Engineering Fracture Mechanics 290 (2023) 109482

Contents lists available at ScienceDirect

Engineering Fracture Mechanics

journal homepage: www.elsevier.com/locate/engfracmech

A novel evaluation method for high cycle and very high cycle fatigue strength

Han Wu [a,b], Chengqi Sun [a,b,*], Wei Xu [c,*], Xin Chen [c], Xiaolei Wu [a,b,*]

[a] State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China
[b] School of Engineering Science, University of Chinese Academy of Sciences, Beijing 100049, China
[c] Beijing Key Laboratory of Aeronautical Materials Testing and Evaluation, Beijing Institute of Aeronautical Materials, Beijing 100095, China

ARTICLE INFO

Keywords:
Continuous runout method
Fatigue strength
Metallic materials
High cycle fatigue
Very high cycle fatigue

ABSTRACT

In this paper, a continuous runout method (CRM) is proposed to evaluate the fatigue strength in high cycle and very high cycle fatigue regimes based on the probability and statistics theory. The CRM features the simultaneous testing of multiple samples such that the testing period could be 1/5–1/3 of that by the common up-and-down method for 16 samples. It is validated by the experimental data for G20Mn5QT steel, 40 Cr steel, and Ti-6Al-4V alloy. The predicted lower limits of fatigue strength by CRM are a little more conservative than those by the up-and-down method.

1. Introduction

Fatigue is one of the main failure modes of materials and components. An effective way to avoid the losses caused by fatigue failure is to evaluate accurately the fatigue life or fatigue strength. Due to the capability of quantitatively assessing the likelihood of a certain fatigue failure [1–3], the probabilistic method has gained great popularity and it has also played an important role in the assessment of fatigue performance in very high cycle fatigue regime [4–6]. The up-and-down method (UDM) is one of the most widely used probabilistic-based methods for evaluating the fatigue strength and it is recommended by ISO [7,8]. However, the UDM provides an inaccurate standard deviation with a small size of samples although it provides accurate estimation of the mean fatigue strength [9,10]. The UDM is not applicable for $D<0.3$ (the meaning of this inequality will be explained in Sec. 3) when evaluating the standard deviation using Dixon-Mood formulas [7,8]. In some cases of $D<0.3$ with three stress levels, the standard deviation cannot even be evaluated by the more general maximum likelihood method [5]. Furthermore, the UDM is time-consuming, especially when evaluating the fatigue strength in very high cycle fatigue regime since the testing stress of the latter sample must be determined according to the result of the prior one [10]. For example, it takes at least 696 d (~1.9 y) for the UDM with sixteen samples to evaluate the fatigue strength at 10^9 cycles under the testing frequency of 100 Hz.

Many modified methods have been proposed to improve the accuracy and efficiency of UDM. For example, Mao et al. [9] proposed a fatigue strength estimation method based on sample expansion and standard deviation correction and validated it by the staircase test of gears. Pollak et al. [11] studied the effectiveness of a bootstrapping algorithm on the standard deviation estimation of traditional Dixon and Mood formulas. They found that the scatter in standard deviation estimation was significantly reduced by incorporating bootstrapping algorithm. Lin et al. [12] compared the staircase, ray-projected, and parallel-projected accelerated methods based on

* Corresponding authors.
E-mail addresses: scq@lnm.imech.ac.cn (C. Sun), wxu621@163.com (W. Xu), xlwu@imech.ac.cn (X. Wu).

https://doi.org/10.1016/j.engfracmech.2023.109482
Received 17 March 2023; Received in revised form 27 June 2023; Accepted 8 July 2023
Available online 14 July 2023

simulations. Their results showed that the parallel-projected method is the best for a regular coefficient of variation of fatigue limits, whereas Dixon and Mood method is better when the coefficient of variation is extremely large.

Some other methods, e.g., the advanced IABG method [10], small sampling up-and-down method (SSUDM) [13], and SSUDM with a new maximum likelihood approach [14], have also been proposed. Nevertheless, all these methods are either too time-consuming or too complicated to utilize in engineering when evaluating the fatigue strength in high cycle and very high cycle fatigue regimes. Therefore, it is urgent to develop a new fatigue strength testing method with high efficiency and convenience to meet the requirement of long-life service, especially the ultra-long life service of materials and components [15–18].

In this article, a continuous runout method (CRM) is proposed for the fatigue strength evaluation based on the probability and statistics theory. The main aim of the paper is to improve the testing efficiency for the fatigue strength evaluation in high cycle and very high cycle fatigue regimes. According to this method, the testing is stopped if all n_p (determined according to the number of samples that could be provided, reliability, etc.) samples tested under a certain stress level do not fail at the given fatigue life. In this method, multiple samples can be tested at the same time, and the testing time is greatly reduced in comparison with UDM. The method is validated by the experimental data of three different metallic materials at 10^7 (or 10^8) cycles. The results of fatigue strength evaluation and testing efficiency are also compared with UDM.

2. Experimental materials and testing methods

Three materials are used for the method validation: the G20Mn5QT steel, the Ti-6Al-4V alloy, and the 40Cr steel. The G20Mn5QT steel was cut from new axle box bodies of a high-speed train [19]. The Ti-6Al-4V alloy was from as-received solid bars with 11 mm diameter. The 40Cr steel was from the solid bars with 11 mm diameter. Two groups of specimens are used. One group was as-received. The other group was first heated at 850 °C for 2 h and then quenched in oil. Finally, they were tempered at 200 °C for 2 h and cooled in air. The tensile strength and yield strength of the tested materials are presented in Table 1. The shapes of fatigue specimens are shown in Fig. 1, respectively.

All the fatigue tests were conducted in air and at room temperature with the stress ratio $R = -1$. The G20Mn5QT steel, Ti-6Al-4V alloy, and as-received 40Cr steel were tested by rotating bending fatigue machines (f=50 Hz). The heat-treated 40Cr steel was tested with the ultrasonic fatigue testing system USF-2000A (f=20 kHz) with an intermittent loading sequence of 200 ms pulse and 1 s pause [20,21]. The surface of the tested section was ground and polished for all the specimens before fatigue tests.

3. Description of UDM

At first, the mean and standard deviation of fatigue strength are estimated at a given fatigue life [7]. Then, the first stress level close to the estimated fatigue strength is chosen, and a stress step near the standard deviation is selected. Afterward, the first sample is tested at the first stress level. If it fails before the given fatigue cycles (e.g., 10^7), the tested stress level is decreased with a stress step; if the sample runs out, a stress step is added to the tested stress level. The samples are tested one after another until all samples have been tested.

Dixon and Mood [8] divided the test results into two sets: failure and runout, and only the data of a smaller set were used in the calculation. The stress levels of the chosen set can be arranged in order as $S_{D-M,0} < S_{D-M,1} < \cdots < S_{D-M,l}$. They then deduced the simple formulas for the evaluation of the mean and standard deviation based on the maximum likelihood method:

$$\hat{\mu}_{D-M} = S_{D-M,\,0} + d\left(\frac{A}{C} \pm \frac{1}{2}\right) \tag{1}$$

$$\hat{\sigma}_{D-M} = 1.62d\left(D + \frac{1}{2}\right),\ D > 0.3 \tag{2}$$

where $S_{D-M,\,0}$ indicates the minimum stress level in the chosen set; d denotes the stress step; "–" is taken if the failure set is selected and "+" is taken for the runout set. Other parameters are calculated as follows:

$$A = \sum_{i=0}^{l} i f_i,\ B = \sum_{i=0}^{l} i^2 f_i,\ C = \sum_{i=0}^{l} f_i,\ D = \frac{BC - A^2}{C^2} \tag{3}$$

where $i = 0, 1, \cdots, l$. The subscript i in $S_{D-M,i}$ denotes the i^{th} stress level, and the subscript l denotes the maximum stress level. f_i denotes the frequency of events at the i^{th} stress level in the selected group (failure or runout). A and B are the parameters used to compute the first two moments of stress levels $S_{D-M,i}$, C is the total frequency of selected events, and D is a linear approximation to determine the

Table 1
Tensile strength and yield strength of the three tested materials.

Material	G20Mn5QT steel [19]	Ti-6Al-4V alloy	40Cr steel[a]	40Cr steel[b]
Tensile strength (MPa)	582	1042	860	1905
Yield strength (MPa)	399	1006	606	1429

[a] As-received. [b] Heat-treated.

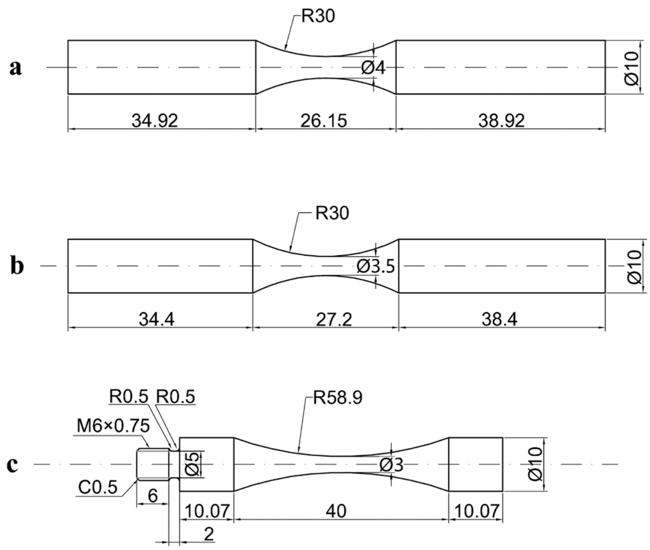

Fig. 1. Shapes of fatigue specimens (mm). a: G20Mn5QT steel; b: 40Cr steel (as-received) and Ti-6Al-4V alloy; c: 40Cr steel (heat-treated).

estimate of σ_{D-M}. A detailed discussion of these parameters can be found in [8].

Eq. (2) is only valid for $D > 0.3$. It can be proved that $D < 0.3$ always holds when the total number of stress levels is equal to three and $D < 0.3$ can also occur in some cases with more than three levels. Dixon and Mood [8] emphasized that the standard deviation should not be estimated by Eq. (2) when $D < 0.3$, and it should be evaluated based on the maximum likelihood method. However, the maximum likelihood method may also fail in some cases of three levels of stress, because the extreme value of the maximum likelihood function does not always exist in this situation.

4. Description of CRM

4.1. Fatigue strength testing

The testing process of CRM is as follows:

Step 1: The mean and standard deviation of fatigue strength are estimated at the given fatigue life, similar to that in UDM [7]. Determine the estimated fatigue strength $S^{'}$ and stress step d between two adjacent stress levels. The stress step should be close to the standard deviation of the fatigue strength. If the standard deviation cannot be obtained before the test, the stress step d could be taken as 5% of the estimated average fatigue strength [7].

Step 2: Fatigue tests are carried out at the estimated fatigue strength and the higher or lower stress levels near it until three data indicated by "*" in Fig. 2 (failure at a higher level and running out at the two adjacent lower levels) are obtained. Denote the stress levels of the three samples in descending order as S_1, S_2, and S_3, respectively.

Step 3: Samples are tested at the stress level of S_3. If all the tested samples run out and the number of runout samples reaches $n_p - 1$, the fatigue tests are accomplished and proceed to *Step 5*. Otherwise, proceed to *Step 4*.

Step 4: Samples are tested at the next lower stress level until all n_p samples run out at this level.

Step 5: The statistical analysis is performed for the tested data at the stress levels S_1, S_2,…, S_N (the stress at which all the tested

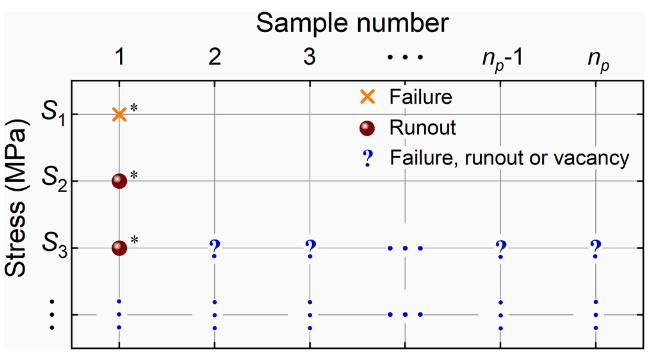

Fig. 2. Sketch map of the CRM.

samples run out and the number is n_p), and the fatigue strength at a given failure probability and confidence is obtained.

As indicated in Fig. 2, $n_p - 1$ samples can be tested at the stress level S_3 simultaneously when the testing capacity is sufficient. If any sample does not reach the given fatigue life, n_p samples can also be tested simultaneously at the next stress level S_4, and so on. Furthermore, the specimens can also be tested simultaneously when determining the stress levels indicated by "*" in Fig. 2 in *step 2*; then $n_p - 1$ samples are tested at the stress level S_3 and/or n_p samples are tested simultaneously at the stress level S_4 if the testing capacity is sufficient. Therefore, the proposed CRM could significantly improve the testing efficiency. The flow chart of the CRM is depicted in Fig. 3.

4.2. Parameter estimation

The maximum likelihood method is used for the parameter estimation [8]. Assuming that the fatigue strength S at a given fatigue life follows normal distribution with mean μ and standard deviation σ, the probability that n_i specimens fail and m_i specimens do not fail for $n_i + m_i$ specimens tested at the i^{th} stress level is [8]:

$$P(n_i, m_i | S_i) = C_{n_i+m_i}^{n_i} p_i^{n_i} q_i^{m_i} \tag{4}$$

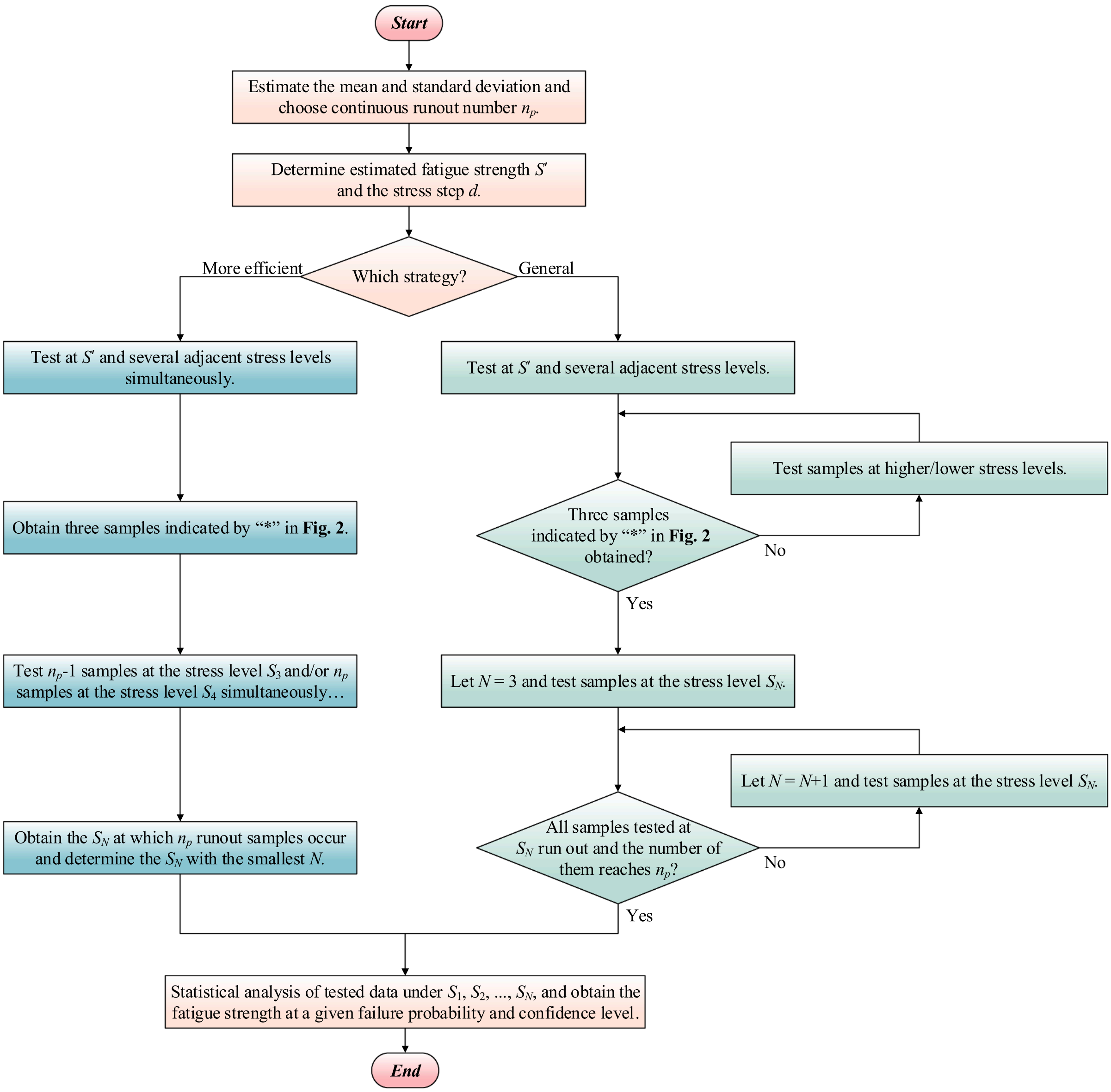

Fig. 3. Flow chart of the CRM.

where p_i and q_i are the probabilities of failure or non-failure of a specimen at stress level S_i, respectively, i.e.,

$$p_i = P(S < S_i) = \int_{-\infty}^{S_i} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt = \int_{-\infty}^{\frac{S_i-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt \tag{5}$$

$$q_i = P(S \geqslant S_i) = 1 - p_i \tag{6}$$

The likelihood function can be derived from Eq. (4) as

$$L(\mu, \sigma) = K \prod_{i=1}^{I} p_i^{n_i} q_i^{m_i} \tag{7}$$

where $K = \prod_{i=1}^{I} C_{n_i+m_i}^{n_i}$ and I is the total number of stress levels. If $K > 1$, the likelihood function corresponds to various possible testing sequences.

If Eq. (7) has the extreme value, taking the natural logarithm of Eq. (7) and the partial derivatives with respect to μ and σ, the following equations are obtained

$$\begin{cases} \sum_{i=1}^{I} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{S_i-\mu}{\sigma}\right)^2} \left(\frac{m_i}{1 - \int_{-\infty}^{\frac{S_i-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt} - \frac{n_i}{\int_{-\infty}^{\frac{S_i-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt} \right) = 0 \\ \sum_{i=1}^{I} \frac{S_i-\mu}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{S_i-\mu}{\sigma}\right)^2} \left(\frac{m_i}{1 - \int_{-\infty}^{\frac{S_i-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt} - \frac{n_i}{\int_{-\infty}^{\frac{S_i-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt} \right) = 0 \end{cases} \tag{8}$$

It is indicated in Eq. (8) that, for a given stress level, μ and σ are only related to the cumulative numbers of failure or non-failure events, and independent of the sequence in which events happen.

Introducing the error function $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$, the following relation is given

$$\int_{-\infty}^{\frac{S_i-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{1}{2} + \frac{1}{2} erf\left(\frac{S_i-\mu}{\sqrt{2}\sigma}\right) \tag{9}$$

Substitution of Eq. (9) into Eq. (8), we have

$$\begin{cases} \sum_{i=1}^{I} e^{-\frac{1}{2}\left[\frac{S_1-(i-1)d-\mu}{\sigma}\right]^2} \left(\frac{m_i}{1 - erf\left(\frac{S_1-(i-1)d-\mu}{\sqrt{2}\sigma}\right)} - \frac{n_i}{1 + erf\left(\frac{S_1-(i-1)d-\mu}{\sqrt{2}\sigma}\right)} \right) = 0 \\ \sum_{i=1}^{I} \frac{S_1-(i-1)d-\mu}{\sigma} e^{-\frac{1}{2}\left[\frac{S_1-(i-1)d-\mu}{\sigma}\right]^2} \left(\frac{m_i}{1 - erf\left(\frac{S_1-(i-1)d-\mu}{\sqrt{2}\sigma}\right)} - \frac{n_i}{1 + erf\left(\frac{S_1-(i-1)d-\mu}{\sqrt{2}\sigma}\right)} \right) = 0 \end{cases} \tag{10}$$

Eq. (10) is a system of nonlinear equations with two independent variables μ and σ. We can easily perceive from Eq. (10) that the solution $(\widehat{\mu}, \widehat{\sigma})$ is not only related to the cumulative numbers of failure or non-failure events at each stress level but also dependent on the values of the stress level S_1 and the stress step d in specific tests. Therefore, Eq. (10) is too specific and fails to reflect the intrinsic characteristics of various testing results.

To deduce more universal equations other than Eq. (10), three dimensionless parameters are defined and introduced:

$$\alpha = \frac{d}{S_1}, \quad k = \frac{\mu - S_1}{\sigma} = \frac{\mu/S_1 - 1}{\sigma/S_1}, \quad r = \frac{d}{\sigma} = \frac{\alpha}{\sigma/S_1} \tag{11}$$

Substitution of Eq. (11) into Eq. (10), a system of equations with respect to k and r can be obtained as follows:

$$\begin{cases} \sum_{i=1}^{I} e^{-\frac{1}{2}[(i-1)r+k]^2} \left(\frac{m_i}{1 + erf\left(\frac{(i-1)r+k}{\sqrt{2}}\right)} - \frac{n_i}{1 - erf\left(\frac{(i-1)r+k}{\sqrt{2}}\right)} \right) = 0 \\ \sum_{i=1}^{I} [(i-1)r+k] e^{-\frac{1}{2}[(i-1)r+k]^2} \left(\frac{m_i}{1 + erf\left(\frac{(i-1)r+k}{\sqrt{2}}\right)} - \frac{n_i}{1 - erf\left(\frac{(i-1)r+k}{\sqrt{2}}\right)} \right) = 0 \end{cases} \tag{12}$$

It is obvious that the solution (k, r) of Eq. (12) only depends on the tested stress levels and the cumulative numbers of failure or non-failure events at each stress level. The values of the stress level S_1 and the stress step d do not influence the solution (k, r) in Eq. (12), indicating that Eq. (12) is a more universal form compared to Eq. (10).

The parameters μ and σ could be estimated by solving Eq. (10) or Eq. (12) through numerical approaches. If the extreme value of Eq. (7) does not exist, one possible method is to take the maximum likelihood estimation of Eq. (7) if the range of values of μ and σ is known, and then find the estimated values of μ and σ. In this situation, maximizing the maximum likelihood function (i.e., Eq. (7)) could be transformed to solving the following constrained optimization problem:

$$\begin{aligned} \max_{\mu,\sigma} \quad & L(\mu,\sigma) \\ s.t. \quad & \mu \in \mathrm{M},\ \sigma \in \Sigma \end{aligned} \tag{13}$$

where M and Σ are intervals of μ and σ, respectively. Apparently, the optimal solutions of (13) should be located on the boundaries of the feasible region, which means that the priori knowledge of μ and σ has a significant influence on the parameter estimation. It is noted that the maximum likelihood method is used here as that for the UDM [8]. Other methods for the parameter estimation can be found in Ref. [10].

4.3. Survival probability

The survival probability of the i^{th} (i=1, 2,..., M, and M is the number of stress levels) stress level S_i is

$$\begin{aligned} 1 - p_i &= 1 - \int_{-\infty}^{S_i} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \\ &= \Phi(k + (i-1)r) \\ &= \frac{1}{2} + \frac{1}{2} erf\left(\frac{k + (i-1)r}{\sqrt{2}}\right) \end{aligned} \tag{14}$$

Note that S_{1-p_0} denotes the stress level at a given survival probability 1 $-p_0$, we have

$$S_{1-p_0} = S_1 + k_{1-p_0} d \tag{15}$$

From Eq. (11), the following expression is obtained

$$S_{1-p_0} = \mu - \left(k - r k_{1-p_0}\right)\sigma \tag{16}$$

The relation between the survival probability 1 $-p_0$ and k_{1-p_0} is written as

$$1 - p_0 = \Phi\left(k - r k_{1-p_0}\right) \tag{17}$$

or

$$k_{1-p_0} = \frac{1}{r}\left[k - \Phi^{-1}(1 - p_0)\right] \tag{18}$$

Substitution of Eq. (18) into Eq. (15), the stress level S_{1-p_0} associated with the survival probability 1 $-p_0$ is expressed as

$$S_{1-p_0} = S_1 + \frac{1}{r}\left[k - \Phi^{-1}(1 - p_0)\right] d \tag{19}$$

or

$$S_{1-p_0} = S_1 \left\{1 + \frac{1}{r}\left[k - \Phi^{-1}(1 - p_0)\right] \alpha \right\} \tag{20}$$

4.4. One-sided tolerance limit

When examining whether the fatigue strength of an individual specimen meets the requirement, the one-sided tolerance limit is usually employed since it encompasses the uncertainty from the parameter estimation.

From Eq. (11), the sample mean $\widehat{\mu}$ and standard deviation $\widehat{\sigma}$ can be derived as

$$\begin{cases} \dfrac{\widehat{\mu}}{S_1} = \alpha \dfrac{\widehat{k}}{\widehat{r}} + 1 \\ \dfrac{\widehat{\sigma}}{S_1} = \alpha \dfrac{1}{\widehat{r}} \end{cases} \tag{21}$$

or

$$\begin{cases} \widehat{\mu} = S_1 + \dfrac{\widehat{k}}{\widehat{r}}d \\ \widehat{\sigma} = \dfrac{d}{\widehat{r}} \end{cases} \tag{22}$$

Then, the lower limit of the fatigue strength $S_{(p,1-\beta,\nu)}$ at a confidence level of $1-\beta$, a failure probability of p and degree of freedom ν can be determined as [7]

$$\frac{S_{(p,1-\beta,\nu)}}{S_1} \begin{aligned} &= \frac{\widehat{\mu}}{S_1} - k^{'}_{(p,1-\beta,\nu)}\frac{\widehat{\sigma}}{S_1} \\ &= \frac{\alpha}{\widehat{r}}\left(\widehat{k} - k^{'}_{(p,1-\beta,\nu)}\right) + 1 \end{aligned} \tag{23}$$

The coefficient $k^{'}_{(p,1-\beta,\nu)}$ in Eq. (23) can be obtained from Table B.1 [7] or calculated by the following formula [22]

$$k^{'}_{(p,1-\beta,\nu)} = \frac{1}{\sqrt{\nu}}F^{-1}_{t^{\nu-1},\sqrt{\nu}z_p}(1-\beta) \tag{24}$$

where z_p is the upper quartile of the standard normal distribution, and $F_{t^{\nu-1},\sqrt{\nu}z_p}$ is the cumulative distribution function of a noncentral t-distribution with the degree of freedom $\nu-1$ and non-centrality parameter $\sqrt{\nu}z_p$.

It is noted that the testing strategy and analysis in Sec. 4.1–4.4 are independent of the fatigue life under the assumption that the fatigue strength follows normal distribution. So, the CRM can evaluate the testing results at any stress range in high cycle and very high cycle fatigue regimes.

5. Several special cases of CRM

Here, further analysis is performed for the case of three and four levels of stress, and detailed results are provided for the case of $n_p = 7$. The reason is that the occurrence of the probability of more than four levels for CRM is relatively small (as shown in Sec. 6.1) and the case of three and four levels of stress with seven continuous runout specimens in CRM could correspond to the number of samples required for exploratory research in UDM [7].

If n_p samples tested at the stress level of S_3 do not fail at the given fatigue life, we have the case of three levels of stress for the CRM in Fig. 2. According to the principle of maximum likelihood, the standard deviation σ derived from Eq. (7) tends to zero, which contradicts the fact that fatigue performance of the material or component generally has a certain dispersion. That is, if the fatigue strength at a given fatigue life approximately follows the normal distribution, the standard deviation is a finite and non-zero value. Therefore, the value of σ should be reasonably estimated or restricted. Otherwise, the values of μ and σ could not be estimated by the maximum likelihood method. In this situation, a special case of four levels of stress in Fig. 4a is introduced for the fatigue strength

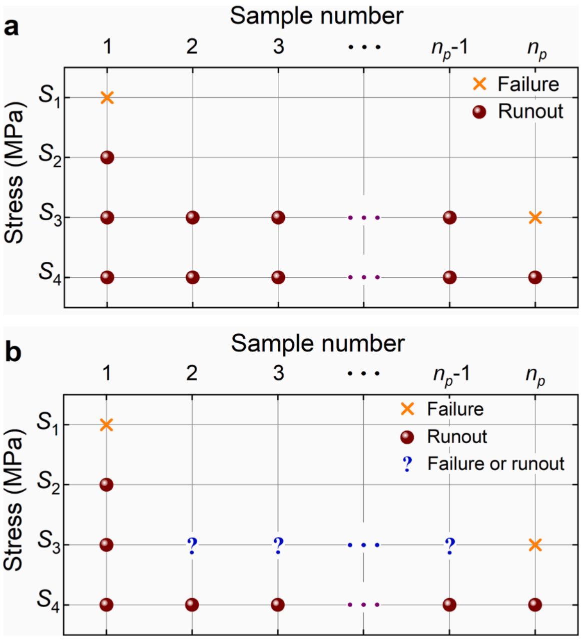

Fig. 4. Sketch maps of the CRM. a: Transformed case of three levels of stress into four levels of stress; b: Case of four levels of stress.

evaluation. In Sec. 6.1, it will be demonstrated that the case of four levels of stress in Fig. 4a does not underestimate the fatigue strength largely due to the much smaller stress step d compared to the fatigue strength. Therefore, the fatigue strength evaluation for the case of three levels of stress can be obtained approximately from the case of four levels of stress in Fig. 4a. It is noted that the conventional UDM [7] also fails to evaluate the standard deviation of fatigue strength when the tested results are three stress levels.

The case of four levels of stress in Fig. 2 will be obtained when at least one specimen fails before the given fatigue life at the stress level S_3 and n_p samples running out tested at the stress level S_4. Without loss of universality, the case of four levels of stress in Fig. 4b is considered, i.e., assume that n_p samples exist at the stress level S_3 even though the tested specimens are fewer than n_p. It is recommended to test n_p samples if the testing resources are sufficient because, in this situation, specific results can be obtained according to the following analysis.

As an example, Table 2 shows the dimensionless mean $\widehat{\mu}/S_1$, dimensionless standard deviation $\widehat{\sigma}/S_1$, and the lower limit of fatigue strength at 95% confidence levels and different failure probabilities (10%, 5%, 1%) for the case of $n_p = 7$ in Fig. 4b, with different numbers of runout specimens m_3 and $d = 0.05S_1$. Table 2 is calculated using Eqs. (11) and (12). By employing Table 2, one can make a conservative estimation according to the actual number of runout specimens tested at the third stress level. For example, if there is one failed specimen and five runout specimens at the stress level S_3 in the test, one can utilize the column $m_3 = 5$ to estimate the fatigue strength. One can also evaluate the fatigue strength by the actually tested results through the approach presented in Sec. 4. The calculated dimensionless mean and standard deviation for one failed specimen and five runout specimens at S_3 is 0.9543 and 0.0445, respectively. The dimensionless lower limits of fatigue strength are 0.8605, 0.8380 and 0.7948, respectively, which is a little bigger than the corresponding results for $m_3 = 5$ in Table 2. The lower limits of the fatigue strength at other confidence levels and failure probabilities can be calculated by substituting the dimensionless mean and standard deviation in Table 2 along with the associated coefficient $k'_{(p,1-\beta,\nu)}$ from Table B.1 [7] or Eq. (24) into Eq. (23).

When stress step d takes other values (e.g., $0.03S_1$), one can at first substitute the dimensionless mean and standard deviation of the corresponding column (according to numbers of runout specimens m_3) in Table 2 into Eq. (25) to determine the mean and standard deviation. Then, obtain the coefficient $k'_{(p,1-\beta,\nu)}$ from Table B.1 [7] or Eq. (24) at the given confidence level and failure probability. Finally, the associated lower limits of the fatigue strength can be calculated by substituting the coefficient together with the mean and standard deviation into Eq. (23).

$$\begin{cases} \widehat{\mu} = 20d\left[\left(\dfrac{\widehat{\mu}}{S_1}\right)_{d=0.05S_1} - 1\right] + S_1 \\ \widehat{\sigma} = 20d\left(\dfrac{\widehat{\sigma}}{S_1}\right)_{d=0.05S_1} \end{cases} \tag{25}$$

It is noted that Eq. (25) applies to any cases in CRM as long as $d = 0.05S_1$ because Eq. (25) is derived by substituting the dimensionless quantity $\alpha = 0.05$ into Eq. (11).

6. Discussion

6.1. Influence of stress step on testing results

In fact, for the given material (or component) and fatigue life, the occurrence of either testing situation (three, four, or more levels of stress) is related to the selection of the stress step d, i.e., the relationship between the stress step d and the standard deviation (fatigue strength is assumed to follow the normal distribution at the given fatigue life). Considering that at least 15 samples are required for exploratory experiments and at least 30 samples for reliability experiments [7], two cases of CRM (i.e., $n_p = 7$ and $n_p = 14$) are chosen for the following analysis since the total sample number for the parameter estimation in CRM could be 16 and 30 respectively in these two situations according to Fig. 2 and Sec. 5. Fig. 5 shows the probabilities of the occurrence of seven continuous runout samples (i.e., $n_p = 7$) and fourteen continuous runout samples (i.e., $n_p = 14$) under the i^{th} ($i = 2,3,4,\ldots,8$) stress level for the CRM under different values of d/σ [8]. In Fig. 5, the mean value and standard deviation of the fatigue strength are estimated by the results of the samples marked by "*" in Fig. 2, and the probabilities of occurrence are obtained by Eq. (14).

It is seen from Fig. 5 that, for the given value of d/σ, the probability of occurrence of both the seven and fourteen continuous runout samples increases with the increase of the number of the stress level i. When the value of d/σ is small (e.g., d/σ=0.75), the probabilities

Table 2
Dimensionless mean, dimensionless standard deviation, and the lower limit of fatigue strength at different confidence levels and failure probabilities for the case of $n_p = 7$ in Fig. 4b with different numbers of runout specimens m_3 tested at the stress level of S_3 and $d = 0.05S_1$.

m_3	1	2	3	4	5	6
$\widehat{\mu}/S_1$	0.8956	0.9053	0.9163	0.9286	0.9420	0.9558
$\widehat{\sigma}/S_1$	0.04127	0.04450	0.04716	0.04873	0.04829	0.04363
$S_{(0.10,0.95,15)}/S_1$	0.8117	0.8148	0.8204	0.8295	0.8439	0.8671
$S_{(0.05,0.95,15)}/S_1$	0.7914	0.7930	0.7973	0.8056	0.8202	0.8457
$S_{(0.01,0.95,15)}/S_1$	0.7526	0.7512	0.7529	0.7598	0.7748	0.8047

of occurrence are 49.0 % and 24.0 % for the seven and fourteen continuous runout samples at the stress level of S_3, respectively. In addition, probabilities are 86.7% and 75.1% at the stress level of S_4 and 98.2% and 96.5% at the stress level of S_5, respectively. This indicates that the case of four levels or five levels of stress will probably occur for the CRM. For the given number of stress level i, the probabilities of occurrence of both the seven and fourteen continuous runout samples increase with an increase of d/σ. For instance, when d/σ=1, the probabilities of occurrence of seven and fourteen continuous runout samples are 68.1% and 46.3% at the stress level of S_3 and 96.9% and 93.9% at the stress level of S_4, respectively. In this case, the case of four levels of stress for the CRM will probably be present. When the value of d/σ is relatively large (e.g., d/σ=1.5), the probabilities of the occurrence of seven and fourteen continuous runout samples become very high at the stress level of S_3, which are 92.4% and 85.4%, respectively. While the probabilities of occurrence are low at the stress level of S_2, which are 18.1% and 3.3% for seven and fourteen continuous runout samples, respectively. In this situation, the case of three levels of stress will probably appear for the CRM.

The results in Fig. 5 also indicate that when the value of d/σ ranges from 0.5 to 2.0, the probabilities of occurrence of both the seven and fourteen continuous runout samples are very low at the stress level of S_2, which ranges from 7.5% to 30.2% and from 0.6% to 9.1%, respectively. This is the reason why the continuous fatigue test begins at the stress level of S_3 rather than S_2 in the CRM. Therefore, for the reasonable value of d/σ, there is usually a high probability for the occurrence of three or four levels of stress, a certain probability for the occurrence of five levels of stress, and a very small probability for the occurrence of more than five levels of stress in the tests by CRM. In other words, CRM does not require testing numerous samples in most situations. For example, after the stress levels indicated by "*" in Fig. 2 are determined, usually only six (thirteen) more samples need to be tested for the case of three (four) levels of stress to occur if $n_p = 7$.

Moreover, for the case of three levels of stress with $n_p = 7$ and $d = 0.05S_1$ in Fig. 2, the lower limit of the fatigue strength at 95% confidence and 5% failure probability is $0.8457S_1$ (i.e., $0.94S_3$) through the transformed case of four levels of stress shown in Fig. 4a. For the case of three levels of stress with $n_p = 14$ and $d = 0.05S_1$ in Fig. 2, the lower limit of the fatigue strength at 95% confidence and 5% failure probability is $0.8758S_1$ (i.e., $0.97S_3$). Based on the consideration that the probabilities of occurrence are very low for both the seven and fourteen continuous runout samples at the stress level S_2 when d/σ ranges from 0.5 to 2.0 (Fig. 5), the transformation from three levels of stress in Fig. 2 to four levels of stress in Fig. 4a would not induce an excessively low mean value when evaluating the fatigue strength, since the stress step d is generally small with respect to the fatigue strength.

6.2. Comparison with UDM

6.2.1. Evaluation of fatigue strength

As examples, the CRM with $n_p = 7$ is used for the fatigue strength evaluation of the G20Mn5QT steel, the Ti-6Al-4V alloy, and the 40Cr steel (as-received and heat-treated). Specimens are considered runout if fatigue lives exceed 10^7 cycles for the G20Mn5QT steel, Ti-6Al-4V alloy and as-received 40Cr steel, and exceed 10^8 cycles for the heat-treated 40Cr steel. The estimated average fatigue strength is 240 MPa for the G20Mn5QT steel, 500 MPa for the Ti-6Al-4V alloy, 400 MPa for the as-received 40Cr steel and 600 MPa for the heat-treated 40Cr steel. Stress steps are all taken as 5% of the estimated average fatigue strength of the tested materials for both CRM and UDM.

Fig. 6 shows the experimental results for the G20Mn5QT steel, Ti-6Al-4V alloy, and 40Cr steel (as-received and heat-treated) by CRM and UDM, in which six specimens were tested simultaneously at the third stress level for Ti-6Al-4V in CRM. It is seen from Fig. 6 that the case of three levels of stress occurs for all three tested materials in CRM. Here, the transformed four levels of stress in Fig. 4a are used to evaluate the fatigue strength for the CRM. For G20Mn5QT steel, $d = 0.05S_1$. Referring to the case $m_3 = 6$ in Table 2, the lower limits of the fatigue strength at different failure probabilities with 95% confidence at 10^7 cycles can be obtained directly and the results are shown in Table 3. For Ti-6Al-4V alloy, $d/S_1 = 0.037 \neq 0.05$. By using Table 2 and substituting $S_1 = 675$ MPa and $d = 25$ MPa into Eq. (25), the mean value and standard deviation of the fatigue strength at 10^7 cycles are obtained as 653 MPa and 22 MPa, respectively.

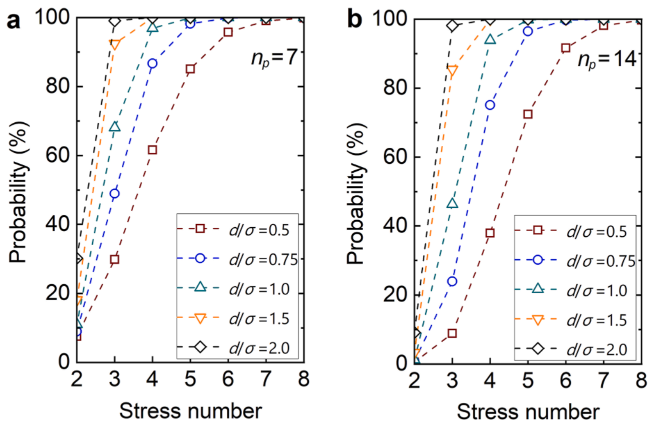

Fig. 5. Probabilities of occurrence of seven and fourteen continuous runout samples under the i^{th} (i=2, 3, 4, …, 8) stress level for the CRM at different values of d/σ.

Taking the calculation of the lower limit of the fatigue strength at the failure probabilities of 10%, 5%, and 1% with 95% confidence as an example, it is to determine the one-sided tolerance limit associated with the confidence and failure probability by referring to Table B.1 in Ref. [7], which are 2.032, 2.523 and 3.463, respectively. Then, substitute these values and the obtained mean value and standard deviation in Eq. (23), and the lower limits of the fatigue strength at the failure probabilities of 10%, 5%, and 1% with 95% confidence are obtained (Table 3). For 40Cr steel, similar to the Ti-6Al-4V alloy, by the employment of Table 2 and substitution of S_1 and d in Eq. (25), the mean value and standard deviation of the fatigue strength are determined as 362 MPa and 17 MPa for the as-received 40Cr steel and 513 MPa and 26 MPa for the heat-treated 40Cr steel, respectively. The lower limits of the fatigue strength at the failure probabilities of 10%, 5% and 1% with 95% confidence are shown in Table 3.

While for the UDM, four levels of stress are obtained for G20Mn5QT steel and the heat-treated 40Cr steel, and three levels of stress are obtained for Ti-6Al-4V alloy and the as-received 40Cr steel, as shown in Fig. 6. For G20Mn5QT steel, the lower limits of the fatigue strength at the failure probabilities of 10%, 5% and 1% with 95% confidence are calculated through the conventional UDM due to the value of $D>0.3$ [7,8], as shown in Table 3. For Ti-6Al-4V alloy and the as-received 40Cr steel, the fatigue strengths cannot be calculated through the conventional UDM due to the value of $D<0.3$ [7,8]. This case can also not be dealt with by the maximum likelihood method [8] since no extreme value exists for the likelihood function Eq. (7). For the heat-treated 40Cr steel ($D<0.3$), the fatigue strengths are calculated by Eq. (12), which can also be calculated through a cut-and-trial method described in Appendix B in Ref. [8]. The results by UDM are shown in Table 3.

Table 3 indicates that, for the same number of samples used for statistical analysis, the lower limits of the fatigue strength obtained

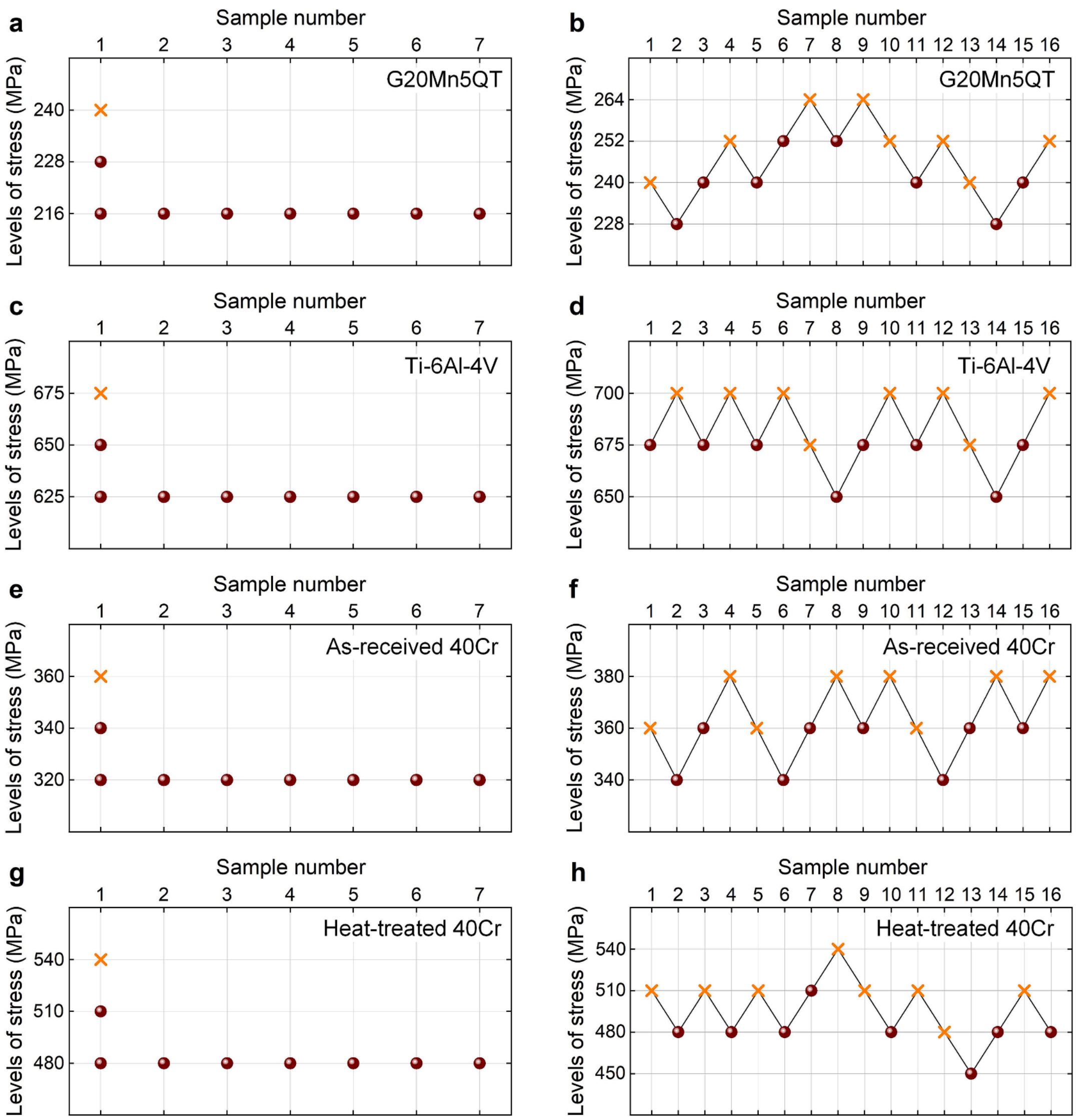

Fig. 6. Testing results by CRM and UDM, in which the stress denotes the nominal stress amplitude in a-f and the stress amplitude in g and h. a, c, e and g: CRM. b, d, f and h: UDM.

Table 3
Lower limits of fatigue strength at different failure probabilities with 95% confidence for different materials at 10^7 (or 10^8) cycles.

Materials	$S_{(0.10,0.95,15)}$/MPa	$S_{(0.05,0.95,15)}$/MPa	$S_{(0.01,0.95,15)}$/MPa
G20Mn5QT steel [a]	208	203	193
G20Mn5QT steel [b]	225	220	210
Ti-6Al-4V alloy [a]	609	598	577
Ti-6Al-4V alloy [b]	–	–	–
40Cr steel (as-received) [a]	307	298	282
40Cr steel (as-received) [b]	–	–	–
40Cr steel (heat-treated) [a]	460	447	423
40Cr steel (heat-treated) [b]	467	460	447

[a] CRM. [b] UDM.

by the CRM are generally a little smaller than those obtained by the UDM at the same confidence and failure probability. One main reason is attributed to the difference in the testing strategy between CRM and UDM. The CRM is toward the low stress, which increases the sample size at relatively low stresses. While the stress levels of samples oscillate around the mean of fatigue strength for the strategy of UDM. Therefore, the CRM gives more reliable evaluations of the fatigue strength because more samples are tested at relatively low-stress levels.

6.2.2. *Testing efficiency*

The testing efficiency between CRM and UDM is also compared. Based on the analysis in Fig. 5, the CRM usually presents the results of three or four levels of stress for the cases of seven continuous runout specimens (i.e., $n_p = 7$) and fourteen continuous runout specimens (i.e., $n_p = 14$) if d/σ is within a reasonable range. Considering that CRM could test multiple samples at the same time, it could be found that only twice the testing time of the given fatigue life is needed to finish the whole tests of the three- or four-stress-level cases when the test condition is allowable. The first part of testing time is the simultaneous test of specimens at the estimated fatigue strength and several stress levels near it to determine the stress levels indicated by "*" in Fig. 2. The other part of the time is the simultaneous test of $n_p - 1$ specimens at the stress level of S_3 or/and n_p specimens at the stress level of S_4. If the same number of samples are used for analysis in the UDM, sixteen and thirty samples are needed corresponding to the case of $n_p = 7$ and $n_p = 14$ in CRM, respectively. Since the stress levels of specimens in UDM depend on the tested result of the previous one, there are at least seven (or fourteen) runout specimens if the tested results are three levels of stress. And there are at least six (or thirteen) runout samples if the tested results are four or five levels of stress. Take the Ti-6Al-4V alloy test in Fig. 6 as an example, the specimens were tested one by one for UDM until the number of valid samples reached sixteen. For CRM, first several samples were tested simultaneously to obtain the three samples by "*" in Fig. 2; then six specimens were tested simultaneously at S_3 and the testing procedure was accomplished because all the n_p ($n_p = 7$) specimens at S_3 ran out. If the testing time of failed samples is excluded, the time consumed for UDM is 18.5 d (the testing time of samples that are not used for analysis is not involved) while that for CRM is only 4.63 d. The testing efficiency of CRM is improved by 75.0% compared with UDM.

Fig. 7 shows a further comparison of the testing time between CRM and UDM. In Fig. 7, the cases of three to five levels of stress are taken into consideration for the tested results. Testing time of samples that fail before the given fatigue life and that are not used for analysis is not involved in UDM. It is seen from Fig. 7a that, if the test condition allows, the testing time of CRM could be independent of the number of test specimens, which significantly reduces the testing time and improves the testing efficiency in comparison with UDM. For example, when sixteen samples are used for the evaluation of the fatigue strength at 10^9 cycles in Fig. 7b, at least 696 d are needed for UDM at the testing frequency of 100 Hz, but only 232 d are needed for CRM. Compared to UDM, the actual testing time is

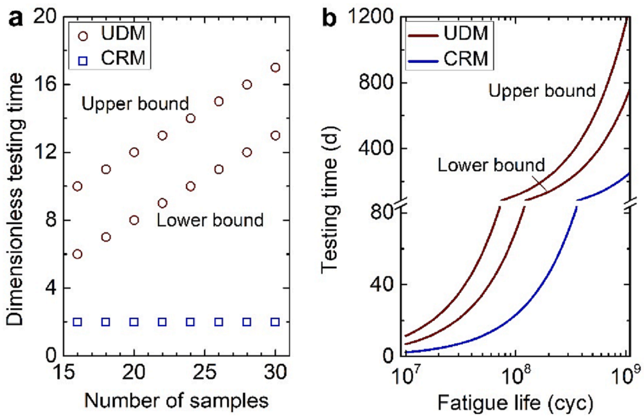

Fig. 7. Comparison of testing time between CRM and UDM. a: Dimensionless testing time vs number of samples, in which the dimensionless testing time denotes the ratio of the total testing time to the given fatigue life; b: Testing time at 100 Hz vs fatigue life for 16 samples.

reduced by 464 d, and the testing efficiency of CRM is improved by at least 66.7%. Fig. 7a also indicates that CRM could have the same dimensionless testing time for the fatigue strength evaluation in high cycle and very high cycle fatigue regimes when the test condition is allowable.

It is noted that the failure mechanism may be different for the samples at different tested stress levels for some materials [23,24]. In this case, the fatigue strength at the given fatigue life may not follow well the normal distribution, and CRM might not give accurately evaluated results.

7. Conclusions

In this paper, the CRM is proposed for the fatigue strength evaluation based on the probability and statistics theory. The main results are as follows:

(1) The new method allows testing multiple samples at the same time, which greatly improves the testing efficiency especially for the evaluation of fatigue strength at ultra-long fatigue life. For example, it could save at least 66.7% of the time with sixteen samples compared with UDM.

(2) The CRM could deal with test results with three levels of stress, while the UDM usually fails in some cases with three levels of stress. It gives a more reliable evaluation of the fatigue strength in high cycle and very high cycle fatigue regimes since the testing stress is towards the low stress and more specimens are tested at relatively lower levels.

(3) This novel method is validated by the experimental data of three kinds of materials, and it is promising for fatigue strength evaluation, especially in very high cycle fatigue regime.

CRediT authorship contribution statement

Han Wu: Writing – original draft, Writing – review & editing, Validation, Formal analysis, Investigation, Data curation. **Chengqi Sun:** Supervision, Resources, Methodology, Conceptualization, Writing – original draft, Writing – review & editing. **Wei Xu:** Conceptualization, Methodology, Writing – review & editing. **Xin Chen:** Conceptualization, Writing – review & editing. **Xiaolei Wu:** Writing – review & editing, Supervision.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgements

The authors acknowledge the support from the National Natural Science Foundation of China Basic Science Center for “Multiscale Problems in Nonlinear Mechanics” (11988102).

References

[1] Tognan A, Salvati E. Probabilistic defect-based modelling of fatigue strength for incomplete datasets assisted by literature data. Int J Fatigue 2023;173.
[2] Tridello A, Niutta C, Berto F, Tedesco M, Plano S, Gabellone D, Paolino D. Design against fatigue failures: lower bound PSN curves estimation and influence of runout data. Int J Fatigue 2022;162.
[3] Patriarca L, Beretta S, Foletti S, Riva A, Parodi S. A probabilistic framework to define the design stress and acceptable defects under combined-cycle fatigue conditions. Eng Fract Mech 2020;224.
[4] Huang Z, Liu H, Wang C, Wang Q. Fatigue life dispersion and thermal dissipation investigations for titanium alloy TC17 in very high cycle regime. Fatigue Fract Eng M 2015;38:1285–93.
[5] Paolino DS, Tridello A, Chiandussi G, Rossetto M. S-N curves in the very-high-cycle fatigue regime: statistical modeling based on the hydrogen embrittlement consideration. Fatigue Fract Eng M 2016;39(11):1319–36.
[6] Sander M, Müller T, Lebahn J. Influence of mean stress and variable amplitude loading on the fatigue behaviour of a high-strength steel in VHCF regime. Int J Fatigue 2014;62:10–20.
[7] ISO 12107:2012, Metallic materials—Fatigue testing—Statistical planning and analysis of data.
[8] Dixon WJ, Mood AM. A method for obtaining and analyzing sensitivity data. J Am Stat Assoc 1948;43(241):109–26.
[9] Mao T, Liu H, Wei P, Chen D, Zhang P, Liu G. An improved estimation method of gear fatigue strength based on sample expansion and standard deviation correction. Int J Fatigue 2022;161.
[10] Müller C, Wächter M, Masendorf R, Esderts A. Accuracy of fatigue limits estimated by the staircase method using different evaluation techniques. Int J Fatigue 2017;100:296–307.
[11] Pollak R, Palazotto A, Nicholas T. A simulation-based investigation of the staircase method for fatigue strength testing. Mech Mater 2006;38(12):1170–81.
[12] Lin S, Lee LM. Evaluation of the staircase and the accelerated test methods for fatigue limit distributions. Int J Fatigue 2001;23(1):75–83.
[13] Gao Z. Applied statistics in fatigue. National Defence Industry Press; 1986.
[14] Zhao Y, Yang B. Probabilistic measurements of the fatigue limit data from a small sampling up-and-down test method. Int J Fatigue 2008;30(12):2094–103.
[15] Stinville JC, Charpagne MA, Cervellon A, Hemery S, Wang F, Callahan PG, et al. On the origins of fatigue strength in crystalline metallic materials. Science 2022;377(6610):1065–71.

[16] Zhu M, Jin L, Xuan F. Fatigue life and mechanistic modeling of interior micro-defect induced cracking in high cycle and very high cycle regimes. Acta Mater 2018;157:259–75.
[17] Li G, Ke L, Ren X, Sun C. High cycle and very high cycle fatigue of TC17 titanium alloy: Stress ratio effect and fatigue strength modeling. Int J Fatigue 2023;166.
[18] Klinger C, Bettge D. Axle fracture of an ICE3 high speed train. Eng Fail Anal 2013;35:66–81.
[19] Zhang Z, Li Z, Wu H, Sun C. Size and shape effects on fatigue behavior of G20Mn5QT steel from axle box bodies in high-speed trains. Metals 2022;12:652.
[20] Nonaka I, Setowaki S, Ichikawa Y. Effect of load frequency on high cycle fatigue strength of bullet train axle steel. Int J Fatigue 2014;60:43–7.
[21] Sun C, Song Q, Hu Y, Wei Y. Effects of intermittent loading on fatigue life of a high strength steel in very high cycle fatigue regime. Int J Fatigue 2018;117:9–12.
[22] Meeker W, Hahn G, Escobar L. Statistical intervals: a guide for practitioners and researchers. John Wiley & Sons; 2017.
[23] Hong Y, Sun C. The nature and the mechanism of crack initiation and early growth for very-high-cycle fatigue of metallic materials – An overview. Theor Appl Fract Mech 2017;92:331–50.
[24] Sakai T, Nakagawa A, Oguma N, Nakamura Y, Ueno A, Kikuchi S, et al. A review on fatigue fracture modes of structural metallic materials in very high cycle regime. Int J Fatigue 2016;60:339–51.