$ be defined by* (6.1) *and* (6.2)*, respectively. For any $0 < \delta < \frac{[[\underline{\epsilon}^{\pm}]]^2}{\overline{M}}$ being with $\overline{M} > 0$ is a fixed constant, if the initial data $U^0 = (\zeta^0, \psi^0)^{\top}$ satisfies*

$$\exists\, h_{\min}^{\pm} > 0\,, \quad \inf_{X\in\mathbb{R}}(H^{\pm} \pm \zeta^0(X)) \geqslant h_{\min}^{\pm}$$

and $\tilde{\mathcal{E}}^N(U^0) < \frac{\delta}{2}$, then there exist $T_\delta > 0$ and a unique solution $U = (\zeta,\psi)^{\top} \in \tilde{E}_T^N$ to the system (1.6) *with the initial condition U^0 such that $\tilde{\mathcal{E}}^N(U) \leqslant \delta$.*

1.3. A simplified model. Theorem 1.2 implies that the tangential electric field can stabilize the system like surface tension. To understand this mechanism and the role of the electric field, we consider a simpler model. Since our point is to understand the electric regularization, for clearness, we assume $\epsilon^- = \rho^- = 0$ and $\epsilon^+ = H^+ = 1$ which reduce the system to a single-layer problem. Furthermore, we retain the leading-order effect arising from the electric field by neglecting quadratic and even higher-order terms in $\tau(\zeta)$, namely, we replace $\tilde{\mathcal{G}}[\zeta]^{-1}$ by $\tilde{\mathcal{G}}[0]$ (i.e., $\tanh(|D|)|D|$, where $|D| = (-\Delta)^{\frac{1}{2}}$). Thus (1.5) becomes

$$\text{(1.7)}\qquad \begin{cases}\partial_t\zeta - \mathcal{G}[\zeta]\psi = 0\,,\\ \partial_t\psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(\mathcal{G}[\zeta]\psi+\nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = -\sigma\kappa(\zeta) - E_0^2\frac{\partial_{X_1}^2}{\tanh(|D|)|D|}\zeta\,.\end{cases}$$

From the simplified system (1.7), one can easily see that the new electric field term should play the same role as the capillary term $\sigma\kappa(\zeta)$ and thereby has a stabilizing effect, since the pseudodifferential operator $-\partial_{X_1}^2(\tanh(|D|)|D|)^{-1}$ is positive, and it is coercive when $d = 1$.

1.4. Main ideas and difficulties. There are various formulations available to handle the local well-posedness of the classical water-wave equations. It is suitable to use the framework of Lannes [14, 15, 16] for our problem. Let us first simply describe Lannes' formulation, which is based on a canonical form of the problem originally proposed by Benjamin and Bridges [6]. In Eulerian coordinates, the interfacial wave problem ($E_0 = 0$) can be written as

$$\text{(1.8)}\qquad \begin{cases}\partial_t\zeta - \mathcal{G}[\zeta]\psi = 0\,,\\ \partial_t\psi + g'\zeta + \frac{1}{2}[[\underline{\rho}^\pm|\nabla\psi^\pm|^2]] - \frac{[[\underline{\rho}^\pm(\mathcal{G}^\pm[\zeta]\psi^\pm+\nabla\zeta\cdot\nabla\psi^\pm)^2]]}{2(1+|\nabla\zeta|^2)} = -\frac{\sigma\kappa(\zeta)}{\rho^++\rho^-}\,,\\ \psi^\pm = \mathcal{G}^\pm[\zeta]^{-1}\mathcal{G}[\zeta]\psi\,.\end{cases}$$

To prove local well-posedness of (1.8), Lannes introduced a "good" unknown

$$U_{(\alpha)} = \big(\zeta_{(\alpha)} := \partial^\alpha\zeta, \psi_{(\alpha)} := \partial^\alpha\psi - \underline{Z}\partial^\alpha\zeta\big)^\top\,,$$

where $\underline{Z} = \underline{\rho}^+\underline{Z}^+ - \underline{\rho}^-\underline{Z}^-$ with $\underline{Z}^\pm = \frac{\mathcal{G}^\pm[\zeta]\psi^\pm+\nabla\zeta\cdot\nabla\psi^\pm}{1+|\nabla\zeta|^2}$, and reduced (1.8) to the quasilinear system

(1.9)

$$\begin{cases}\partial_t\zeta_{(\alpha)} + \text{transport terms} - \mathcal{G}[\zeta]\psi_{(\alpha)} + \text{subprincipal terms} = \text{lower-order terms}\,,\\ \partial_t\psi_{(\alpha)} + \text{transport terms} - \frac{\sigma\nabla\cdot\mathcal{K}[\nabla\zeta]\nabla\zeta_{(\alpha)}}{\rho^++\rho^-} + \text{subprincipal terms} = \text{lower-order terms}\,,\end{cases}$$

where $\mathcal{K}[\nabla\zeta] = \frac{(1+|\nabla\zeta|^2)Id - \nabla\zeta\bigotimes\nabla\zeta}{(1+|\nabla\zeta|^2)^{\frac{3}{2}}}$. It is noted that this quasilinear system is symmetrizable; thus by using the energy method, Lannes obtained the local well-posedness in [16].

For our problem, according to section 1.3, the quasilinearization of (1.5) should take the same form as (1.9) except that there is an additional term in the second equation owing to the electric field. This term plays the same role as the term $-\frac{\sigma\nabla\cdot\mathcal{K}[\nabla\zeta]\nabla\zeta_{(\alpha)}}{\rho^++\rho^-}$. More precisely, this extra term, which is denoted by $\mathcal{L}_e[\zeta]\zeta_{(\alpha)}$ hereafter, arises from $\partial^\alpha\tau(\zeta)$, where $\mathcal{L}_e[\zeta]$ is a first-order operator. Hence our problem boils down to linearzing $\partial^\alpha\tau(\zeta)$. Although $\mathcal{L}_e[\zeta]\zeta_{(\alpha)}$ is subprincipal, we must show that it is a self-adjoint operator to ensure a symmetrization of the quasilinear system. It is not difficult to understand that the dominated term in $\tau(\zeta)$ is $-[[\underline{\epsilon}^\pm]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}[\zeta]\partial_{X_1}\zeta$, but the other terms in $\tau(\zeta)$ cannot be neglected when we linearize $\partial^\alpha\tau(\zeta)$ since all terms in $\tau(\zeta)$ have the same order. This is a nontrivial task since there is no evidence that the linearized operator $\mathcal{L}_e[\zeta]$ is self-adjoint at the first sight of the involved expression of $\tau(\zeta)$. This is the main difficulty of the problem. Fortunately, although $\tau(\zeta)$ is very intricate, we can obtain the following linearization formula through careful analysis (see Proposition 3.4):

$$\partial^\alpha\tau(\zeta) = \mathcal{L}_e[\zeta]\zeta_{(\alpha)} + \text{lower-order terms}$$

with

$$\begin{aligned}\mathcal{L}_e[\zeta]\bullet = &- [[\underline{\epsilon}^{\pm}]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\bullet - [[\underline{\epsilon}^{\pm}]]\left(\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\bullet) + \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\bullet\right)\\ &- \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\bullet) + \underline{\epsilon}^{+}\underline{\epsilon}^{-}[[\underline{\tilde{Z}}^{\pm}]]\mathcal{G}^{+}\tilde{\mathcal{G}}^{-1}\mathcal{G}^{-}([[\underline{\tilde{Z}}^{\pm}]]\bullet)\\ := &- [[\underline{\epsilon}^{\pm}]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\bullet + \tilde{\mathcal{L}}_e[\zeta]\bullet,\end{aligned}$$

where $\underline{\tilde{Z}}$ and $\underline{W}^{\pm}$ are defined in (1.3) and (1.4), respectively, and $\underline{W} = \underline{\epsilon}^{+}\underline{W}^{+} - \underline{\epsilon}^{-}\underline{W}^{-}$. This formula is the main contribution of our paper. Furthermore, from Remark 3.6, we know that $\mathcal{L}_e[\zeta]$ is a self-adjoint operator. Thus, we can obtain the well-posedness result shown in Theorem 1.1. Noting that $-\partial_{X_1}^2(\tanh(|D|)|D|)^{-1}$ (or $-[[\underline{\epsilon}^{\pm}]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}$) is a first-order positive operator and is coercive when $d = 1$, it is not surprising that Theorem 1.2 holds when one neglects the surface tension. We remark that the small data condition is imposed due to the presence of the operator $\tilde{\mathcal{L}}_e[\zeta]$ and the instability operator $\widetilde{Ins}[U]$ (see (6.4)).

The rest of the paper is organized as follows. In section 2, we give some preliminary results that will be used later. Section 3 is our main new ingredient: the linearization formula of $\tau(\zeta)$ will be established; this is the main contribution of this paper. The next parts are relatively standard. In section 4, we will quasilinearize the system (1.5). In section 5, we will prove Theorem 1.1. Finally, in section 6, we focus on the proof of Theorem 1.2.

Since our interest is the local well-posedness, we take $E_b = W_e = H^{\pm} = 1$ in the following sections for convenience.

2. Preliminary results.

In this part, we introduce some operators together with their properties, which were mostly given by [15, 16]. We first introduce some notations. Denote by $L^p(\mathbb{R}^d)$ the standard Lebesgue space with associated norm $\|u\|_p = (\int_{\mathbb{R}^d}|u|^p dx)^{\frac{1}{p}}$ when $1 \leqslant p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{R}^d}|f|$, and $H^s(\mathbb{R}^d)$ ($s \in \mathbb{R}$) the usual Sobolev space $H^s(\mathbb{R}^d) = \{u \in \mathcal{S}', \|u\|_{H^s} < \infty\}$, where $\|u\|_{H^s} = \|(1-\Delta)^{\frac{s}{2}}u\|_2$. In addition, the space $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ is defined as

$$\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d) = \{u \in L^2_{loc}(\mathbb{R}^d), \nabla u \in H^{s-\frac{1}{2}}(\mathbb{R}^d)^d\}$$

endowed with the norm $\|u\|_{\dot{H}^{s+\frac{1}{2}}} = \|\nabla u\|_{H^{s-\frac{1}{2}}}$.

2.1. Dirichlet–Neumann operators.

Since the Dirichlet–Neumann operator plays an important role in the free-surface/interfacial wave problem, we start with its precise definition.

Let $t_0 > \frac{d}{2}$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$. Consider the following boundary value problem of the Laplace equation:

$$\begin{cases}\Delta_{X,z}\Phi^{\pm} = 0 \quad \text{in } \Omega^{\pm},\\ \Phi^{\pm}|_{z=\zeta} = \psi^{\pm}, \quad \partial_z\Phi^{\pm}|_{z=\mp H^{\pm}} = 0,\end{cases} \tag{2.1}$$

where

$$\Omega^{+} = \{(X,z)^\top \in \mathbb{R}^{d+1}, -H^{+} < z < \zeta(X)\},\ \Omega^{-} = \{(X,z)^\top \in \mathbb{R}^{d+1}, \zeta(X) < z < H^{-}\}.$$

It is well known that for $\psi^{\pm} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$, there exist unique solutions $\Phi^{\pm} \in \dot{H}^2(\Omega^{\pm}) = \{u \in L^2_{loc}(\Omega^{\pm}), \nabla_{X,z}u \in H^1(\Omega^{\pm})^{d+1}\}$ to (2.1). Therefore, we can define the Dirichlet–Neumann operators $\mathcal{G}^{\pm}[\zeta, H^{\pm}] : \dot{H}^{\frac{3}{2}}(\mathbb{R}^d) \mapsto \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ as follows:

$$\mathcal{G}^{\pm}[\zeta, H^{\pm}]\psi^{\pm} := \sqrt{1+|\nabla\zeta|^2}\frac{\partial\Phi^{\pm}}{\partial\mathbf{n}}\,. \tag{2.2}$$

In the following, without loss of generality, we suppress the dependency of Dirichlet–Neumann operators on ζ and $H^{\pm}$ for simplicity of notations, namely, we denote $\mathcal{G}^{\pm} = \mathcal{G}^{\pm}[\zeta, 1]$. For the operators $\mathcal{G}^{\pm}$, we always assume that the following condition holds:

$$\exists\, h^{\pm}_{\min} > 0\,, \quad \inf_{X\in\mathbb{R}^d}(1 \pm \zeta(X)) \geqslant h^{\pm}_{\min}\,. \tag{2.3}$$

We also introduce a constant M defined as

$$M := C\left(\frac{1}{h^{\pm}_{\min}}, \|\zeta\|_{H^{t_0+2}}\right)$$

as well as $M(s) := C(M, \|\zeta\|_{H^s})$. Hereafter, $C(\cdot)$ denotes generically a nondecreasing, positive function of its arguments.

For the self-adjoint operators $\mathcal{G}^{\pm}$, we have the following estimates (see [15, Theorem 3.15] and [16, Inequality (2.23)]):

(2.4)

$$\|\mathcal{G}^{\pm}\psi\|_{H^{s-\frac12}} \leqslant M\left(s+\frac12\right)\|\mathscr{B}\psi\|_{H^s}\,, \quad 0 \leqslant s \leqslant t_0 + \frac32, \psi \in \dot{H}^{s+\frac12}(\mathbb{R}^d)\,,$$
$$\|\mathcal{G}^{\pm}\psi\|_{H^{s-\frac12}} \leqslant M\left(\|\mathscr{B}\psi\|_{H^s} + \|\zeta\|_{H^{s+\frac12}}\|\mathscr{B}\psi\|_{H^{t_0+\frac32}}\right)\,, s > t_0 + \frac32, \psi \in \dot{H}^{s+\frac12}(\mathbb{R}^d)\,,$$

(2.5)

$$|(\Lambda^s\mathcal{G}^{\pm}\psi_1, \Lambda^s\psi_2)| \leqslant M\|\mathscr{B}\psi_1\|_{H^s}\|\mathscr{B}\psi_2\|_{H^s}\,, 0 \leqslant s \leqslant t_0+1, \psi_1, \psi_2 \in \dot{H}^{s+\frac12}(\mathbb{R}^d),$$

where $\mathscr{B} := \frac{|D|}{(1+|D|)^{\frac12}}$ and $\Lambda := (-\Delta)^{\frac12}$.

To establish the linearization formula of $\tau(\zeta)$, we need some commutator estimates.

Proposition 2.1. *For any $\psi_1, \psi_2 \in \dot{H}^{\frac12}(\mathbb{R}^d)$, one has*

$$|([\mathcal{G}^+, \nabla]\psi_1, \psi_2)| \leqslant M(t_0+3)\|\mathscr{B}\psi_1\|_2\|\mathscr{B}\psi_2\|_2\,, \tag{2.6}$$

$$|([\mathcal{G}^+, \mathcal{G}^-]\psi_1, \psi_2)| \leqslant M(t_0+3)\|\mathscr{B}\psi_1\|_2\|\mathscr{B}\psi_2\|_2\,. \tag{2.7}$$

Proof. We only give the proof of (2.7), and the proof of (2.6) is simpler. For any $\psi \in \dot{H}^{\frac12}$ and $0 \leqslant s \leqslant t_0 + 1$, from Theorem 3.10 of [14] or Remark 18 of [16], one has

$$\|\mathcal{G}^{\pm}\psi \mp g(X,D)\psi\|_{H^{s+\frac12}} \leqslant M(t_0+3)\|\mathscr{B}\psi\|_{H^s} \tag{2.8}$$

where

$$g(X, \vec{\xi}) = \sqrt{|\vec{\xi}|^2 + |\nabla\zeta|^2|\vec{\xi}|^2 - (\nabla\zeta\cdot\vec{\xi})^2}\,. \tag{2.9}$$

Denoting $R^{\pm} = \mathcal{G}^{\pm} \mp g(X,D)$, one then has

$$\begin{aligned}([\mathcal{G}^+, \mathcal{G}^-]\psi_1, \psi_2) =& (\mathcal{G}^-\psi_1, \mathcal{G}^+\psi_2) - (\mathcal{G}^-\psi_2, \mathcal{G}^+\psi_1)\\ =& \big(R^-\psi_1, \mathcal{G}^+\psi_2\big) - \big(R^-\psi_2, \mathcal{G}^+\psi_1\big)\\ &- \Big((1+|D|)^{-\frac12}g(X,D)\psi_1, (1+|D|)^{\frac12}R^+\psi_2\Big)\\ &+ \Big((1+|D|)^{-\frac12}g(X,D)\psi_2, (1+|D|)^{\frac12}R^+\psi_1\Big).\end{aligned}$$

Thus, one can obtain (2.7) from (2.5) and (2.8) with $s = 0$.

2.2. Some inverse operators. It follows from [16] that $(\mathcal{G}^\pm)^{-1}$ are well defined on the range of $\mathcal{G}^\pm$ and with values in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$. Furthermore, according to [16, Remarks 8 and 9] and [16, 2], one has, for $0 \leqslant s \leqslant t_0+1$,

$$\|\mathscr{B}(\mathcal{G}^\pm)^{-1}\nabla\psi\|_{H^s} \leqslant M\|\mathscr{B}\psi\|_{H^s}\,, \quad \|\mathscr{B}(\tilde{\mathcal{G}})^{-1}\nabla\psi\|_{H^s} \leqslant M\|\mathscr{B}\psi\|_{H^s}\,, \tag{2.10}$$

where $\tilde{\mathcal{G}} := \underline{\epsilon}^+\mathcal{G}^+ - \underline{\epsilon}^-\mathcal{G}^-$, and

$$\|\mathscr{B}(\mathcal{G}^\pm)^{-1}\mathcal{G}^\mp\psi\|_{H^s} \leqslant M\|\mathscr{B}\psi\|_{H^s}\,, \quad \|\mathscr{B}(\tilde{\mathcal{G}})^{-1}\mathcal{G}^\pm\psi\|_{H^s} \leqslant M\|\mathscr{B}\psi\|_{H^s} \tag{2.11}$$

and

$$\|\mathscr{B}(\underline{\rho}^+\mathcal{G}^- - \underline{\rho}^-\mathcal{G}^+)^{-1}\mathcal{G}^+\psi\|_{H^s} \leqslant M\|\mathscr{B}\psi\|_{H^s}\,. \tag{2.12}$$

In addition, one can deduce from (2.6) and (2.7) that

$$\|\mathscr{B}(\mathcal{G}^\pm)^{-1}[\tilde{\mathcal{G}},\nabla]\psi\|_2 \leqslant M(t_0+3)\|\mathscr{B}\psi\|_2\,, \quad \|\mathscr{B}(\tilde{\mathcal{G}})^{-1}[\tilde{\mathcal{G}},\mathcal{G}^\pm]\psi\|_2 \leqslant M(t_0+3)\|\mathscr{B}\psi\|_2\,. \tag{2.13}$$

Finally, in order to show that the operator $\mathcal{L}_e[\zeta]$ is coercive, we need the following fact.

PROPOSITION 2.2. *Let $d=1$, $k\in\mathbb{N}$, and $k\geqslant 1$. For any $f\in H^{k+\frac{1}{2}}(\mathbb{R})$, one has*

$$(\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f, \partial_x\partial_x^k f) \geqslant \frac{1}{2}\|\partial_x^k f\|^2_{H^{\frac{1}{2}}} - M(t_0+3)\|\partial_x^k f\|_2^2\,, \tag{2.14}$$

where we set $x := X_1$.

Proof. From (2.8), if we set $\tilde{R} = \tilde{\mathcal{G}} - g(x,D)$, then

$$\|\tilde{R}f\|_{H^{\frac{1}{2}}} \leqslant M(t_0+3)\|\mathscr{B}f\|_2\,. \tag{2.15}$$

Since $d=1$, from (2.9), we have $g(x,D)=|D|$, and therefore

$$\begin{aligned}(\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f, \partial_x\partial_x^k f) &= (|D|^{-1}(\tilde{\mathcal{G}}-\tilde{R})\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f, \partial_x\partial_x^k f)\\ &= (|D|^{-1}\partial_x\partial_x^k f, \partial_x\partial_x^k f) - (|D|^{-1}\tilde{R}\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f, \partial_x\partial_x^k f)\\ &= \||D|^{\frac{1}{2}}\partial_x^k f\|_2^2 + (|D|^{\frac{1}{2}}\tilde{R}\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f, |D|^{\frac{1}{2}}\partial_x^{k-1}f)\,.\end{aligned} \tag{2.16}$$

By (2.10), (2.15), and the interpolation theorem, one gets

$$\begin{aligned}|(|D|^{\frac{1}{2}}\tilde{R}\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f, |D|^{\frac{1}{2}}\partial_x^{k-1}f)| &\leqslant \||D|^{\frac{1}{2}}\tilde{R}\tilde{\mathcal{G}}^{-1}\partial_x\partial_x^k f\|_2\||D|^{\frac{1}{2}}\partial_x^{k-1}f\|_2\\ &\leqslant M(t_0+3)\|\mathscr{B}\partial_x^k f\|_2\||D|^{\frac{1}{2}}\partial_x^{k-1}f\|_2\\ &\leqslant \frac{1}{2}\||D|^{\frac{1}{2}}\partial_x^k f\|_2^2 + M(t_0+3)\|\partial_x^k f\|_2^2\,.\end{aligned} \tag{2.17}$$

One can obtain (2.14) from (2.16) and (2.17).

2.3. Shape derivatives. It is useful to define derivatives of the Dirichlet–Neumann operator on ζ. More accurately, we let $\psi^\pm \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ for $0\leqslant s\leqslant t_0+1$, considering the mapping

$$\zeta \mapsto \mathcal{G}^\pm\psi^\pm = \mathcal{G}^\pm[\zeta,1]\psi^\pm : \quad H^{t_0+2}(\mathbb{R}^d) \mapsto H^{s-\frac{1}{2}}(\mathbb{R}^d)\,.$$

Let $j \in \mathbb{N}$ and $\mathbf{h} = (h_1, \dots, h_j)^\top \in (H^{t_0+2}(\mathbb{R}^d))^j$. We denote by $d^j\mathcal{G}^\pm(\mathbf{h})\psi^\pm$ its jth derivative at ζ and in the direction of $\mathbf{h} = (h_1, \dots, h_j)^\top$. These derivatives are called the *shape derivatives*.

Next, we list an exact formula for the first-order shape derivative for the operators $\mathcal{G}^\pm$,

$$d\mathcal{G}^\pm(h)\psi^\pm = -\mathcal{G}^\pm(h\underline{Z}^\pm) - \nabla \cdot (h\underline{V}^\pm) \tag{2.18}$$

with

$$\underline{Z}^\pm = \frac{\mathcal{G}^\pm\psi^\pm + \nabla\zeta \cdot \nabla\psi^\pm}{1 + |\nabla\zeta|^2} \quad \text{and} \quad \underline{V}^\pm = \nabla\psi^\pm - \underline{Z}^\pm\nabla\zeta\,. \tag{2.19}$$

For the shape derivative of $\mathcal{G}^\pm$, one has the following useful estimates ([16, Inequalities (2.31) and (2.35)]): If $0 \leqslant s \leqslant t_0 + 1$ and $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$, then

$$\|d^j\mathcal{G}^\pm(\mathbf{h})\psi\|_{H^{s-\frac{1}{2}}} \leqslant M \prod_{m=1}^{j} \|h_m\|_{H^{s\vee t_0+1}} \|\mathscr{B}\psi\|_{H^s}\,, \tag{2.20}$$

where $a \vee b$ stands for $\max\{a, b\}$. If $0 \leqslant s \leqslant t_0 + 1$ and $\psi_1, \psi_2 \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$, then

$$|(\Lambda^s d^j\mathcal{G}^\pm(h)\psi_1, \Lambda^s\psi_2)| \leqslant M \prod_{m=1}^{j} \|h_m\|_{H^{s\vee t_0+1}} \|\mathscr{B}\psi_1\|_{H^s} \|\mathscr{B}\psi_2\|_{H^s}\,. \tag{2.21}$$

In the end of this section, we introduce two new operators which will be utilized in the following sections (see also [16]).

- Define $\mathcal{I}[U]$ by

$$\mathcal{I}[U](\bullet) := \nabla \cdot (\underline{V}^+\bullet) + \underline{\rho}^-\mathcal{G}(\mathcal{G}^-)^{-1}(\nabla \cdot ([[\underline{V}^\pm]]\bullet))\,. \tag{2.22}$$

- Define the operator $\mathscr{E}[\zeta]$ (which is associated with the Kelvin–Helmholtz instabilities) by

$$\mathscr{E}[\zeta](\bullet) = \nabla \circ \left(\underline{\rho}^+\mathcal{G}^- - \underline{\rho}^-\mathcal{G}^+\right)^{-1} \circ \nabla^\top(\bullet)\,. \tag{2.23}$$

3. Linearization formula of $\boldsymbol{\tau(\zeta)}$. As argued in sections 1.3 and 1.4, the essence of the proof is to get the linearization formula of $\tau(\zeta)$. In this part, we will focus on this point.

Before stating the result, we first give some definitions. Following [16], we define the energy $\mathcal{E}^N(U)$ of the system (1.5) as

$$\mathcal{E}^N(U) = \|\nabla\psi\|^2_{H^{t_0+2}} + \sum_{\alpha\in\mathbb{N}^{1+d}, |\alpha|\leqslant N} \left(\|\zeta_{(\alpha)}\|^2_{H^1} + \|\mathscr{B}\psi_{(\alpha)}\|^2_2\right), \tag{3.1}$$

where $N \in \mathbb{N}$ and $U = (\zeta, \psi)^\top$. It is noted that the energy can be used to measure the size of the residual. For all $T > 0$, we define the space E^N_T as

$$E^N_T = \left\{U \in C([0, T]; H^{t_0+2}(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d), \sup_{0\leqslant t\leqslant T} \mathcal{E}^N(U(t)) < \infty\right\}. \tag{3.2}$$

Moreover, we set

$$\mathfrak{m}^N(U) = C\left(M, \mathcal{E}^N(U)\right).$$

To linearize $\tau(\zeta)$, we first give a linearization formula of $\mathcal{G}^\pm w$. The proof is very similar to [16, Proposition 6] (a linearization formula for $\mathcal{G}\psi$), and therefore we omit the details here.

PROPOSITION 3.1. *Let $T > 0$, $t_0 > \frac{d}{2}$, and $N \in \mathbb{N}$ be such that*

$$[(N+1)/2] \geqslant 1 \vee t_0 + \frac{1}{2} \quad \text{and} \quad N \geqslant t_0 + \frac{7}{2}\,.$$

Then for all $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)^\top \in \mathbb{N}^{1+d}$ with $1 \leqslant |\alpha| \leqslant N$, one has

$$\begin{aligned}
&\partial^\alpha(\mathcal{G}^\pm w) = \mathcal{G}^\pm \partial^\alpha w + S_\alpha && \text{if} \quad |\alpha| \leqslant N-1\,,\\
&\partial^\alpha(\mathcal{G}^\pm w) = \mathcal{G}^\pm \partial^\alpha w + d\mathcal{G}^\pm(\partial^\alpha \zeta) w + S_\alpha &&\\
&\qquad = \mathcal{G}^\pm \partial^\alpha w - \mathcal{G}^\pm(\partial^\alpha \zeta \underline{\tilde{Z}}) - \nabla \cdot (\underline{W} \partial^\alpha \zeta) + S_\alpha && \text{if} \quad |\alpha| = N\,,
\end{aligned}$$

where $\underline{\tilde{Z}}^\pm$ and $\underline{W}^\pm$ are defined in (1.3) *and* (1.4)*, respectively, and S_α satisfies*

$$\|\mathscr{B} S_\alpha\|_2 \leqslant \mathfrak{m}^N(U)\,.$$

We then focus on the linearization formula of $\partial^\alpha \tau(\zeta)$. For the sake of clearness, we first establish the following two lemmas.

LEMMA 3.2. *If $f \in H^1(\mathbb{R}^d)$, then one has*

$$\mathcal{G}^\pm \tilde{\mathcal{G}}^{-1} \nabla f \sim \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm \nabla f \sim \tilde{\mathcal{G}}^{-1} \nabla \mathcal{G}^\pm f \sim \nabla \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm f\,, \tag{3.3}$$

$$\mathcal{G}^+ \tilde{\mathcal{G}}^{-1} \mathcal{G}^- f = \mathcal{G}^- \tilde{\mathcal{G}}^{-1} \mathcal{G}^+ f \sim \tilde{\mathcal{G}}^{-1} \mathcal{G}^+ \mathcal{G}^- f \sim \tilde{\mathcal{G}}^{-1} \mathcal{G}^- \mathcal{G}^+ f\,, \tag{3.4}$$

$$\tilde{\mathcal{G}}^{-1} \nabla \tilde{\mathcal{G}} f \sim \nabla f\,, \quad \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm \tilde{\mathcal{G}} f \sim \mathcal{G}^\pm f\,, \quad \tilde{\mathcal{G}}^{-1} \partial_{X_i} \partial_{X_j} f \sim \partial_{X_i} \tilde{\mathcal{G}}^{-1} \partial_{X_j} f\,, \tag{3.5}$$

where $a \sim b$ means

$$\|\mathscr{B}(a-b)\|_2 \leqslant M(t_0+3) \|\mathscr{B} f\|_2\,.$$

Proof. We only need to prove (3.3) and the first identity of (3.4), since the other cases are similar to (3.3). Actually, one has

$$\mathcal{G}^\pm \tilde{\mathcal{G}}^{-1} \nabla = \tilde{\mathcal{G}}^{-1} \tilde{\mathcal{G}} \mathcal{G}^\pm \tilde{\mathcal{G}}^{-1} \nabla = \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm \nabla + \tilde{\mathcal{G}}^{-1} [\tilde{\mathcal{G}}, \mathcal{G}^\pm] \tilde{\mathcal{G}}^{-1} \nabla\,,$$

$$\tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm \nabla = \tilde{\mathcal{G}}^{-1} \nabla \mathcal{G}^\pm + \tilde{\mathcal{G}}^{-1} [\mathcal{G}^\pm, \nabla]\,,$$

$$\tilde{\mathcal{G}}^{-1} \nabla \mathcal{G}^\pm = \tilde{\mathcal{G}}^{-1} (\nabla \tilde{\mathcal{G}}) \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm = \nabla \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm + \tilde{\mathcal{G}}^{-1} [\nabla, \tilde{\mathcal{G}}] \tilde{\mathcal{G}}^{-1} \mathcal{G}^\pm\,.$$

Thus (3.3) follows from (2.10)–(2.13). The first identity of (3.4) follows from the following calculation:

$$\begin{aligned}
&\underline{\epsilon}^+ \underline{\epsilon}^- \mathcal{G}^- \tilde{\mathcal{G}}^{-1} \mathcal{G}^+ - \underline{\epsilon}^+ \underline{\epsilon}^- \mathcal{G}^+ \tilde{\mathcal{G}}^{-1} \mathcal{G}^-\\
&\quad = [(\underline{\epsilon}^- \mathcal{G}^- - \underline{\epsilon}^+ \mathcal{G}^+) + \underline{\epsilon}^+ \mathcal{G}^+] \tilde{\mathcal{G}}^{-1} (\underline{\epsilon}^+ \mathcal{G}^+) - \underline{\epsilon}^+ \underline{\epsilon}^- \mathcal{G}^+ \tilde{\mathcal{G}}^{-1} \mathcal{G}^-\\
&\quad = -\underline{\epsilon}^+ \mathcal{G}^+ + \underline{\epsilon}^+ \mathcal{G}^+ \tilde{\mathcal{G}}^{-1} (\underline{\epsilon}^+ \mathcal{G}^+) - \underline{\epsilon}^+ \mathcal{G}^+ \tilde{\mathcal{G}}^{-1} (\underline{\epsilon}^- \mathcal{G}^-)\\
&\quad = -\underline{\epsilon}^+ \mathcal{G}^+ + \underline{\epsilon}^+ \mathcal{G}^+ \tilde{\mathcal{G}}^{-1} (\underline{\epsilon}^+ \mathcal{G}^+ - \underline{\epsilon}^- \mathcal{G}^-) = 0\,.
\end{aligned}$$

LEMMA 3.3. *If $g \in H^{\frac{1}{2}}(\mathbb{R}^d)$ and $f \in H^{t_0+1}(\mathbb{R}^d)$, then one has*

$$\|\mathscr{B} [\mathcal{G}^\pm \tilde{\mathcal{G}}^{-1} \nabla, f] g\|_2 \leqslant M(t_0+3) \|f\|_{H^{t_0+1}} \|g\|_{H^{\frac{1}{2}}}\,.$$

Proof. Similar to the proof of Proposition 2.1, denoting $R^\pm = \mathcal{G}^\pm \mp g(X,D)$ and $\tilde{R} = \tilde{\mathcal{G}} - g(X,D)$, one has

$$\begin{aligned}\mathcal{G}^{\pm}\tilde{\mathcal{G}}^{-1}\nabla &= \pm\, g(X,D)\tilde{\mathcal{G}}^{-1}\nabla + R^{\pm}\tilde{\mathcal{G}}^{-1}\nabla\\ &= \pm\,(\tilde{\mathcal{G}}-\tilde{R})\tilde{\mathcal{G}}^{-1}\nabla + R^{\pm}\tilde{\mathcal{G}}^{-1}\nabla\\ &= \pm\,\nabla + (\mp\tilde{R}+R^{+})\tilde{\mathcal{G}}^{-1}\nabla\,.\end{aligned}$$

Thus,

$$\begin{aligned}[\mathcal{G}^{\pm}\tilde{\mathcal{G}}^{-1}\nabla, f]g =&[\pm\nabla + (\mp\tilde{R}+R^{+})\tilde{\mathcal{G}}^{-1}\nabla, f]g\\ =& \pm\, g\nabla f + (\mp\tilde{R}+R^{+})\tilde{\mathcal{G}}^{-1}\nabla(fg) + f(\mp\tilde{R}+R^{+})\tilde{\mathcal{G}}^{-1}\nabla g\,.\end{aligned}$$

Using (2.8), (2.10), and the Sobolev multiplication law (see, for example, [30]), one obtains

$$\|\mathscr{B}[\mathcal{G}^{\pm}\tilde{\mathcal{G}}^{-1}\nabla, f]g\|_2 \leqslant M(t_0+3)\|f\|_{H^{t_0+1}}\|g\|_{H^{\frac{1}{2}}}\,.$$

We can now state our main result as follows.

PROPOSITION 3.4. *Let $T>0$, $t_0>\frac{d}{2}$, and $N\in\mathbb{N}$ be such that*

$$[(N+1)/2] \geqslant 1\vee t_0 + \frac{1}{2} \quad \textit{and} \quad N\geqslant t_0+\frac{7}{2}\,.$$

Then for all $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_d)^\top\in\mathbb{N}^{1+d}$ with $1\leqslant|\alpha|\leqslant N$, one has

$$\begin{aligned}\partial^\alpha\tau(\zeta) =&S_\alpha && \textit{if}\quad |\alpha|\leqslant N-1\,,\\ \partial^\alpha\tau(\zeta) =&\mathcal{L}_e[\zeta]\partial^\alpha\zeta + S_\alpha && \textit{if}\quad |\alpha| = N\,,\end{aligned}$$

where S_α satisfies

$$\|\mathscr{B}S_\alpha\|_2 \leqslant \mathfrak{m}^N(U)\,,$$

and

$$\begin{aligned}\mathcal{L}_e[\zeta]\bullet =& -[[\underline{\epsilon}^{\pm}]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\bullet - [[\underline{\epsilon}^{\pm}]]\left(\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\bullet) + \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\bullet\right)\\ &-\underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\bullet) + \underline{\epsilon}^{+}\underline{\epsilon}^{-}[[\underline{\tilde{Z}}]]\mathcal{G}^{+}\tilde{\mathcal{G}}^{-1}\mathcal{G}^{-}([[\underline{\tilde{Z}}]]\bullet)\,,\end{aligned}$$

where $\underline{W}=\underline{\epsilon}^{+}\underline{W}^{+}-\underline{\epsilon}^{-}\underline{W}^{-}$, and $\underline{\tilde{Z}}$, $\underline{W}^{\pm}$ are defined in (1.3) *and* (1.4)*, respectively.*

Remark 3.5. From Proposition 3.1 and the following proof, it is easy to check that if $\alpha=(\alpha_1,\dots,\alpha_d)^\top\in\mathbb{N}^d$, one can get $\|\mathscr{B}S_\alpha\|_2\leqslant\tilde{\mathfrak{m}}^N(U)$, where $\tilde{\mathfrak{m}}^N(U)$ is defined in (6.3).

Remark 3.6. Since $\mathcal{G}^{+}\tilde{\mathcal{G}}^{-1}\mathcal{G}^{-}=\mathcal{G}^{-}\tilde{\mathcal{G}}^{-1}\mathcal{G}^{+}$ (see (3.4)), we know that the operator $\mathcal{L}_e[\zeta]$ is self-adjoint.

Proof. In the following, for convenience, we write $a=b+\cdots$ if $\|\mathscr{B}(a-b)\|_2\leqslant\mathfrak{m}^N(U)$.

For the sake of clearness, before establishing the linearization formula of $\partial^\alpha\tau(\zeta)$, we list the expression of $\tau(\zeta)$:

$$\tau(\zeta)=\frac{[[\underline{\epsilon}^{\pm}(\mathcal{G}^{\pm}w)^2]]}{1+|\nabla\zeta|^2}-\frac{1}{2}\left([[\underline{\epsilon}^{\pm}|\underline{W}^{\pm}|^2]]+[[\underline{\epsilon}^{\pm}|\underline{\tilde{Z}}^{\pm}|^2]]\right)-\frac{[[\underline{\epsilon}^{\pm}]]\zeta_{X_1}^2}{1+|\nabla\zeta|^2}-[[\underline{\epsilon}^{\pm}\underline{W}_1^{\pm}]]\,,$$

where

$$\underline{\tilde{Z}}^{\pm}=\frac{\mathcal{G}^{\pm}w+\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\,,\quad \underline{W}^{\pm}=\nabla w-\underline{\tilde{Z}}^{\pm}\nabla\zeta,\quad w=[[\underline{\epsilon}^{\pm}]]\tilde{\mathcal{G}}^{-1}\partial_{X_1}\zeta\,.$$

In order to obtain the linearization formula of $\partial^\alpha \tau(\zeta)$, we first give the following three facts.

(a) It is easy to check that

$$\partial^\alpha \left(\frac{1}{1+|\nabla\zeta|^2} \right) = -\frac{2\nabla\zeta\cdot\nabla\partial^\alpha\zeta}{(1+|\nabla\zeta|^2)^2} + \cdots .$$

(b) One has the following fact:

$$\begin{aligned}\nabla\partial^\alpha w =&[[\underline{\epsilon}^\pm]]\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta + \underline{\tilde{Z}}^+\nabla\partial^\alpha\zeta \\ &+ \epsilon^-[[\underline{\tilde{Z}}^\pm]]\nabla\tilde{\mathcal{G}}^{-1}\mathcal{G}^-\partial^\alpha\zeta + \nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta) + \cdots \\ \overset{or}{=}&[[\underline{\epsilon}^\pm]]\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta + \underline{\tilde{Z}}^-\nabla\partial^\alpha\zeta \\ &+ \epsilon^+[[\underline{\tilde{Z}}^\pm]]\nabla\tilde{\mathcal{G}}^{-1}\mathcal{G}^+\partial^\alpha\zeta + \nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta) + \cdots .\end{aligned} \tag{3.6}$$

Actually, due to $\tilde{\mathcal{G}}w = [[\underline{\epsilon}^\pm]]\zeta_{X_1}$, from (2.10) and Proposition 3.1, one has

$$\begin{aligned}\nabla\partial^\alpha w =&\tilde{\mathcal{G}}^{-1}\tilde{\mathcal{G}}\nabla\partial^\alpha w \\ =&\tilde{\mathcal{G}}^{-1}\nabla\tilde{\mathcal{G}}\partial^\alpha w + \tilde{\mathcal{G}}^{-1}[\tilde{\mathcal{G}},\nabla]\partial^\alpha w \\ =&\tilde{\mathcal{G}}^{-1}\nabla\partial^\alpha\left(\tilde{\mathcal{G}}w\right) - \tilde{\mathcal{G}}^{-1}\nabla d\tilde{\mathcal{G}}(\partial^\alpha\zeta)w + \tilde{\mathcal{G}}^{-1}[\tilde{\mathcal{G}},\nabla]\partial^\alpha w + \cdots \\ =&[[\underline{\epsilon}^\pm]]\tilde{\mathcal{G}}^{-1}\nabla\partial_{X_1}\partial^\alpha\zeta - \tilde{\mathcal{G}}^{-1}\nabla d\tilde{\mathcal{G}}(\partial^\alpha\zeta)w + \tilde{\mathcal{G}}^{-1}[\tilde{\mathcal{G}},\nabla]\partial^\alpha w + \cdots ,\end{aligned}$$

and from (2.10) and (2.13), one has

$$\|\mathscr{B}\tilde{\mathcal{G}}^{-1}[\tilde{\mathcal{G}},\nabla]\partial^\alpha w\|_2 \leqslant M(t_0+3)\|\mathscr{B}\partial^\alpha w\|_2 \leqslant M\left(N+\frac{1}{2}\right)\|\zeta\|_{H^{N+\frac{1}{2}}} \leqslant \mathfrak{m}^N(U).$$

From this, one has

$$\nabla\partial^\alpha w = [[\underline{\epsilon}^\pm]]\tilde{\mathcal{G}}^{-1}\nabla\partial_{X_1}\partial^\alpha\zeta - \tilde{\mathcal{G}}^{-1}\nabla d\tilde{\mathcal{G}}(\partial^\alpha\zeta)w + \cdots .$$

Noticing that $\tilde{\mathcal{G}} = \underline{\epsilon}^+\mathcal{G}^+ - \underline{\epsilon}^-\mathcal{G}^-$, one has

$$\begin{aligned}-d\tilde{\mathcal{G}}(\partial^\alpha\zeta)w =& \left(\underline{\epsilon}^+\mathcal{G}^+(\underline{\tilde{Z}}^+\partial^\alpha\zeta) - \underline{\epsilon}^-\mathcal{G}^-(\underline{\tilde{Z}}^-\partial^\alpha\zeta)\right) + \left(\underline{\epsilon}^+\nabla\cdot(\underline{W}^+\partial^\alpha\zeta) - \underline{\epsilon}^-\nabla\cdot(\underline{W}^-\partial^\alpha\zeta)\right) \\ =& \tilde{\mathcal{G}}(\underline{\tilde{Z}}^+\partial^\alpha\zeta) + \epsilon^-\mathcal{G}^-([[\underline{\tilde{Z}}^\pm]]\partial^\alpha\zeta) + \nabla\cdot(\underline{W}\partial^\alpha\zeta) \\ \overset{or}{=}& \tilde{\mathcal{G}}(\underline{\tilde{Z}}^-\partial^\alpha\zeta) + \epsilon^+\mathcal{G}^+([[\underline{\tilde{Z}}^\pm]]\partial^\alpha\zeta) + \nabla\cdot(\underline{W}\partial^\alpha\zeta).\end{aligned} \tag{3.7}$$

Now, one can obtain (3.6) from the above facts and Lemmas 3.2 and 3.3.

(c) Similar to (b), one has

$$\begin{aligned}\mathcal{G}^\pm\partial^\alpha w =& \tilde{\mathcal{G}}^{-1}\tilde{\mathcal{G}}\mathcal{G}^\pm\partial^\alpha w \\ =& \tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\tilde{\mathcal{G}}\partial^\alpha w + \tilde{\mathcal{G}}^{-1}[\tilde{\mathcal{G}},\mathcal{G}^\pm]\partial^\alpha w \\ =& \tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial^\alpha\tilde{\mathcal{G}}w - \tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm d\tilde{\mathcal{G}}(\partial^\alpha\zeta)w + \cdots \\ =& [[\underline{\epsilon}^\pm]]\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial_{X_1}\partial^\alpha\zeta - \tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm d\tilde{\mathcal{G}}(\partial^\alpha\zeta)w + \cdots ;\end{aligned}$$

from (3.7) and Lemmas 3.2 and 3.3, one gets

$$\begin{aligned}\mathcal{G}^\pm\partial^\alpha w =&[[\underline{\epsilon}^\pm]]\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial_{X_1}\partial^\alpha\zeta + \mathcal{G}^\pm(\underline{\tilde{Z}}^\pm\partial^\alpha\zeta) + \epsilon^\mp\mathcal{G}^\pm\tilde{\mathcal{G}}^{-1}\mathcal{G}^\mp([[\underline{\tilde{Z}}^\pm]]\partial^\alpha\zeta) \\ &+ \underline{W}\cdot\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\nabla\partial^\alpha\zeta + \cdots .\end{aligned}$$

Next, we calculate $\partial^\alpha\tau(\zeta)$.

Step 1. The first step is to show that

$$\partial^\alpha \tau(\zeta) = \left[\left[\frac{\underline{\epsilon}^\pm \mathcal{G}^\pm w \partial^\alpha (\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2}\right]\right] + \left[\left[\underline{\epsilon}^\pm \left(-\underline{W}^\pm \cdot \nabla\partial^\alpha w + \underline{\tilde{Z}}^\pm \underline{W}^\pm \cdot \nabla\partial^\alpha \zeta\right)\right]\right] + [[\underline{\epsilon}^\pm]] \left(\frac{(\nabla\zeta\cdot\nabla w)\partial^\alpha \zeta_{X_1}}{1+|\nabla\zeta|^2} - \partial^\alpha \partial_{X_1} w\right) + \cdots . \tag{3.8}$$

Actually, by the Sobolev multiplication law, (2.4), and (2.10), it is easy to get

$$\begin{aligned}
&\partial^\alpha \left(\frac{(\mathcal{G}^\pm w)^2}{1+|\nabla\zeta|^2} - \frac{1}{2}(|\underline{W}^\pm|^2 + |\underline{\tilde{Z}}^\pm|^2)\right)\\
&= \frac{2\mathcal{G}^\pm w \partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2} + \partial^\alpha((1+|\nabla\zeta|^2)^{-1})(\mathcal{G}^\pm w)^2 - \underline{W}^\pm \cdot \partial^\alpha(\nabla w - \underline{\tilde{Z}}^\pm \nabla\zeta) - \underline{\tilde{Z}}^\pm \partial^\alpha \underline{\tilde{Z}}^\pm + \cdots\\
&= \frac{2\mathcal{G}^\pm w \partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2} + \partial^\alpha((1+|\nabla\zeta|^2)^{-1})(\mathcal{G}^\pm w)^2 - \underline{W}^\pm \cdot \nabla\partial^\alpha w + \underline{\tilde{Z}}^\pm \underline{W}^\pm \cdot \nabla\partial^\alpha\zeta - \mathcal{G}^\pm w \partial^\alpha \underline{\tilde{Z}}^\pm\\
&\quad + \cdots
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}^\pm w \partial^\alpha \underline{\tilde{Z}}^\pm &= \mathcal{G}^\pm w \partial^\alpha \left(\frac{\mathcal{G}^\pm w + \nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right)\\
&= \frac{\mathcal{G}^\pm w \partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2} + \partial^\alpha((1+|\nabla\zeta|^2)^{-1})(\mathcal{G}^\pm w)^2 + \mathcal{G}^\pm w \partial^\alpha \left(\frac{\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right) + \cdots .
\end{aligned}$$

Hence

$$\begin{aligned}
&\partial^\alpha \left(\frac{(\mathcal{G}^\pm w)^2}{1+|\nabla\zeta|^2} - \frac{1}{2}(|\underline{W}^\pm|^2 + |\underline{\tilde{Z}}^\pm|^2)\right)\\
&= \frac{\mathcal{G}^\pm w \partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2} - \underline{W}^\pm \cdot \nabla\partial^\alpha w + \underline{\tilde{Z}}^\pm \underline{W}^\pm \cdot \nabla\partial^\alpha\zeta - \mathcal{G}^\pm w \partial^\alpha \left(\frac{\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right) + \cdots .
\end{aligned}$$

Noticing $(\underline{\epsilon}^+ \mathcal{G}^+ - \underline{\epsilon}^- \mathcal{G}^-)\, w = \tilde{\mathcal{G}} w = [[\underline{\epsilon}^\pm]]\zeta_{X_1}$, one has

$$\begin{aligned}
&\left[\left[\epsilon^\pm \partial^\alpha \left(\frac{(\mathcal{G}^\pm w)^2}{1+|\nabla\zeta|^2} - \frac{1}{2}(|\underline{W}^\pm|^2 + |\underline{\tilde{Z}}^\pm|^2)\right)\right]\right]\\
&\quad = \left[\left[\epsilon^\pm \frac{\mathcal{G}^\pm w \partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2}\right]\right] + \left[\left[\underline{\epsilon}^\pm \left(-\underline{W}^\pm \cdot \nabla\partial^\alpha w + \underline{\tilde{Z}}^\pm \underline{W}^\pm \cdot \nabla\partial^\alpha\zeta\right)\right]\right]\\
&\quad\quad - [[\underline{\epsilon}^\pm]]\zeta_{X_1} \partial^\alpha \left(\frac{\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right) + \cdots .
\end{aligned} \tag{3.9}$$

On the other hand, from the Sobolev multiplication law and (2.4), one has

$$\begin{aligned}
\partial^\alpha \underline{W}_1^\pm &= \partial^\alpha \partial_{X_1} w - \partial^\alpha \left(\zeta_{X_1} \frac{\mathcal{G}^\pm w + \nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right)\\
&= \partial^\alpha \partial_{X_1} w - \frac{\mathcal{G}^\pm w \partial^\alpha \zeta_{X_1}}{1+|\nabla\zeta|^2} + \frac{2\zeta_{X_1} \mathcal{G}^\pm w \nabla\zeta\cdot\nabla\partial^\alpha\zeta}{(1+|\nabla\zeta|^2)^2}\\
&\quad - \frac{\zeta_{X_1}\partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2} - \frac{\partial^\alpha \zeta_{X_1}(\nabla\zeta\cdot\nabla w)}{1+|\nabla\zeta|^2} - \zeta_{X_1}\partial^\alpha \left(\frac{\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right) + \cdots ,
\end{aligned}$$

and then

$$
\begin{aligned}
&-\partial^\alpha\left(\frac{[[\underline{\epsilon}^\pm]]\zeta_{X_1}^2}{1+|\nabla\zeta|^2}+[[\underline{\epsilon}^\pm\underline{W}_1^\pm]]\right)\\
&\quad=-[[\underline{\epsilon}^\pm]]\left(\frac{2\zeta_{X_1}\partial^\alpha\zeta_{X_1}}{1+|\nabla\zeta|^2}-\frac{2\zeta_{X_1}^2\nabla\zeta\cdot\nabla\partial^\alpha\zeta}{(1+|\nabla\zeta|^2)^2}\right)-[[\underline{\epsilon}^\pm]]\partial^\alpha\partial_{X_1}w\\
&\qquad+\left[\left[\frac{\underline{\epsilon}^\pm\mathcal{G}^\pm w\partial^\alpha\zeta_{X_1}}{1+|\nabla\zeta|^2}\right]\right]-\left[\left[\underline{\epsilon}^\pm\zeta_{X_1}\frac{2\mathcal{G}^\pm w\nabla\zeta\cdot\nabla\partial^\alpha\zeta}{(1+|\nabla\zeta|^2)^2}\right]\right]+\left[\left[\underline{\epsilon}^\pm\zeta_{X_1}\frac{\partial^\alpha\mathcal{G}^\pm w}{1+|\nabla\zeta|^2}\right]\right]\\
&\qquad+[[\underline{\epsilon}^\pm]]\frac{\partial^\alpha\zeta_{X_1}(\nabla\zeta\cdot\nabla w)}{1+|\nabla\zeta|^2}+[[\underline{\epsilon}^\pm]]\zeta_{X_1}\partial^\alpha\left(\frac{\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right)+\cdots.
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
&\left[\left[\frac{\underline{\epsilon}^\pm\mathcal{G}^\pm w\partial^\alpha\zeta_{X_1}}{1+|\nabla\zeta|^2}\right]\right]-\left[\left[\underline{\epsilon}^\pm\zeta_{X_1}\frac{2\mathcal{G}^\pm w\nabla\zeta\cdot\nabla\partial^\alpha\zeta}{(1+|\nabla\zeta|^2)^2}\right]\right]+\left[\left[\underline{\epsilon}^\pm\zeta_{X_1}\frac{\partial^\alpha(\mathcal{G}^\pm w)}{1+|\nabla\zeta|^2}\right]\right]\\
&\quad=[[\underline{\epsilon}^\pm]]\frac{2\zeta_{X_1}\partial^\alpha\zeta_{X_1}}{1+|\nabla\zeta|^2}-[[\underline{\epsilon}^\pm]]\frac{2\zeta_{X_1}^2\nabla\zeta\cdot\nabla\partial^\alpha\zeta}{(1+|\nabla\zeta|^2)^2}.
\end{aligned}
$$

Therefore

(3.10)

$$
\begin{aligned}
&-\partial^\alpha\left(\frac{[[\underline{\epsilon}^\pm]]\zeta_{X_1}^2}{1+|\nabla\zeta|^2}+[[\underline{\epsilon}^\pm\underline{W}_1^\pm]]\right)\\
&\quad=-[[\underline{\epsilon}^\pm]]\partial^\alpha\partial_{X_1}w+[[\underline{\epsilon}^\pm]]\frac{\partial^\alpha\zeta_{X_1}(\nabla\zeta\cdot\nabla w)}{1+|\nabla\zeta|^2}+[[\underline{\epsilon}^\pm]]\zeta_{X_1}\partial^\alpha\left(\frac{\nabla\zeta\cdot\nabla w}{1+|\nabla\zeta|^2}\right)+\cdots.
\end{aligned}
$$

Adding (3.9) and (3.10), one can get (3.8).

Step 2. The second step is to show that

$$
\begin{aligned}
\partial^\alpha\tau(\zeta)=&-[[\underline{\epsilon}^\pm]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta+\left[\left[\frac{\underline{\epsilon}^\pm\mathcal{G}^\pm w\partial^\alpha(\mathcal{G}^\pm w-\zeta_{X_1})}{1+|\nabla\zeta|^2}\right]\right]\\
&-[[\underline{\epsilon}^\pm]]\left(\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta)+\underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta\right)\\
&+\epsilon^+\epsilon^-[[([[\underline{\tilde{Z}}^\pm]])\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial_{X_1}\partial^\alpha\zeta]]-\underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta)\\
&-\epsilon^+\epsilon^-[[([[\underline{\tilde{Z}}^\pm]])\underline{W}^\pm\cdot\tilde{\mathcal{G}}^{-1}\mathcal{G}^\mp\nabla\partial^\alpha\zeta]].
\end{aligned}\tag{3.11}
$$

Firstly, by (b), one can get

$$
\begin{aligned}
&\left[\left[\underline{\epsilon}^\pm\left(-\underline{W}^\pm\cdot\nabla\partial^\alpha w+\underline{\tilde{Z}}^\pm\underline{W}^\pm\cdot\nabla\partial^\alpha\zeta\right)\right]\right]\\
&\quad=-[[\underline{\epsilon}^\pm]]\underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta-\epsilon^+\epsilon^-[[([[\underline{\tilde{Z}}^\pm]])\underline{W}^\pm\cdot\nabla\tilde{\mathcal{G}}^{-1}\mathcal{G}^\mp\partial^\alpha\zeta]]\\
&\qquad-\underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta)+\cdots.
\end{aligned}\tag{3.12}
$$

On the other hand, due to (b), one has

$$
\begin{aligned}
\partial_{X_1}\partial^\alpha w=&[[\underline{\epsilon}^\pm]]\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta+\underline{\tilde{Z}}^+\partial_{X_1}\partial^\alpha\zeta\\
&+\epsilon^-[[\underline{\tilde{Z}}^\pm]]\partial_{X_1}\tilde{\mathcal{G}}^{-1}\mathcal{G}^-(\partial^\alpha\zeta)+\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta)+\cdots\\
\overset{or}{=}&[[\underline{\epsilon}^\pm]]\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta+\underline{\tilde{Z}}^-\partial_{X_1}\partial^\alpha\zeta\\
&+\epsilon^+[[\underline{\tilde{Z}}^\pm]]\partial_{X_1}\tilde{\mathcal{G}}^{-1}\mathcal{G}^+(\partial^\alpha\zeta)+\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta)+\cdots.
\end{aligned}
$$

Upon noting

$$\underline{\epsilon}^+ \underline{\tilde{Z}}^+ \partial_{X_1}\partial^\alpha\zeta - \underline{\epsilon}^- \underline{\tilde{Z}}^- \partial_{X_1}\partial^\alpha\zeta = \left[\left[\frac{\underline{\epsilon}^\pm \mathcal{G}^\pm w \partial_{X_1}\partial^\alpha\zeta}{1+|\nabla\zeta|^2}\right]\right] + [[\underline{\epsilon}^\pm]]\frac{\partial^\alpha\zeta_{X_1}(\nabla\zeta\cdot\nabla w)}{1+|\nabla\zeta|^2},$$

one has

$$\begin{aligned}
&[[\underline{\epsilon}^\pm]]\left(\frac{\partial^\alpha\zeta_{X_1}(\nabla\zeta\cdot\nabla w)}{1+|\nabla\zeta|^2} - \partial^\alpha\partial_{X_1} w\right)\\
&\quad = -[[\underline{\epsilon}^\pm]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta - \left[\left[\frac{\underline{\epsilon}^\pm \mathcal{G}^\pm w\partial_{X_1}\partial^\alpha\zeta}{1+|\nabla\zeta|^2}\right]\right] + \epsilon^+\epsilon^-[[([[\underline{\tilde{Z}}^\pm]])\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial_{X_1}\partial^\alpha\zeta]]\\
&\qquad - [[\underline{\epsilon}^\pm]]\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta) + \cdots .
\end{aligned} \tag{3.13}$$

Thus (3.11) follows from (3.8), (3.12), and (3.13).

Step 3. The third step is to show that

$$\begin{aligned}
&\left[\left[\frac{\underline{\epsilon}^\pm \mathcal{G}^\pm w\partial^\alpha\left(\mathcal{G}^\pm w - \zeta_{X_1}\right)}{1+|\nabla\zeta|^2}\right]\right]\\
&\quad = \underline{\epsilon}^+\underline{\epsilon}^-[[\underline{\tilde{Z}}^\pm]]\mathcal{G}^+\tilde{\mathcal{G}}^{-1}\mathcal{G}^-([[\underline{\tilde{Z}}^\pm]]\partial^\alpha\zeta) - \underline{\epsilon}^+\underline{\epsilon}^-[[([[\underline{\tilde{Z}}^\pm]])\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial_{X_1}\partial^\alpha\zeta]]\\
&\qquad + \underline{\epsilon}^+\underline{\epsilon}^-[[([[\underline{\tilde{Z}}^\pm]])\underline{W}^\pm\cdot\tilde{\mathcal{G}}^{-1}\mathcal{G}^\mp\nabla\partial^\alpha\zeta]] + \cdots .
\end{aligned} \tag{3.14}$$

First of all, one has

$$\begin{aligned}
\underline{\epsilon}^+\mathcal{G}^+ w\partial^\alpha(\mathcal{G}^+ w) - \underline{\epsilon}^+\mathcal{G}^+ w\partial^\alpha\zeta_{X_1} &= \underline{\epsilon}^+\mathcal{G}^+ w\partial^\alpha(\mathcal{G}^+ w) - \frac{\underline{\epsilon}^+}{[[\underline{\epsilon}^+]]}\mathcal{G}^+ w\partial^\alpha\left(\underline{\epsilon}^+\mathcal{G}^+ w - \underline{\epsilon}^-\mathcal{G}^- w\right)\\
&= \frac{\underline{\epsilon}^+\underline{\epsilon}^-\mathcal{G}^+ w}{[[\underline{\epsilon}^+]]}\partial^\alpha\left(\mathcal{G}^- w - \mathcal{G}^+ w\right)
\end{aligned}$$

and

$$\begin{aligned}
\underline{\epsilon}^-\mathcal{G}^- w\partial^\alpha(\mathcal{G}^- w) - \underline{\epsilon}^-\mathcal{G}^- w\partial^\alpha\zeta_{X_1} &= \underline{\epsilon}^-\mathcal{G}^- w\partial^\alpha(\mathcal{G}^- w) - \frac{\underline{\epsilon}^-}{[[\underline{\epsilon}^+]]}\mathcal{G}^- w\partial^\alpha\left(\underline{\epsilon}^+\mathcal{G}^+ w - \underline{\epsilon}^-\mathcal{G}^- w\right)\\
&= \frac{\underline{\epsilon}^+\underline{\epsilon}^-\mathcal{G}^- w}{[[\underline{\epsilon}^+]]}\partial^\alpha\left(\mathcal{G}^- w - \mathcal{G}^+ w\right).
\end{aligned}$$

Since $[[\underline{\tilde{Z}}^\pm]] = \frac{\mathcal{G}^+ w - \mathcal{G}^- w}{1+|\nabla\zeta|^2}$, one has

$$\left[\left[\frac{\underline{\epsilon}^\pm \mathcal{G}^\pm w\partial^\alpha\left(\mathcal{G}^\pm w - \zeta_{X_1}\right)}{1+|\nabla\zeta|^2}\right]\right] = \frac{\underline{\epsilon}^+\underline{\epsilon}^-}{[[\underline{\epsilon}^+]]}[[\underline{\tilde{Z}}^\pm]]\partial^\alpha(\mathcal{G}^- w - \mathcal{G}^+ w). \tag{3.15}$$

Next, by Proposition 3.1 and Lemma 3.2 and (c), one can get that

$$\begin{aligned}
\partial^\alpha(\mathcal{G}^\pm w) &= \mathcal{G}^\pm\partial^\alpha w - \mathcal{G}^\pm(\partial^\alpha\zeta\underline{\tilde{Z}}^\pm) - \nabla\cdot(\underline{W}^\pm\partial^\alpha\zeta) + \cdots\\
&= \mathcal{G}^\pm\partial^\alpha w - \mathcal{G}^\pm(\partial^\alpha\zeta\underline{\tilde{Z}}^\pm) - \underline{W}^\pm\cdot\tilde{\mathcal{G}}^{-1}\tilde{\mathcal{G}}\nabla\partial^\alpha\zeta + \cdots\\
&= \epsilon^\mp\mathcal{G}^\pm\tilde{\mathcal{G}}^{-1}\mathcal{G}^\mp([[\underline{\tilde{Z}}^\pm]]\partial^\alpha\zeta) + [[\underline{\epsilon}^\pm]]\tilde{\mathcal{G}}^{-1}\mathcal{G}^\pm\partial_{X_1}\partial^\alpha\zeta\\
&\quad - \underline{\epsilon}^\mp\underline{W}^-\cdot\tilde{\mathcal{G}}^{-1}\mathcal{G}^+\nabla\partial^\alpha\zeta + \underline{\epsilon}^\mp\underline{W}^+\cdot\tilde{\mathcal{G}}^{-1}\mathcal{G}^-\nabla\partial^\alpha\zeta + \cdots .
\end{aligned} \tag{3.16}$$

One can obtain (3.14) from (3.15) and (3.16).

Step 4. Finally, inserting (3.14) into (3.11), one has

$$\begin{aligned}\partial^\alpha\tau(\zeta)\\ &= -[[\underline{\epsilon}^\pm]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta - [[\underline{\epsilon}^\pm]]\left(\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\partial^\alpha\zeta\underline{W}) + \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\partial^\alpha\zeta\right)\\ &\quad - \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\partial^\alpha\zeta) + \underline{\epsilon}^+\underline{\epsilon}^-[[\underline{\tilde{Z}}^\pm]]\mathcal{G}^+\tilde{\mathcal{G}}^{-1}\mathcal{G}^-(\partial^\alpha\zeta([[\underline{\tilde{Z}}^\pm]])) + \cdots\\ &= \mathcal{L}_e[\zeta]\zeta_{(\alpha)} + \cdots .\end{aligned}$$

Thus, we complete the proof of the proposition.

4. Quasilinearization. Having obtained the linearization formula of $\tau(\zeta)$, the next procedures are standard. We quasilinearize the system in this part and then establish the energy estimates in the next part. To this end, we first introduce some notations. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)^\top \in \mathbb{N}^{1+d}$; we denote by ∂^{α_0} the time derivative and by ∂^{α_j} ($j \neq 0$) the spatial derivatives. We define the "good" unknowns as

$$\zeta_{(\alpha)} = \partial^\alpha\zeta\,, \quad \psi_{(\alpha)} = \partial^\alpha\psi - \underline{Z}\partial^\alpha\zeta\,, \tag{4.1}$$

where $\underline{Z} = \underline{\rho}^+\underline{Z}^+ - \underline{\rho}^-\underline{Z}^-$ and $\underline{Z}^\pm$ are defined by (2.19). To describe the subprincipal part of some quantities, as in [16], we also introduce the following notations:

$$\zeta_{\langle\check{\alpha}\rangle} = \left(\zeta_{(\check{\alpha}^0)}, \ldots, \zeta_{(\check{\alpha}^d)}\right)^\top, \quad \psi_{\langle\check{\alpha}\rangle} = \left(\psi_{(\check{\alpha}^0)}, \ldots, \psi_{(\check{\alpha}^d)}\right)^\top,$$

where $\check{\alpha}^j \in \mathbb{N}^{1+d}$ satisfy $\check{\alpha}^j + \mathbf{e}_j = \alpha$. Following [16], we introduce the operator $Ins[U]$ as follows:

$$Ins[U]\bullet = \mathfrak{a}\bullet - \underline{\rho}^+\underline{\rho}^-[[\underline{V}^\pm]]\cdot\mathscr{E}[\zeta]([[V^\pm]]\bullet) - \nabla\cdot\mathcal{K}[\nabla\zeta]\nabla\bullet\,,$$

where $\mathfrak{a} = g' + [[\underline{\rho}^\pm(\partial_t + [[\underline{V}]]\cdot\nabla)\underline{Z}^\pm]]$, $\mathscr{E}[\zeta]$ is defined by (2.23), and

$$\mathcal{K}[\nabla\zeta] = \frac{(1+|\nabla\zeta|^2)Id - \nabla\zeta\bigotimes\nabla\zeta}{(1+|\nabla\zeta|^2)^{\frac{3}{2}}}\,. \tag{4.2}$$

We also introduce the matrix operators

$$\mathcal{A}[U] = \begin{pmatrix} 0 & -\mathcal{G} \\ Ins[U] + \mathcal{L}_e[\zeta] & 0 \end{pmatrix}, \quad \mathcal{B}[U] = \begin{pmatrix} \mathcal{I}[U] & 0 \\ 0 & -\mathcal{I}[U]^* \end{pmatrix},$$

and

$$\mathcal{C}_\alpha[U] = \begin{pmatrix} 0 & -\mathcal{G}_{(\alpha)} \\ \mathcal{K}_{(\alpha)}[\nabla\zeta] & 0 \end{pmatrix},$$

where $\mathcal{G} := \mathcal{G}^-\left(\underline{\rho}^+\mathcal{G}^- - \underline{\rho}^-\mathcal{G}^+\right)^{-1}\mathcal{G}^+$, $\mathcal{I}[U]$ is defined by (2.22), $\mathcal{I}[U]^*$ is the dual operator of $\mathcal{I}[U]$, $\mathcal{G}_{(\alpha)}$ is given by

$$\mathcal{G}_{(\alpha)}\psi_{\langle\check{\alpha}\rangle} = \sum_{j=0}^{d}\alpha_j d\mathcal{G}(\partial_j\zeta)\psi_{(\check{\alpha}^j)}\,, \tag{4.3}$$

and $\mathcal{K}_{(\alpha)}[\nabla\zeta]$ is defined by

$$\mathcal{K}_{(\alpha)}[\nabla\zeta]F = -\nabla\cdot\left[\sum_{j=0}^{d}\left(d\mathcal{K}\left(\nabla\partial_j\zeta\right)\nabla f_j + d\mathcal{K}\left(\nabla f_j\right)\nabla\partial_j\zeta\right)\right] \tag{4.4}$$

for all $F = (f_0, f_1, \ldots, f_d)^\top$. The following proposition indicates that the system (1.5) can be quasilinearized.

PROPOSITION 4.1. *Let $T > 0$, $t_0 > \frac{d}{2}$, and N be chosen as in Proposition* 6 *in* [16]. *If $U = (\zeta, \psi)^\top \in E_T^N$ satisfies* (2.3) *uniformly on* $[0, T]$ *and solves* (1.5), *then for all*

$$\alpha = (\alpha_0, \ldots, \alpha_d)^\top \in \mathbb{N}^{1+d}$$

with $1 \leqslant |\alpha| \leqslant N$, $U_{(\alpha)} = (\zeta_{(\alpha)}, \psi_{(\alpha)})^\top$ solves

$$\begin{aligned} \partial_t U_{(\alpha)} + \mathcal{A}[U]U_{(\alpha)} &= (R_{(\alpha)}, S_{(\alpha)})^\top, \quad |\alpha| < N, \\ \partial_t U_{(\alpha)} + \mathcal{A}[U]U_{(\alpha)} + \mathcal{B}[U]U_{(\alpha)} + C_\alpha[U]U_{\langle\check{\alpha}\rangle} &= (R_{(\alpha)}, S_{(\alpha)})^\top, \quad |\alpha| = N, \end{aligned} \tag{4.5}$$

where $U_{\langle\check{\alpha}\rangle} = (\zeta_{\langle\check{\alpha}\rangle}, \psi_{\langle\check{\alpha}\rangle})^\top$, and the residuals R_α and S_α satisfy the estimate

$$\|R_\alpha\|_{H^1} + \|\mathscr{B} S_\alpha\|_2 \leqslant \mathfrak{m}^N(U) \,. \tag{4.6}$$

Proof. We consider the most difficult case, i.e., $|\alpha| = N$. Firstly, by [16, Proposition 6], one has

$$\partial_t \zeta_{(\alpha)} - \mathcal{G}\psi_{(\alpha)} + \mathcal{I}[U] - \mathcal{G}_{(\alpha)}\psi_{\langle\check{\alpha}\rangle} = R_\alpha \,. \tag{4.7}$$

Secondly, if $\alpha = \beta + \gamma \in \mathbb{N}^{1+d}$ with $|\gamma| = 1$, then it is easy to check that

$$\partial_t \partial^\gamma \psi + \partial^\gamma \zeta + [[\underline{\rho}^\pm \underline{V}^\pm \cdot (\nabla \partial^\gamma \psi^\pm - \underline{Z}^\pm \nabla \partial^\gamma \zeta^\pm)]] - [[\underline{\rho}^\pm \underline{Z}^\pm \partial^\gamma (\mathcal{G}\psi)]] = -\partial^\gamma \left(\kappa(\zeta) + \tau(\zeta)\right) .$$

In the following, $a \sim b$ means $\|\mathscr{B}(a-b)\|_2 \leqslant \mathfrak{m}^N(U)$. By [16, Lemma 9], one has

$$\partial_t \partial^\alpha \psi + \partial^\alpha \zeta + [[\underline{\rho}^\pm \underline{V}^\pm \cdot (\nabla \psi^\pm_{(\alpha)} + \partial^\alpha \zeta \nabla \underline{Z}^\pm)]] - [[\underline{\rho}^\pm \underline{Z}^\pm \partial^\alpha (\mathcal{G}\psi)]] \sim -\partial^\alpha \left(\kappa(\zeta) + \tau(\zeta)\right)$$

based on the fact that

$$\underline{\rho}^\pm \underline{V}^\pm \cdot \{\partial^\beta, \underline{Z}^\pm\} \nabla \partial^\gamma \zeta - \underline{\rho}^\pm \{\partial^\beta, \underline{Z}^\pm\} \partial^\gamma (\mathcal{G}\psi) \sim \underline{\rho}^\pm \{\partial^\beta, \underline{Z}^\pm\} (\partial^\gamma (\underline{V}^\pm \cdot \nabla \partial^\gamma \zeta + \mathcal{G}\psi)) \sim 0 \,.$$

Noting that $\partial^\alpha(\mathcal{G}\psi) = \partial_t \partial^\alpha \zeta$, one obtains

$$\partial_t \partial^\alpha \psi + \mathfrak{a} \partial^\alpha \zeta + [[\underline{\rho}^\pm \underline{V}^\pm \cdot \nabla \psi^\pm_{(\alpha)}]] \sim -\partial^\alpha \left(\kappa(\zeta) + \tau(\zeta)\right) ,$$

where $\mathfrak{a} = g' + [[\underline{\rho}^\pm (\partial_t + \underline{V}^\pm \cdot \nabla) \underline{w}^\pm]]$. Since

$$[[f^\pm g^\pm]] = \langle f^\pm \rangle [[g^\pm]] + [[f^\pm]] \langle g^\pm \rangle \,, \tag{4.8}$$

one has

$$\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \langle \underline{V}^\pm \rangle \cdot \nabla \psi_{(\alpha)} + [[\underline{V}^\pm]] \cdot \langle \underline{\rho}^\pm \psi^\pm_{(\alpha)} \rangle \sim -\partial^\alpha \left(\kappa(\zeta) + \tau(\zeta)\right) .$$

Next, [16, Lemma 10] implies that

$$\partial_t \psi_{(\alpha)} - \mathcal{I}[U]^* \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta - \underline{\rho}^+ \underline{\rho}^- [[\underline{V}^\pm]] \cdot \mathscr{E}[\zeta] \left(\zeta_{(\alpha)} [[\underline{V}^\pm]]\right) \sim -\partial^\alpha \left(\kappa(\zeta) + \tau(\zeta)\right) . \tag{4.9}$$

Note that

$$\partial^\alpha \kappa(\zeta) \sim -\nabla \cdot \mathcal{K}[\nabla\zeta] \nabla \partial^\alpha \zeta + \mathcal{K}_{(\alpha)}[\nabla\zeta] \zeta_{(\check{\alpha})} \,; \tag{4.10}$$

see, for example, [15, equation (9.17)]. From (4.10) and Proposition 3.4, one has

$$\partial^\alpha \left(\kappa(\zeta) + \tau(\zeta)\right) \sim -\nabla \cdot \mathcal{K}[\nabla\zeta] \nabla \zeta_{(\alpha)} + \mathcal{L}_e[\zeta] \zeta_{(\alpha)} \,. \tag{4.11}$$

Thus Proposition 4.1 can be obtained from (4.7), (4.9), and (4.11).

5. Proof of Theorem 1.1. Since the energy method is applied for local well-posedness, we will prove that $\mathcal{E}^N(U)$ in a short time interval is controlled by the energy at $t=0$. Noting that $\mathcal{E}^N(U)$ involves time derivatives, we must specify in what sense the initial energy $\mathcal{E}^N(U^0)$ holds. Hence we must choose initial data $U^0_{(\alpha)}$ for $(U_{(\alpha)})|_{t=0}$ when $\alpha_0>0$, in terms of U^0 and its spatial derivatives. As in [16], we achieve this via a finite induction. When $\alpha_0=0$, we take

$$U^0_{(\alpha)} = (\partial^\alpha\zeta^0, \partial^\alpha\psi^0 - \underline{Z}^0\partial^\alpha\zeta^0)^\top \quad \text{with} \quad \underline{Z}^0 = \left[\!\!\left[\underline{\rho}^\pm \frac{\mathcal{G}^\pm\psi^{\pm,0} + \nabla\zeta\cdot\nabla\psi^{\pm,0}}{1+|\nabla\zeta|^2}\right]\!\!\right].$$

Let $1\leqslant n\leqslant N$, and assume $U_{(\beta)}|_{t=0} = U^0_{(\beta)}$ has been chosen for all

$$\beta = (\beta_0,\beta_1,\ldots,\beta_d)^\top \in \mathbb{N}^{1+d}$$

with $\beta_0<n$. We remark that for all α with $\alpha_0=n$ we have

$$U_{(\alpha)}|_{t=0} = (\partial_t\zeta_{(\alpha')}, \partial_t\psi_{(\alpha')} + \partial_t\underline{Z}\partial^{\alpha'}\zeta)^\top|_{t=0},$$

where $\alpha' = (\alpha_0-1,\alpha_1,\ldots,\alpha_d)^\top$, and therefore we can set up initial conditions for $\partial_t U_{(\alpha')}$ by using Proposition 4.1.

The initial energy, which we denote slightly abusively by $\mathcal{E}^N(U^0)$ as in [16], is therefore defined as

$$\mathcal{E}^N(U^0) = \|\nabla\psi^0\|^2_{H^{t_0+2}} + \sum_{\alpha\in\mathbb{N}^{1+d},|\alpha|\leqslant N} \left(\|\zeta^0_{(\alpha)}\|^2_{H^1} + \|\mathscr{B}\psi^0_{(\alpha)}\|^2_2\right), \tag{5.1}$$

with $U^0_{(\alpha)}$ constructed as above.

5.1. The mollified quasilinear system. Following [16], let $\chi:\mathbb{R}\to\mathbb{R}$ be a smooth, compactly supported function, which equals one in a neighborhood of the origin. For all $0<\iota<1$, we denote by J^ι the mollifier $J^\iota = \chi(\iota|D|)$. Consider the mollified system

(5.2)
$$\begin{cases}\partial_t\zeta - J^\iota\mathcal{G}\psi = 0, \\ \partial_t\psi + g'J^\iota\zeta + \frac{1}{2}J^\iota\left([[\underline{\rho}^\pm|\nabla\psi^\pm|^2]] - (1+|\nabla\zeta|^2)[[\underline{\rho}^\pm(\underline{Z}^\pm)^2]]\right) = -J^\iota\left(\kappa(\zeta)+\tau(\zeta)\right).\end{cases}$$

Since J^ι is a smoothing operator, from the Cauchy-Lipschitz theorem of ODE, we know (5.2) has a unique maximal solution $U^\iota = (\zeta^\iota,\psi^\iota)$ with initial data (ζ^0,ψ^0) on a time interval $[0,T^\iota_{max}]$. Proceeding exactly as in the proof of Proposition 4.1, one can check that for all $\alpha = (\alpha_0,\alpha_1,\ldots,\alpha_d)^\top\in\mathbb{N}^{1+d}$, $1\leqslant|\alpha|\leqslant N$, $U^\iota_{(\alpha)} = (\zeta^\iota_{(\alpha)},\psi^\iota_{(\alpha)})^\top$ solves

$$\partial_t U_{(\alpha)} + J^\iota\mathcal{A}[U]U_{(\alpha)} + J^\iota\mathcal{B}[U]U_{(\alpha)} + J^\iota\mathcal{C}_\alpha[U]U_{\langle\check{\alpha}\rangle} = (J^\iota R_\alpha, J^\iota S_\alpha + S'_\alpha)^\top, \tag{5.3}$$

where R_α and S_α satisfy (4.6) and S'_α is given by

$$S'_\alpha = -(1-J^\iota)\left([[\underline{\rho}^\pm\partial_t\underline{Z}^\pm]]\zeta_{(\alpha)}\right) + \left[[[\underline{\rho}^\pm\underline{Z}^\pm]], J^\iota\right]\partial^\alpha(\mathcal{G}\psi).$$

5.2. Symmetrizer and energy. We denote a symmetrizer of the system $(5.3)_\alpha$ by

$$\mathcal{S}[U] = \mathcal{S}^1[U] + \bar{\mathcal{S}}^1[U] + \mathcal{S}^2_\alpha[U],$$

where

$$\mathcal{S}^1[U] = diag(Ins[U],\mathcal{G})\,,\ \bar{\mathcal{S}}^1[U] = diag(\mathcal{L}_e[\zeta],0)\,,\ \mathcal{S}^2_\alpha[U] = diag\left(\mathcal{K}_{(\alpha)}[\nabla\zeta],\mathcal{G}_{(\alpha)}\right)\,. \tag{5.4}$$

The corresponding energy $\mathcal{F}^l(U)$ is defined as

$$\mathcal{F}^l(U) = \sum_{0\leqslant|\alpha|\leqslant l}\mathcal{F}^\alpha(U)\,,\quad 1\leqslant l\leqslant N\,,$$

with

$$\mathcal{F}^\alpha(U) = \frac{1}{2}([\mathcal{S}^1[U]+\bar{\mathcal{S}}^1[U]]U_{(\alpha)},U_{(\alpha)})\quad\text{if}\quad\alpha\neq 0\,,$$
$$\mathcal{F}^0(U) = \mathfrak{m}^1(U)[|\zeta|^2_{H^1}+(\psi,\mathcal{G}[0]\psi)]\quad\text{if}\quad\alpha=0\,,$$

where $U_{(\alpha)} = \left(\zeta_{(\alpha)},\psi_{(\alpha)}\right)^\top$.

We now give a lemma which implies that the energy $\mathcal{F}^N(U)$ is equivalent to $\mathcal{E}^N(U)$.

LEMMA 5.1. *Let $T>0$, and assume $U=(\zeta,\psi)^\top$ solves* (1.5) *on* $[0,T]$ *and satisfies* (2.3). *Then one has, for all $0\leqslant j\leqslant N$,*

$$\mathcal{E}^j(U)\leqslant\mathfrak{m}^1(U)\mathcal{F}^j(U)\quad\textit{and}\quad\mathcal{F}^j(U)\leqslant\mathfrak{m}^1(U)\mathcal{E}^j(U)\,. \tag{5.5}$$

Proof. The definition of $\mathcal{F}^j(U)$ yields

$$\mathcal{F}^j(U) = \sum_{1\leqslant|\alpha|\leqslant j}\left[\left(Ins[U]\zeta_{(\alpha)}+\mathcal{L}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)}\right)+\left(\mathcal{G}\psi_{(\alpha)},\psi_{(\alpha)}\right)\right]$$
$$+\mathfrak{m}^1(U)\left(\|\zeta\|^2_{H^1}+(\psi,\mathcal{G}[0]\psi)\right)\,.$$

It is easy to check that

$$\frac{\|\nabla\zeta_{(\alpha)}\|_2^2}{(1+\|\nabla\zeta\|^2_\infty)^{\frac{3}{2}}}\leqslant\left(\mathcal{K}[\nabla\zeta]\nabla\zeta_{(\alpha)},\nabla\zeta_{(\alpha)}\right)\leqslant\|\nabla\zeta_{(\alpha)}\|_2^2\,.$$

On the other hand, from [16, Proposition 5] for the estimate of the operator $\mathscr{E}[\zeta]$, the following inequality holds:

$$|(\mathfrak{a}\zeta_{(\alpha)}-\underline{\rho}^+\underline{\rho}^-[[\underline{V}^\pm]]\cdot\mathscr{E}[\zeta]([[V^\pm]]\zeta_{(\alpha)}),\zeta_{(\alpha)})|\leqslant\frac{\mathfrak{m}^1(U)}{4N}\|\zeta\|^2_{H^1}+\frac{\|\zeta_{(\alpha)}\|^2_{H^1}}{4(1+\|\nabla\zeta\|^2_\infty)^{\frac{3}{2}}}\,.$$

Next, using (2.10) and (2.11), one has

$$\begin{aligned}|((1+|D|)^{-\frac{1}{2}}\mathcal{L}_e[\zeta]\zeta_{(\alpha)},(1+|D|)^{\frac{1}{2}}\zeta_{(\alpha)})|\leqslant&M(1+\|\underline{W}\|^2_{H^{t_0}}+\|[[\underline{\tilde{Z}}]]\|^2_{H^{t_0}})\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}}\\ \leqslant&\frac{\mathfrak{m}^1(U)}{4N}\|\zeta\|^2_{H^1}+\frac{\|\zeta_{(\alpha)}\|^2_{H^1}}{4(1+\|\nabla\zeta\|^2_\infty)^{\frac{3}{2}}}\,.\end{aligned}$$

Thus, one has

$$\frac{\|\zeta_{(\alpha)}\|^2_{H^1}}{\mathfrak{m}^1(U)}-\frac{\mathfrak{m}^1(U)}{2N}\|\zeta\|^2_{H^1}\leqslant|\left(Ins[U]\zeta_{(\alpha)}+\mathcal{L}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)}\right)|\leqslant\mathfrak{m}^1(U)\|\zeta_{(\alpha)}\|^2_{H^1}.$$

In addition, it follows from Proposition 3 and Lemma 7 of [16] that

$$M^{-1}\|\mathscr{B}\psi_{(\alpha)}\|_2^2\leqslant\left(\mathcal{G}\psi_{(\alpha)},\psi_{(\alpha)}\right)\leqslant M\|\mathscr{B}\psi_{(\alpha)}\|_2^2\,.$$

Therefore, we have the estimates (5.5).

5.3. Energy estimates. The proof is similar to [15, 16], and we just focus on the new terms arising from the electric field. We first consider the case $\alpha \neq 0$. Taking the L^2-scalar product of (5.3) with $(\mathcal{S}^1+\bar{\mathcal{S}}^1)U_{(\alpha)}+\mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}$ and noting that

$$(J^\iota \mathcal{A}U_{(\alpha)}, (\mathcal{S}^1+\bar{\mathcal{S}}^1)U_{(\alpha)})=0$$

and

$$(J^\iota \mathcal{A}U_{(\alpha)}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle})+(J^\iota \mathcal{C}_\alpha U_{\langle\check{\alpha}\rangle}, (\mathcal{S}^1+\bar{\mathcal{S}}^1)U_{(\alpha)})=0\,,$$

one can get

$$\big((\mathcal{S}^1+\bar{\mathcal{S}}^1)\partial_t U_{(\alpha)}, U_{(\alpha)}\big)+\big(\partial_t U_{(\alpha)}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\big)+(J^\iota \mathcal{C}_\alpha U_{\langle\check{\alpha}\rangle}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle})+\big((\mathcal{S}^1+\bar{\mathcal{S}}^1)J^\iota\mathcal{B}U_{(\alpha)}, U_{(\alpha)}\big)$$
$$+\big(J^\iota\mathcal{B}U_{(\alpha)}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\big)=\Big(J^\iota(R_\alpha, S_\alpha+S'_\alpha)^\top, (\mathcal{S}^1+\bar{\mathcal{S}}^1)U_{(\alpha)}+\mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\Big),$$

where the superscript ι of U is omitted for convenience's sake. Hence, one has

$$\frac{d}{dt}\big(\mathcal{F}^\alpha(U)+(U_{(\alpha)}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle})\big)=\sum_{j=1}^{7}A_j+\sum_{j=1}^{3}B_j\,, \tag{5.6}$$

where

$$A_1=\frac{1}{2}\big([\partial_t,\mathcal{S}^1]U_{(\alpha)}, U_{(\alpha)}\big),\quad A_2=\big(U_{(\alpha)}, \partial_t(\mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle})\big),\qquad A_3=-\big(J^\iota\mathcal{C}_\alpha U_{\langle\check{\alpha}\rangle}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\big),$$
$$A_4=-\big(\mathcal{S}^1 J^\iota\mathcal{B}U_{(\alpha)}, U_{(\alpha)}\big),\quad A_5=-\big(J^\iota\mathcal{B}U_{(\alpha)}, \mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\big),\quad A_6=\frac{1}{2}\big(([\partial_t,\bar{\mathcal{S}}^1]U_{(\alpha)}, U_{(\alpha)}\big),$$
$$A_7=-\big(\bar{\mathcal{S}}^1 J^\iota\mathcal{B}U_{(\alpha)}, U_{(\alpha)}\big),$$

and

$$B_1=\big(J^\iota(R_\alpha,S_\alpha)^\top, \mathcal{S}^1U_{(\alpha)}+\mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\big),\quad B_2=\big(J^\iota(0,S'_\alpha)^\top, \mathcal{S}^1U_{(\alpha)}+\mathcal{S}^2_\alpha U_{\langle\check{\alpha}\rangle}\big),$$
$$B_3=\big(J^\iota(R_\alpha,S_\alpha)^\top, \bar{\mathcal{S}}^1U_{(\alpha)}\big).$$

Next, we estimate A_j $(j=1,\dots,7)$ and B_j $(j=1,2,3)$.

Estimate of A_1. Similar to the control of A_1 in [16, section 5.5.4], one can get $|A_1|\leqslant \mathfrak{m}^N(U)$.

Estimate of A_2. It follows from the definitions of $\mathcal{S}^2_\alpha$ and $\mathcal{E}^N(U)$ that $|A_2|\leqslant \mathfrak{m}^N(U)$.

Estimate of A_3. It follows from $\|\mathcal{G}_{(\alpha)}\psi_{\langle\check{\alpha}\rangle}\|_2\leqslant \mathfrak{m}^N(U)$ and $\|\mathcal{K}_{(\alpha)}[\nabla\zeta]\zeta_{\langle\check{\alpha}\rangle}\|_2\leqslant \mathfrak{m}^N(U)$ that $|A_3|\leqslant \mathfrak{m}^N(U)$.

Estimate of A_4. From Appendix C of [16], one has $|A_4|\leqslant \mathfrak{m}^N(U)$.

Estimate of A_5. It is noted that

$$A_5=(J^\iota\mathcal{I}\zeta_{(\alpha)}, \mathcal{K}_{(\alpha)}[\nabla\zeta]\zeta_{(\check{\alpha})})+(J^\iota\mathcal{I}^*\psi_{(\alpha)}, \mathcal{G}_{(\alpha)}\psi_{(\check{\alpha})})\,.$$

Thus, from (2.20) for $\mathcal{G}$, [16, Proposition 4] and the definitions of $\mathcal{K}_{(\alpha)}[\nabla\zeta]\zeta_{(\check{\alpha})}$ and $\mathcal{G}_{(\alpha)}\psi_{(\check{\alpha})}$, one can get $|A_5|\leqslant \mathfrak{m}^N(U)$.

Estimate of A_6. First, from

$$\partial_t\tilde{\mathcal{G}}^{-1}\partial_{X_1}=\tilde{\mathcal{G}}^{-1}\tilde{\mathcal{G}}\partial_t\tilde{\mathcal{G}}^{-1}\partial_{X_1}=\tilde{\mathcal{G}}^{-1}\partial_t\partial_{X_1}+\tilde{\mathcal{G}}^{-1}[\partial_t,\tilde{\mathcal{G}}]\tilde{\mathcal{G}}^{-1}\partial_{X_1}\,,$$

one has

$$[\partial_t,\tilde{\mathcal{G}}^{-1}\partial_{X_1}]=\tilde{\mathcal{G}}^{-1}[\partial_t,\tilde{\mathcal{G}}]\tilde{\mathcal{G}}^{-1}\partial_{X_1}=\tilde{\mathcal{G}}^{-1}d\tilde{\mathcal{G}}(\partial_t\zeta)\tilde{\mathcal{G}}^{-1}\partial_{X_1}\,.$$

From this, (2.10), and (2.21), one has

$$\left|(\partial_{X_1}[\partial_t, \tilde{\mathcal{G}}^{-1}\partial_{X_1}]\zeta_\alpha, \zeta_\alpha)\right| = |(d\tilde{\mathcal{G}}(\partial_t\zeta)\tilde{\mathcal{G}}^{-1}\partial_{X_1}\zeta_{(\alpha)}, \tilde{\mathcal{G}}^{-1}\partial_{X_1}\zeta_\alpha)| \leqslant \mathfrak{m}^N(U)\,.$$

Similarly, one can get

$$\left|\left(\left[\partial_t, \left(\partial_{X_1}\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\bullet) + \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\partial_{X_1}\bullet\right)\right]\zeta_{(\alpha)}, \zeta_{(\alpha)}\right)\right| \leqslant \mathfrak{m}^N(U)\,,$$

$$\left|\left(\left[\partial_t, \underline{W}\cdot\nabla\tilde{\mathcal{G}}^{-1}\nabla\cdot(\underline{W}\bullet)\right]\zeta_{(\alpha)}, \zeta_{(\alpha)}\right)\right| \leqslant \mathfrak{m}^N(U)\,,$$

and

$$\left|\left(\left[\partial_t, [[\underline{\tilde{Z}}^\pm]]\mathcal{G}^+\tilde{\mathcal{G}}^{-1}\mathcal{G}^-([[\underline{\tilde{Z}}^\pm]]\bullet)\right]\zeta_{(\alpha)}, \zeta_{(\alpha)}\right)\right| \leqslant \mathfrak{m}^N(U)\,.$$

Thus, one obtains $|A_6| \leqslant \mathfrak{m}^N(U)$.

Estimate of A_7. By Proposition 2.1, (2.10), and (2.11), one can get

$$\|\mathcal{L}_e[\zeta]\zeta_{(\alpha)}\|_2 \leqslant \mathfrak{m}^N(U).$$

From this estimate and [16, Proposition 4] for the estimates of the operator $\mathcal{I}$, one has

$$|A_7| = |(\mathcal{L}_e[\zeta]\mathcal{I}J^\iota\zeta_{(\alpha)}, \zeta_{(\alpha)})| = |(\mathcal{I}J^\iota\zeta_{(\alpha)}, \mathcal{L}_e[\zeta]\zeta_{(\alpha)})| \leq \mathfrak{m}^N(U).$$

We remark that the above proof is not applicable to the case $\sigma = 0$. However, by (2.8), as in the proof of Lemma 3.3, one can deduce that $\mathcal{L}_e[\zeta]\mathcal{I}J^\iota$ is a second-order operator with a skew symmetric principal symbol; therefore, for all $f \in H^{\frac{1}{2}}(\mathbb{R}^d)$, one can obtain

$$|(\mathcal{L}_e[\zeta]\mathcal{I}J^\iota f, f)| = \frac{1}{2}|(\mathcal{L}_e[\zeta]\mathcal{I}J^\iota + (\mathcal{L}_e[\zeta]\mathcal{I}J^\iota)^*)f, f)| \leqslant M(t_0+3)\|\nabla\psi\|_{H^{t_0+1}}\|f\|_{H^{\frac{1}{2}}}\,,$$

which can be used to estimate $\tilde{A}_4$ in the next section.

Estimate of B_1. Similar to the control of B_1 in [16, section 5.5.4], one can get $|B_1| \leqslant \mathfrak{m}^N(U)$.

Estimate of B_2. This is similar to the control of B_2 in [16, section 5.5.4], and one has $|B_1| \leqslant \mathfrak{m}^N(U)$.

Estimate of B_3. Combining Proposition 2.1, (2.10), (2.11), and (4.6) yields

$$|B_3| = |((1+|D|)^{\frac{1}{2}}J^\iota R_\alpha, (1+|D|)^{-\frac{1}{2}}\mathcal{L}_e[\zeta]\zeta_{(\alpha)})| \leqslant \mathfrak{m}^N(U)\,.$$

Now, collecting the above estimates, one can deduce from (5.6) that for any $1 \leqslant |\alpha| \leqslant N$,

$$\frac{d}{dt}\left(\mathcal{F}^\alpha(U) + (U_{(\alpha)}, \mathcal{S}_\alpha^2 U_{\langle\check{\alpha}\rangle})\right) \leq \mathfrak{m}^N(U)\,. \tag{5.7}$$

When $\alpha = 0$, one can rewrite the system as follows:

$$\begin{cases}\zeta_t - J^\iota\mathcal{G}[0]\psi + J^\iota\mathcal{N}_1(U) = 0\,,\\ \psi_t + J^\iota(1-\Delta)\zeta + J^\iota\mathcal{N}_2(U) = 0\,,\end{cases} \tag{5.8}$$

where $\mathcal{N}_1(U)$ and $\mathcal{N}_2(U)$ are given by

$$\mathcal{N}_1(U) = \mathcal{G}[0]\psi - \mathcal{G}[\zeta]\psi\,,$$
$$\mathcal{N}_2(U) = \frac{1}{2}\left([[\rho^{\pm}|\nabla\psi|^2]] - (1+|\nabla\zeta|^2)[[\underline{\rho}^{\pm}(\underline{Z}^{\pm})^2]]\right) + \tau(\zeta) + \nabla\cdot\left(\nabla\zeta - \frac{\nabla\zeta}{\sqrt{1+|\nabla\zeta|^2}}\right) + (g'-1)\zeta\,.$$

Taking the L^2 product of (5.8) with $((1-\Delta)\zeta, \mathcal{G}[0]\psi)^\top$ yields

$$\frac{d}{dt}\mathcal{F}^0(U) \leqslant \mathfrak{m}^1(U)\|\mathcal{N}_1(U)\|_{H^1}\|\zeta\|_{H^1} + \mathfrak{m}^1(U)\|\mathscr{B}\mathcal{N}_2(U)\|_2\|\mathscr{B}\psi\|_2 \leqslant \mathfrak{m}^N(U)\,. \tag{5.9}$$

From (5.7) and (5.9), one has that for all $0<\iota<1$,

$$\frac{d}{dt}\left(\mathcal{F}^{N-1}(U^\iota)\right) \leqslant \mathfrak{m}^N(U^\iota)\,, \quad \frac{d}{dt}\left(\mathcal{F}^N(U^\iota) + \sum_{|\alpha|=N}(U^\iota_{(\alpha)}, \mathcal{S}^2_\alpha U^\iota_{\langle\check{\alpha}\rangle})\right) \leqslant \mathfrak{m}^N(U^\iota)\,.$$

Set

$$\tilde{\mathcal{F}}^N(U^\iota) := \mathcal{F}^N(U^\iota) + M\mathcal{F}^{N-1}(U^\iota) + \sum_{|\alpha|=N}(U^\iota_{(\alpha)}, \mathcal{S}^2_\alpha U^\iota_{\langle\check{\alpha}\rangle});$$

then one has

$$\frac{d}{dt}\tilde{\mathcal{F}}^N(U^\iota) \leqslant \mathfrak{m}^N(U^\iota)\,. \tag{5.10}$$

Noting that

$$\sum_{|\alpha|=N}|(U^\iota_{(\alpha)}, \mathcal{S}^2_\alpha U^\iota_{\langle\check{\alpha}\rangle})| \leqslant \frac{1}{2}\mathcal{F}^N(U^\iota) + M\mathcal{F}^{N-1}(U^\iota)\,,$$

one has

$$\frac{1}{2}\mathcal{F}^N(U^\iota) \leqslant \tilde{\mathcal{F}}^N(U^\iota) \leqslant M\mathcal{F}^N(U^\iota).$$

From this fact and Lemma 5.1, (5.10) is surely the energy estimate. Once the energy estimate is established, the rest of the proof is standard, and we omit the detail here (the interested reader is referred to [1, 15]).

6. Case without surface tension. In the case without surface tension and $d=1$, we introduce the energy $\tilde{\mathcal{E}}^N(U)$ of the system (1.6) for all $N\in\mathbb{N}$ and $U=(\zeta,\psi)^\top$ as follows:

$$\tilde{\mathcal{E}}^N(U) = \|\nabla\psi\|^2_{H^{t_0+2}} + \sum_{\alpha\in\mathbb{N},|\alpha|\leqslant N}\left(\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}} + \|\mathscr{B}\psi_{(\alpha)}\|_2^2\right)\,,$$

and we denote the initial energy by $\tilde{\mathcal{E}}^N(U^0)$, which is defined as

$$\tilde{\mathcal{E}}^N(U^0) = \|\nabla\psi^0\|^2_{H^{t_0+2}} + \sum_{\alpha\in\mathbb{N},|\alpha|\leqslant N}\left(\|\zeta^0_{(\alpha)}\|^2_{H^{\frac{1}{2}}} + \|\mathscr{B}\psi^0_{(\alpha)}\|_2^2\right)\,, \tag{6.1}$$

where $U^0_{(\alpha)} = (\partial^\alpha\zeta^0, \partial^\alpha\psi^0 - \underline{Z}^0\partial^\alpha\zeta^0)^\top$ with

$$\underline{Z}^0 = \left[\left[\underline{\rho}^{\pm}\frac{\mathcal{G}^{\pm}\psi^{\pm,0} + \nabla\zeta\cdot\nabla\psi^{\pm,0}}{1+|\nabla\zeta|^2}\right]\right]\,.$$

We remark that the summation is over $\alpha = \alpha_1 \in \mathbb{N}$ rather than $\alpha = (\alpha_0, \alpha_1)^\top \in \mathbb{N}^{1+1}$ in this case. For all $T > 0$, we define

$$\tilde{E}_T^N = \left\{ U \in C([0,T]; H^{t_0+2}(\mathbb{R}) \times \dot{H}^2(\mathbb{R}), \sup_{0\leqslant t\leqslant T} \tilde{\mathcal{E}}^N(U(t)) < \infty \right\} . \tag{6.2}$$

Moreover, we set

$$\tilde{\mathfrak{m}}^N(U) = C\left(M, \tilde{\mathcal{E}}^N(U)\right) . \tag{6.3}$$

Similar to the proof of Proposition 6 in [16], with a slight modification, we have the following fact.

PROPOSITION 6.1. *Set $d = 1$. Let $T > 0$, $t_0 > \frac{1}{2}$ and $N \in \mathbb{N}$ be such that*

$$[(N+1)/2] \geqslant 1 \vee t_0 + \frac{1}{2} \quad \textit{and} \quad N \geqslant t_0 + \frac{7}{2} .$$

Then for all $\alpha \in \mathbb{N}$ with $1 \leqslant \alpha \leqslant N$, one has

$$\begin{aligned} \partial^\alpha(\mathcal{G}\psi) =& \mathcal{G}\psi_{(\alpha)} + R_\alpha && \textit{if} \quad \alpha \leqslant N-1 , \\ \partial^\alpha(\mathcal{G}\psi) =& \mathcal{G}\psi_{(\alpha)} - \mathcal{I}[U]\partial^\alpha\zeta + R_\alpha && \textit{if} \quad \alpha = N , \end{aligned}$$

where the linear operators $\mathcal{I}[U]$ are defined in (2.22)*, while R_α satisfies the estimate*

$$\|R_\alpha\|_{H^{\frac{1}{2}}} \leqslant \tilde{\mathfrak{m}}^N(U) .$$

In the same vein as section 4, we introduce the operator $\widetilde{Ins}[U]$ as

$$\widetilde{Ins}[U]\bullet = \mathfrak{a}\bullet - \underline{\rho}^+ \underline{\rho}^- [[\underline{V}^\pm]] \cdot \mathscr{E}[\zeta]([[V^\pm]]\bullet) , \tag{6.4}$$

and the matrix operators as

$$\tilde{\mathcal{A}}[U] = \begin{pmatrix} 0 & -\mathcal{G} \\ \widetilde{Ins}[U] + \mathcal{L}_e[\zeta] & 0 \end{pmatrix} , \quad \mathcal{B}[U] = \begin{pmatrix} \mathcal{I}[U] & 0 \\ 0 & -\mathcal{I}[U]^* \end{pmatrix} .$$

As in the proof of Proposition 4.1, by using Remark 3.5 and Proposition 6.1, one has the following proposition.

PROPOSITION 6.2. *Set $d = 1$. Let $T > 0$, $t_0 > \frac{1}{2}$ and N be chosen as in Proposition* 6 *in* [16]*. If $U = (\zeta, \psi)^\top \in E_T^N$ satisfies* (2.3) *uniformly on $[0, T]$ and solves* (1.6)*, then for all $\alpha \in \mathbb{N}$ with $1 \leqslant \alpha \leqslant N$, $U_{(\alpha)} = (\zeta_{(\alpha)}, \psi_{(\alpha)})^\top$ solves*

$$\begin{aligned} \partial_t U_{(\alpha)} + \tilde{\mathcal{A}}[U]U_{(\alpha)} &= (R_{(\alpha)}, S_{(\alpha)})^\top \quad \textit{if} \quad \alpha < N , \\ \partial_t U_{(\alpha)} + \tilde{\mathcal{A}}[U]U_{(\alpha)} + \mathcal{B}[U]U_{(\alpha)} &= (R_{(\alpha)}, S_{(\alpha)})^\top \quad \textit{if} \quad \alpha = N , \end{aligned}$$

where the residuals R_α and S_α satisfy the inequality

$$\|R_\alpha\|_{H^{\frac{1}{2}}} + \|\mathscr{B}S_\alpha\|_2 \leqslant \tilde{\mathfrak{m}}^N(U) . \tag{6.5}$$

Now, we consider the following modified system:

(6.6)
$$\begin{cases} \partial_t\zeta - J^\iota\mathcal{G}\psi = 0 , \\ \partial_t\psi + g'J^\iota\zeta + \frac{1}{2}J^\iota\left([[\underline{\rho}^\pm|\nabla\psi^\pm|^2]] - (1+|\nabla\zeta|^2)[[\underline{\rho}^\pm(\underline{Z}^\pm)^2]]\right) = -J^\iota(\tau(\zeta) + \nu\Delta\zeta) . \end{cases}$$

The Cauchy-Lipschitz theorem of ODE, indicates that there exists a unique maximal solution $U^{\iota,\nu} = (\zeta^{\iota,\nu}, \psi^{\iota,\nu})^\top$ to the system (6.6) with initial data (ζ^0, ψ^0) on the time interval $[0, T^{\iota,\nu}_{max}]$. Proceeding exactly as in the proof of Proposition 6.2, one can check that for all $\alpha \in \mathbb{N}$, $1 \leqslant \alpha \leqslant N$, $U^{\iota,\nu}_{(\alpha)} = (\zeta^{\iota,\nu}_{(\alpha)}, \psi^{\iota,\nu}_{(\alpha)})^\top$ solves

$$\partial_t U_{(\alpha)} + J^\iota \Big(\tilde{\mathcal{A}}[U] - \nu\mathcal{C}(D)\Big) U_{(\alpha)} + J^\iota \mathcal{B}[U] U_{(\alpha)} = (J^\iota R_\alpha, J^\iota S_\alpha + S'_\alpha)^\top \tag{6.7}$$

with

$$\mathcal{C}(D) = \begin{pmatrix} 0 & 0 \\ \Delta & 0 \end{pmatrix},$$

and where R_α and S_α satisfy the estimate (6.5) and S'_α is given by

$$S'_\alpha = -(1 - J^\iota)\left([[\underline{\rho}^\pm \partial_t \underline{w}^\pm]]\zeta_{(\alpha)}\right) + \left[[[\underline{\rho}^\pm \underline{Z}^\pm]], J^\iota\right] \partial^\alpha(\mathcal{G}\psi)\,.$$

A symmetrizer for the system $(6.7)_\alpha$ is given by

$$\mathcal{S}_\nu[U] = \tilde{\mathcal{S}}^1[U] + \nu \mathrm{diag}(-\Delta, 0) + \bar{\mathcal{S}}^1[U]\,,$$

where $\bar{\mathcal{S}}^1[U]$ is defined as (5.4) and $\tilde{\mathcal{S}}^1[U] = diag(\widetilde{Ins}[U], \mathcal{G})$.

The corresponding energy $\tilde{\mathcal{F}}^l(U)$ is defined as

$$\tilde{\mathcal{F}}^l(U) = \sum_{0 \leqslant |\alpha| \leqslant l} \tilde{\mathcal{F}}^\alpha(U)\,, \quad 1 \leqslant l \leqslant N\,,$$

with

$$\begin{aligned} \tilde{\mathcal{F}}^\alpha(U) =& \frac{1}{2}([\tilde{\mathcal{S}}^1[U] + \bar{\mathcal{S}}^1[U]]U_{(\alpha)}, U_{(\alpha)}) && \text{if} \quad \alpha \neq 0\,, \\ \tilde{\mathcal{F}}^0(U) =& M(t_0+3)(\zeta,\zeta) + [[\underline{\epsilon}^\pm]]^2(\tilde{\mathcal{G}}^{-1}[0]\partial_{X_1}\zeta, \partial_{X_1}\zeta) + (\psi, \mathcal{G}[0]\psi) && \text{if} \quad \alpha = 0\,, \end{aligned}$$

where $U_{(\alpha)} = \big(\zeta_{(\alpha)}, \psi_{(\alpha)}\big)^\top$.

The following lemma is crucial in the proof, which tells us $\tilde{\mathcal{F}}^l(U)$ is true energy.

LEMMA 6.3. *Set $d = 1$. Let $T > 0$, and assume that $U = (\zeta, \psi)^\top$ solves (1.6) on $[0, T]$ and satisfies (2.3). We then have, for all $0 \leqslant j \leqslant N$,*

$$\Big([[\underline{\epsilon}^\pm]]^2 - M(t_0+3)\tilde{\mathcal{E}}^1(U)\Big)\,\mathcal{E}^j(U) \leqslant M\mathcal{F}^j(U) \text{ and } \mathcal{F}^j(U) \leqslant \tilde{\mathfrak{m}}^1(U)\mathcal{E}^j(U)\,. \tag{6.8}$$

Proof. By the definition of $\mathcal{F}^j(U)$, one has

$$\begin{aligned} \mathcal{F}^j(U) = \sum_{1 \leqslant |\alpha| \leqslant j} &\Big[\Big(\widetilde{Ins}[U]\zeta_{(\alpha)} + \mathcal{L}_e[\zeta]\zeta_{(\alpha)}, \zeta_{(\alpha)}\Big) + \big(\mathcal{G}\psi_{(\alpha)}, \psi_{(\alpha)}\big)\Big] \\ &+ \Big(M(t_0+3)(\zeta,\zeta) + [[\underline{\epsilon}^\pm]]^2(\tilde{\mathcal{G}}^{-1}[0]\partial_{X_1}\zeta, \partial_{X_1}\zeta) + (\psi, \mathcal{G}[0]\psi)\Big)\,. \end{aligned}$$

First, it is noted that

$$(\mathcal{L}_e[\zeta]\zeta_{(\alpha)}, \zeta_{(\alpha)}) = [[\underline{\epsilon}^\pm]]^2(\tilde{\mathcal{G}}^{-1}\partial_{X_1}\zeta_{(\alpha)}, \partial_{X_1}\zeta_{(\alpha)}) + (\tilde{\mathcal{L}}_e[\zeta]\zeta_{(\alpha)}, \zeta_{(\alpha)})\,,$$

where we decompose $\mathcal{L}_e[\zeta]$ as $\mathcal{L}_e[\zeta] := -[[\underline{\epsilon}^\pm]]^2\partial_{X_1}\tilde{\mathcal{G}}^{-1}\partial_{X_1} + \tilde{\mathcal{L}}_e[\zeta]$. Since $d = 1$, Proposition 2.2 gives

$$(\tilde{\mathcal{G}}^{-1}\partial_{X_1}\zeta_{(\alpha)},\partial_{X_1}\zeta_{(\alpha)}) \geqslant \frac{1}{2}\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}} - \frac{M(t_0+3)}{2N}\|\zeta\|_2^2\,,$$

and it is easy to know that

$$|(\tilde{\mathcal{G}}^{-1}\partial_{X_1}\zeta_{(\alpha)},\partial_{X_1}\zeta_{(\alpha)})| \leqslant \mathfrak{m}^1(U)\|\zeta_{(\alpha)}\|^2_{\dot{H}^{\frac{1}{2}}}\,.$$

From Proposition 2.1, (2.10), and (2.11), one has

$$\begin{aligned}|(\tilde{\mathcal{L}}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)})| \leqslant& |((1+|D|)^{-\frac{1}{2}}\tilde{\mathcal{L}}_e[\zeta]\zeta_{(\alpha)},(1+|D|)^{\frac{1}{2}}\zeta_{(\alpha)})|\\ \leqslant& M(\|\underline{W}\|_{H^{t_0+1}}+\|\underline{W}\|^2_{H^{t_0+1}}+\|[[\underline{\tilde{Z}}^{\pm}]]\|^2_{H^{t_0+1}})\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}}\\ \leqslant& \frac{1}{4}M(t_0+3)\tilde{\mathcal{E}}^1(U)\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}}\,.\end{aligned}$$

Hence, one has

$$(\mathcal{L}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)}) \geqslant \frac{1}{2}\left([[\underline{\epsilon}^{\pm}]]^2 - \frac{M(t_0+3)}{2}\tilde{\mathcal{E}}^1(U)\right)\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}} - \frac{M(t_0+3)}{2N}\|\zeta\|_2^2\,,$$

$$|(\mathcal{L}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)})| \leqslant \frac{1}{2}\tilde{\mathfrak{m}}^1(U)\|\zeta_{(\alpha)}\|_2^2\,.$$

Next, by [16, Proposition 5] for the estimates of the operator $\mathcal{E}[\zeta]$, it is easy to obtain

$$|(\widetilde{Ins}[U]\zeta_{(\alpha)},\zeta_{(\alpha)})| \leqslant M\tilde{\mathcal{E}}^1(U)\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}} \leqslant \frac{M(t_0+3)}{4}\tilde{\mathcal{E}}^1(U)\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}}\,.$$

Thus, one can get

$$\frac{1}{2}\left([[\underline{\epsilon}^{\pm}]]^2 - M(t_0+3)\tilde{\mathcal{E}}^1(U)\right)\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}} - \frac{M(t_0+3)}{2N}\|\zeta\|_2^2 \leqslant \left(\widetilde{Ins}[U]\zeta_{(\alpha)}+\mathcal{L}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)}\right),$$

$$\left|\left(\widetilde{Ins}[U]\zeta_{(\alpha)}+\mathcal{L}_e[\zeta]\zeta_{(\alpha)},\zeta_{(\alpha)}\right)\right| \leqslant \tilde{\mathfrak{m}}^1(U)\|\zeta_{(\alpha)}\|^2_{H^{\frac{1}{2}}}.$$

In addition, it follows from Proposition 3 and Lemma 7 of [16] that

$$M^{-1}\|\mathscr{B}\psi_{(\alpha)}\|_2^2 \leqslant \left(\mathcal{G}\psi_{(\alpha)},\psi_{(\alpha)}\right) \leqslant M\|\mathscr{B}\psi_{(\alpha)}\|_2^2.$$

Therefore, we have the estimates (6.8).

Next, we establish the energy estimates. The argument is similar to section 5.3, and hence we just give a sketch. If $\alpha \neq 0$, taking the L^2-scalar product of (6.7) with $\mathcal{S}_\nu U_{(\alpha)}$, and noting that

$$\left(J^\iota\left(\mathcal{A}-\nu\mathcal{C}(D)\right)U_{(\alpha)},\mathcal{S}_\nu U_{(\alpha)}\right)=0,$$

one can get

$$\begin{aligned}&\left(\mathcal{S}_\nu\partial_t U_{(\alpha)},U_{(\alpha)}\right)+\left(\mathcal{S}_\nu U_{(\alpha)}J^\iota\mathcal{B}U_{(\alpha)},U_{(\alpha)}\right)\\ &\qquad=\left(J^\iota(R_\alpha,S_\alpha+S'_\alpha)^\top,\mathcal{S}_\nu U_{(\alpha)}\right)\,,\end{aligned}$$

where the superscripts ι, ν of U are omitted for convenience's is sake. Hence,

$$\frac{d}{dt}\left(\mathcal{F}^\alpha(U)+\nu\|\nabla\zeta_{(\alpha)}\|_2^2\right)=\sum_{j=1}^4\tilde{A}_j+\sum_{j=1}^3\tilde{B}_j\,,$$

where

$$\tilde{A}_1 = \frac{1}{2}\left([\partial_t, \tilde{\mathcal{S}}^1]U_{(\alpha)}, U_{(\alpha)}\right),\ \tilde{A}_2 = -\left(\tilde{\mathcal{S}}^1 J^\iota \mathcal{B} U_{(\alpha)}, U_{(\alpha)}\right) + \nu\left(\mathrm{diag}(\Delta, 0) J^\iota \mathcal{B} U_{(\alpha)}, U_{(\alpha)}\right),$$
$$\tilde{A}_3 = \frac{1}{2}\left(([\partial_t, \bar{\mathcal{S}}^1]U_{(\alpha)}, U_{(\alpha)}\right),\ \tilde{A}_4 = -\left(\bar{\mathcal{S}}^1 J^\iota \mathcal{B} U_{(\alpha)}, U_{(\alpha)}\right),$$

and

$$\tilde{B}_1 = \left(J^\iota (R_\alpha, S_\alpha)^\top, \tilde{\mathcal{S}}^1 U_{(\alpha)}\right) - \nu\left(J^\iota (R_\alpha, S_\alpha)^\top, \mathrm{diag}(\Delta, 0) U_{(\alpha)}\right),$$
$$\tilde{B}_2 = \left(J^\iota (0, S'_\alpha)^\top, \mathcal{S}_\nu U_{(\alpha)}\right) = (J^\iota S'_\alpha, \mathcal{G}\psi_{(\alpha)}),\ \tilde{B}_3 = \left(J^\iota (R_\alpha, S_\alpha)^\top, \bar{\mathcal{S}}^1 U_{(\alpha)}\right).$$

Similar to the estimates of A_1, A_4, A_6, A_7, B_1, B_2, B_3 in section 5.3, one can obtain

$$\frac{d}{dt}\left(\mathcal{F}^\alpha(U) + \nu\|\nabla\zeta_{(\alpha)}\|_2\right) \leqslant \tilde{\mathfrak{m}}^N(U)(1 + \nu\|\zeta\|_{H^{N+1}}), \quad 1 \leqslant \alpha \leqslant N, \tag{6.9}$$

where $\tilde{\mathfrak{m}}^N(U)$ is independent of ν. The case $\alpha = 0$ is easier. Following section 5.3, one has

$$\frac{d}{dt}\left(\mathcal{F}^0(U) + \nu\|\nabla\zeta\|_2^2\right) \leqslant \tilde{\mathfrak{m}}^N(U). \tag{6.10}$$

Collecting (6.9) and (6.10), one has that for all $0 < \iota < 1$,

$$\frac{d}{dt}\left(\mathcal{F}^N(U^{\iota,\nu}) + \nu\|\nabla\zeta^{\iota,\nu}_{(\alpha)}\|_2\right) \leqslant \tilde{\mathfrak{m}}^N(U^{\iota,\nu})(1 + \nu\|\zeta^{\iota,\nu}\|_{H^{N+1}}). \tag{6.11}$$

Choose $M(t_0+3) > 1$, and let $\delta < \frac{[[\underline{\epsilon}^\pm]]^2}{M(t_0+3)} \leq [[\underline{\epsilon}^\pm]]^2$; define $T_\delta^{\iota,\nu}$ as

$$T_\delta^{\iota,\nu} = \sup\{t \in [0, T^{\iota,\nu}] : \tilde{\mathcal{E}}^N(U^{\iota,\nu}(t)) + \nu\|\nabla\zeta^{\iota,\nu}_{(\alpha)}(t)\|_2 \leqslant \delta\}.$$

Due to Lemma 6.3, for all $0 < \nu < 1$, (6.11) implies that if $\tilde{\mathcal{E}}^N(U^0) + \nu\|\nabla\zeta^0_{(\alpha)}\|_2 \leqslant \frac{\delta}{2}$, then there exists a constant T_δ independent of ι, ν such that $T_\delta^{\iota,\nu} \geqslant T_\delta$. Finally, the standard compactness argument gives Theorem 1.2 (see [1, 15] for more details).

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