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Bayesian model averaging for probabilistic *S-N* curves with probability distribution model form uncertainty

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ABSTRACT

Reliability analysis of engineering components or structures heavily relies on accurately estimating the fatigue properties of materials. However, significant uncertainty exists regarding the distribution form and value in fatigue data, posing significant challenges in constructing a robust probability fatigue model. To address this challenge, we propose a Bayesian model averaging (BMA) method to incorporate model form uncertainty into the estimation of the probability density of fatigue life. The performance of BMA was verified through numerical experiments using both simulated and experimental data. The results highlight the robustness and reliability of BMA compared to individual models, as it effectively incorporates model form uncertainty. The proposed BMA model offers a general framework for developing probabilistic fatigue models with high robustness and accuracy in their predictions. This model contributes to advancing the field of reliability analysis by addressing the challenges posed by uncertainty and enhancing the understanding of fatigue properties for engineering components and structures.

1. Introduction

Engineering components often experience fatigue failure when they are exposed to repeated stress over an extended period of time. Therefore, in the reliability design, it is crucial to understand and depict the relationship between the fatigue life of engineering components and the applied stress [1–5]. The *S-N* curve is commonly employed in safe-life design as an essential criterion for describing the relationship between fatigue life and cyclic stress [6–11]. Due to the origin of fatigue in materials from microscopic flaws that propagate into macroscopic cracks under cyclic stress conditions, fatigue data exhibits high scatter and requires uncertainty analysis [12,13]. Therefore, it becomes necessary to extend the median *S-N* curve to assess both the deterministic and probabilistic aspects of the composite's fatigue behavior [14,15].

Due to the inherent scattering in fatigue data, the variability in stress-life datasets is typically described using probability distributions to enable the construction of the *P-S-N* curves for reliability and safety design. Consequently, the specification of the probability distribution is of primary importance. However, deriving the distribution function of fatigue data based on physical argument is challenging [16]. In practice, the distribution function is often assumed or fitted to experimental data. Such as, the Lognormal and Weibull distributions being the two most commonly employed models for analyzing the probabilistic behavior of fatigue failure [17]. This inevitably introduces uncertainty since the true underlying probability distributions are unknown. Furthermore, if the incorrect model is fit to data, estimates of life quantiles will be inaccurate or biased [18]. Additionally, in practical engineering, the life distribution sometimes consists of a mixture of multiple distributions, with the failure rate being represented by the well-known bathtub curve [19,20].

The practical engineering materials exhibit significant uncertainty in fatigue life performance due to the randomness of microstructures and defects. The 'classical' method, like ISO-12107 [21], assumes specific distributions and estimate their parameters, which can be limited by the assumptions made and might not fully capture the underlying variability in the fatigue life data. Selecting a "best" model without considering the uncertainty between different models can lead to biased predictions [22]. Moreover, the process of fatigue testing is both time-consuming and expensive, leading to limited availability of experimental observations. This limitation poses challenges in accurately determining the true underlying life distribution of practical engineering materials or components. Therefore, it becomes imperative to consider the uncertainty

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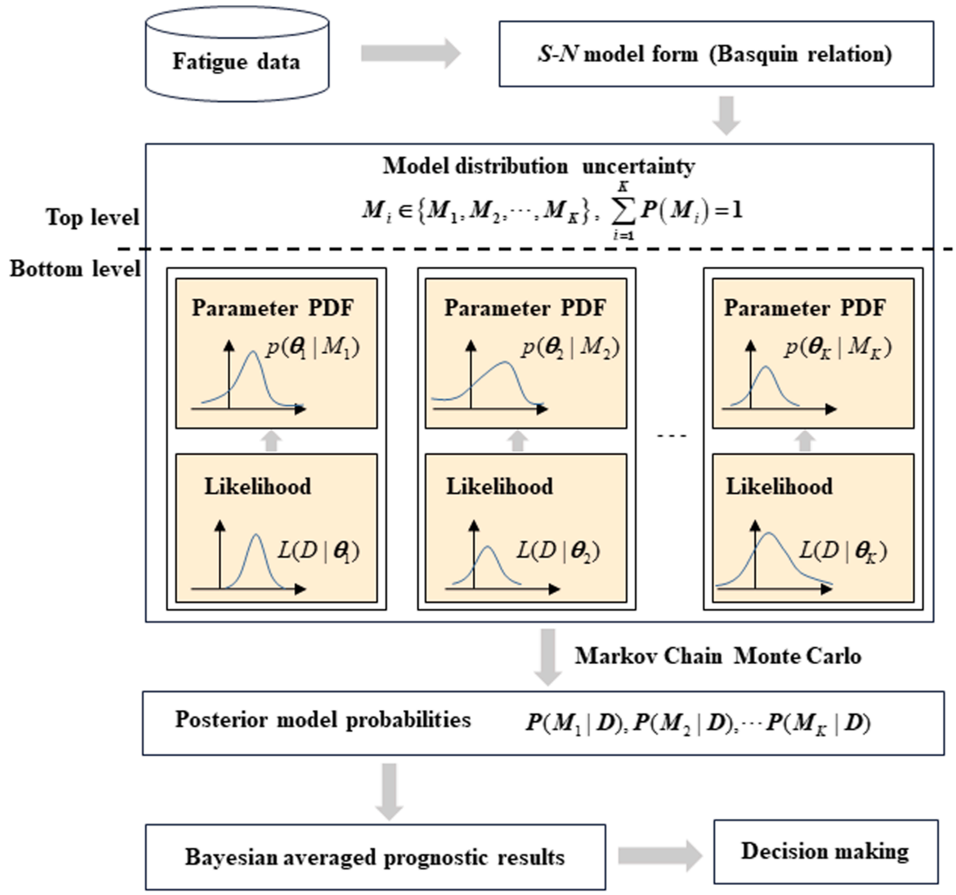

Fig. 1. Schematic description of the hierarchical Bayesian framework for uncertainty quantification of fatigue *S-N* curves. $p(\theta_i|M_i), i = 1, 2, \cdots K$ denotes the prior distribution of parameter vector of the candidate model M_i. $L(D|\theta_i), i = 1, 2, \cdots, K$ is the likelihood function under model M_i, where D is the observed fatigue life data containing the applied stress level, the fatigue life (or right censored time for cases where failure hasn't occurred), and an indicator that distinguishes between failure and runout.

between models in decision-making, especially when multiple candidate distributions are deemed plausible but yield different *P-S-N* estimations.

To mitigate the risk of neglecting uncertainty between multiple potential distribution models, we utilized the Bayesian model averaging (BMA) method [23–26]. Unlike assuming a single distribution, the BMA method takes into account a range of candidate distributions, such as the Weibull, Lognormal, or other possible distributions that could describe the fatigue life data. Each candidate distribution is assigned a weight based on its posterior probability, reflecting the level of support derived from the observed data. The BMA method then combines the information from all candidate distributions to obtain a more comprehensive and robust estimate of the *P-S-N* curve.

The paper is organized as follows. The hierarchical Bayesian framework involving multiple models are firstly introduced in Section 2. The specific Bayesian inference is performed in Section 3. The effectiveness of the BMA method is validated through simulated and experimental fatigue data analysis in Section 4. Conclusions are finally presented in Section 5.

2. Probabilistic modeling for hierarchical uncertainties

We consider a fatigue life experiment involving a total of n specimens. Failures are seen only if they occur before a particular time t_C, which is named runout time. A unit surviving longer than that time t_C is considered a right-censored observation or runout. The fatigue data is represented as $\{(S_i, t_i, \delta_i) | i = 1, 2, \cdots, n\}$. In this representation, S_i represents the applied stress level for the i-th test specimen. The fatigue life $t_i = \min(T_i, t_c)$ is defined as the minimum value between the actual fatigue life T_i and the right censoring time t_C. The variable δ_i indicating whether the test of i-th specimen is runout ($\delta_i = 0$) or not ($\delta_i = 1$), and it is given by Eq. (1) as

$$\delta_i = \begin{cases} 1, & \text{if specimen failure,} \\ 0, & \text{if specimen runout.} \end{cases} \tag{1}$$

The likelihood function for runout (right-censored) observations has been extensively studied [27,28], which is deduced as

$$L(t) = \prod_{i=1}^{n} [f(t_i)]^{\delta_i} [R(t_i)]^{1-\delta_i}, \tag{2}$$

where $f(\cdot)$ and $R(\cdot)$ are the probability density function and reliability function, respectively.

2.1. General regression model

The determination of fatigue properties in products or materials involves testing a set of specimens under varying stress levels to establish a relationship between fatigue life and the applied stress levels. In engineering applications, it is commonly assumed that this relationship follows a power stress-life curve [29], known as the Basquin relation [6] shown in Eq. (3).

$$T_i = A S_i^{-B}, \tag{3}$$

where $A > 0$ and $B > 0$ are the parameters which are shared over stress levels.

Fatigue life is the duration that it takes for a material or component to fail under repeated or cyclic loading. In practical scenarios, it is inherently stochastic and displays variability influenced by various factors, including material properties, manufacturing variations, environmental conditions, and operational parameters. The stochastic nature of fatigue life implies that identical specimens subjected to identical loading conditions can exhibit different fatigue lives. A statistical model can be used to characterize the variability or scatter in fatigue life as

$$T \sim F(T|\boldsymbol{\theta}) \tag{4}$$

where T denotes the fatigue life, the distribution function F can be given as different forms (such as: Lognormal distribution, Weibull distribution, Gamma distribution, and Extreme value distribution *etc.*) from its special characteristic of the research object in experimental observations. $\boldsymbol{\theta}$ means the distribution parameters involved in the given distribution model F. The fatigue of products or materials can be expressed by *S-N* curves depicting the relationship between the load cycles up to failure and the applied stress. Here, the power relationship between life and stress can be reflected by modeling the distribution parameters $\boldsymbol{\theta}$. Through this modeling process, we can assess the effect of stress on the probability distribution of fatigue life.

2.2. Candidate distribution models

The Lognormal and Weibull distributions are two most widely used log-location-scale distributions for description of the fatigue life-stress relationship [17,30]. For example, [31] applied the Lognormal distribution and 3-parameter Weibull distribution to describe the fatigue life data of T700/MTM46 composite laminates. Lognormal and Weibull distributions are both utilized in Bayesian inference and model comparison for analyzing metallic fatigue data [32]. Therefore, we consider merely the Lognormal and Weibull distribution as the candidate distribution models.

3. The Bayesian inference framework

In this section, we provide a detailed explanation of the Bayesian inference framework, taking the Bayesian Model Averaging (BMA) model integrated with the Lognormal regression model (LNR) and the Weibull regression model (WBR) as an illustrative example. It is worth noting that the proposed framework can be easily extended to contain more candidate distributions. The hierarchical Bayesian framework for uncertainty quantification of fatigue *S-N* curves is shown in Fig. 1. Starting from fatigue life data, the *S-N* model for depicting the relationship between the stress level and the fatigue life can be firstly assumed (i.e. the Basquin relation) or fitted from the fatigue life data at several stress levels. For mapping the scatter rule of fatigue life at every stress level, the distribution model is then constructed. By considering the uncertainty of model distribution, some candidate distributions for fatigue life are assumed, and the stress level are incorporated into the model parameters. For each model, the classic Bayesian inference is carried out according to the fatigue life data used in this paper. In specific, the prior knowledge or beliefs (prior probability) is combined with observed data (likelihood) based on Bayes' theorem to obtain a posterior distribution. Then the candidate models are combined through Bayesian model averaging (BMA) to tackle model distribution uncertainty issue. The idea behind BMA is to weight these models based on their posterior probabilities, which reflect the relative support provided by the observed data. That is the posterior probabilities are used to assign weights to the models, and the final prediction or inference is obtained by averaging the predictions or estimates from the weighted models.

3.1. Bayesian Lognormal regression model for S-N curves (LRN)

Suppose the fatigue lives $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)^{'}$ for n specimens are independently and identically distributed (*i.i.d.*), and each $\ln(t_i), i = 1, 2, \cdots, n$ follows a normal distribution $N(\mu, \sigma^2)$. That is each t_i follows the Lognormal distribution $LN(\mu, \sigma^2)$ with parameters μ and σ^2. Then the probability density function (PDF) and reliability function are as follows:

$$f(t_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma t_i}\exp\left\{-\frac{1}{2\sigma^2}[\ln(t_i) - \mu]^2\right\}, \tag{5}$$

$$R(t_i|\mu, \sigma^2) = 1 - \Phi\left[\frac{\ln(t_i) - \mu}{\sigma}\right]. \tag{6}$$

$\Phi(t)$ is the cumulative probability function (CDF) of the standard normal distribution. Let's denote $\mathbf{x_i} = (1, \ln(S_i))^{'}, \boldsymbol{\beta_1} = (\beta_{10}, \beta_{11})^{'}$. The stress level is introduced by $\mu_i = \mathbf{x}_i^{'}\boldsymbol{\beta_1}$ to construct the Lognormal regression model. Eq. (2) provides the likelihood function for right-censored observations under the general distribution. Then the joint likelihood function is derived by substituting Eqs. (5) and (6) into Eq. (2)

$$\begin{aligned} L(D|\boldsymbol{\beta}_1, \sigma) = & (2\pi\sigma^2)^{-\frac{1}{2}\sum_{i=1}^{n}\delta_i}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\delta_i\left[\ln(t_i) - \mathbf{x}_i^{'}\boldsymbol{\beta}_1\right]^2\right\} \\ & \times\prod_{i=1}^{n}t_i^{-\delta_i}\left\{1 - \Phi\left(\frac{\ln(t_i) - \mathbf{x}_i^{'}\boldsymbol{\beta_1}}{\sigma}\right)\right\}^{1-\delta_i}. \end{aligned} \tag{7}$$

To develop Bayesian inference, the prior distributions for σ and $\boldsymbol{\beta}_1$ are required. Firstly, we consider that prior distributions are independent, i.e., $p(\sigma, \boldsymbol{\beta_1}) = p(\sigma)p(\boldsymbol{\beta_1})$. A typical choice for σ is a Half-Cauchy prior distribution. The choice of a Half-Cauchy prior [33,34] is mainly motivated by the following two reasons: (I) The Half-Cauchy distribution has heavy tails, which allows for the possibility of extreme values or outliers in the standard deviation parameter. This can be useful in cases where there is substantial uncertainty or variability in the fatigue data, and the heavy-tailed prior accommodates the potential for extreme values. (II) The Half-Cauchy distribution is often considered non-informative or weakly informative, as it does not impose strong assumptions on the prior knowledge about the standard deviation parameter. It allows the data have a greater influence on the posterior estimates. Here, we assume $\sigma \sim halfcauchy(\sigma_0)$ with hyper-parameter σ_0, where its probability density function is $f(\sigma|\sigma_0) = \begin{cases} \frac{2}{\pi\sigma_0}\frac{1}{1+\sigma^2/\sigma_0^2}, \sigma > 0, \\ 0, \text{ otherwise.} \end{cases}$. In addition, we assume a conjugate multi-normal prior distribution $N(\boldsymbol{\mu_0}, \boldsymbol{\Sigma_0})$ for $\boldsymbol{\beta_1}$. Then the joint posterior distribution is given by

$$\begin{aligned} p(\tau, \beta_1|D) \propto & L(D|\tau, \beta_1) \times p(\sigma|\sigma_0) \times p(\beta_1|\mu_0, \Sigma_0) \propto \sigma^{-}\left(\sum_{i=1}^{n}\delta_i + 1\right) \\ & \times\left(\sigma^2 + \sigma_0^2\right)^{-1} \times \prod_{i=1}^{n}t_i^{-\delta_i}\{1 - \Phi[\frac{\ln(t_i) - x_i^{'}\beta_1}{\sigma}]\}^{1-\delta_i} \\ & \times\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\delta_i[\ln(t_i) - x^{'}\beta_1]^2 + (\beta_1 - \mu_0)^{'}\Sigma_0^{-1}(\beta_1 - \mu_0)\} \end{aligned} \tag{8}$$

Then the Bayesian inference can be carried out via the joint posterior samples, which can be drawn with a Markov Chain Monte Carlo (MCMC). The fundamental idea is to construct a Markov chain such that it converges to the posterior distribution [25,35,36]. Among many existed MCMC algorithms, the Metropolis-Hastings (M−H) algorithm [37] and the Gibbs sampler [38–40] are the two most extensively used methods. However, only when the full conditional distributions are known, the Gibbs sampler can be used [41]. M−H algorithm is a generalization form of Gibbs sampler. In standard M−H, a proposal distribution is used to generate candidate samples, which is then accepted or rejected based on the acceptance probability. However, selecting an appropriate proposal distribution can be challenging, especially for high-dimensional or complex target distributions. Poor choices of the proposal distribution can lead to slow convergence or high rejection rates. Therefore, we employed the Adaptive Metropolis [42] a variant of the Metropolis-Hastings algorithm, which aims to improve the efficiency and convergence of the sampling process by dynamically adapting the proposal distribution during the sampling iterations.

3.2. Bayesian Weibull regression model for S-N curves (WBR)

Suppose the fatigue lives $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)'$ for n specimens are *i.i.d.*, and each one follows the Weibull distribution $Weibull(a,b)$, in which the shape parameter $a > 0$ and the scale parameter $b > 0$. The PDF and the reliability are as follows:

$$f(t_i|a,b) = \frac{a}{b}\left(\frac{t_i}{b}\right)^{a-1}\exp\left[-\left(\frac{t_i}{b}\right)^a\right], \tag{9}$$

$$R(t_i|a,b) = 1 - F(t_i|a,b) = \exp\left[-\left(\frac{t_i}{b}\right)^a\right]. \tag{10}$$

Based on the observation data $D = \{(S_i, t_i, \delta_i)|i = 1, 2, \cdots, n\}$, and Eqs. (2), 9–10), then the joint likelihood function can be obtained as

$$L(D|a,b) = \frac{a^{\sum_{i=1}^n \delta_i}}{b^a \sum_{i=1}^n \delta_i}\exp\left[-\sum_{i=1}^n\left(\frac{t_i}{b}\right)^a\right]\prod_{i=1}^n t_i^{(a-1)\delta_i}. \tag{11}$$

Let's denote $\mathbf{x_i} = (1, \ln(S_i))'$, $\boldsymbol{\beta_2} = (\beta_{20}, \beta_{21})'$. The shape parameter a is empirically found to be dependent on the failure mechanism [43,44]. In the context of low-cycle fatigue, where the failure mechanisms are assumed to be consistent across different stress levels, the shape parameter is considered fixed, meaning it is independent of the stress level. To construct the *S-N* curves in Basquin relation form, we introduce the stress by the scale parameter b, i.e., $\ln b = \mathbf{x_i}'\boldsymbol{\beta_2}$. Then the likelihood function for the Weibull regression model is as

$$L(D|a,\boldsymbol{\beta_2}) = a^{\sum_{i=1}^n \delta_i}\exp\left\{\sum_{i=1}^n\left[a\delta_i\mathbf{x_i}'\boldsymbol{\beta_2} - t_i^a\exp(-a\mathbf{x_i}'\boldsymbol{\beta_2})\right]\right\}\prod_{i=1}^n t_i^{(a-1)\delta_i}. \tag{12}$$

To develop Bayesian inference, the prior distributions for a and $\boldsymbol{\beta_2}$ are required. Firstly, we take that priors are independent, i.e., $p(a, \boldsymbol{\beta_2}) = p(a)p(\boldsymbol{\beta_2})$. In addition, we take the Half-Cauchy distribution as the prior distribution of a, the multi-normal distribution as the prior distribution of $\boldsymbol{\beta_2}$. That is,

$$a \sim halfcauchy(a_0),$$

$$\boldsymbol{\beta_2} \sim N(\boldsymbol{\mu_0}, \Sigma_0).$$

Then according the Bayesian theorem, the joint posterior distribution for $(\alpha,\boldsymbol{\beta_2})$ on which inference is based is given by

$$p(a,\boldsymbol{\beta_2}|D) \propto L(D|a,\boldsymbol{\beta_2}) \times p(a|a_0) \times p(\boldsymbol{\beta_2}|\boldsymbol{\mu_0},\Sigma_0) \tag{13}$$

Similarly, the Bayesian inference based on Eq. (13) is carried out via the joint posterior samples, which can be drawn with a component-wise Adaptive Metropolis algorithm.

3.3. Bayesian model averaging for S-N curves

In engineering applications, parametric mechanism models are commonly constructed to depict the underlying mechanism of the system response to infer the future behavior of the system [22]. Meanwhile, the uncertainty is introduced during the inference process because of model uncertainty, which mainly comes from three aspects: (I) model expression, (II) the probability distribution for the measurement, and (III) model parameter. And the first two points are collectively referred to as model form.

In fatigue life analysis, a well-established empirical model [45] or a physical model, such as the Basquin relation, is usually used to describe the relationship between fatigue mechanism and the stress variable. Therefore, we mainly consider the latter two aspects of uncertainty, which are handled by a two-level hierarchical Bayesian inference structure. Specifically, the firstly level is used to treat the distribution uncertainty (as discussed in Section 2.2, only two candidate models, M_1 and M_2), and the second level is used to dispose of the model parameter uncertainty.

Generally, when multiple distribution forms are suitable for describing the quantity of interest, the BMA method can be used to propagate the distribution uncertainty into the prediction of system response [46]. The basic idea of BMA is firstly to estimate some quantity under each model $M_i, i = 1,2$, and then to average the estimates based on posterior probabilities of models [47,48]. In this paper, Bayesian model averaging is employed to derive the probabilistic *S-N* curves. Based on the observation data $D = \{(S_i, t_i, \delta_i)|i = 1, 2, \cdots, n\}$, it gives the posterior distribution of the $t_p(s)$, i.e., the p quantile of the life at stress level s as

$$p(t_p(s)|D) = \sum_{i=1}^2 p(t_p(s)|D, M_i)Pr(M_i|D) \tag{14}$$

where $p(t_p(s)|D, M_i)$ is the posterior density of $t_p(s)$ by assuming that M_i, $i = 1, 2$ is the true model. $p(t_p(s)|D, M_1)$ and $p(t_p(s)|D, M_2)$ can be obtained according to the illustration in Sections 3.1 and 3.2, respectively. Besides, $Pr(M_i|D)$ is the posterior probability that M_i is true, which can be given by

$$Pr(M_i|D) \propto Pr(M_i)p(\boldsymbol{t}|M_i) \tag{15}$$

where $p(\boldsymbol{t}|M_i)$ is density of $\boldsymbol{t}$ under M_i, and $Pr(M_i)$ is the prior probabilities of model M_i. As is commonly done, we choose $Pr(M_i) = 1/2$ for no extra available information. Based on the Bayes' theorem, the posterior for model M_i can be given by

$$Pr(M_i|D) = \frac{m_i}{\sum_{j=1}^2 m_j} \tag{16}$$

where

$$m_r = \int L_r(\boldsymbol{\theta}_r)\pi(\boldsymbol{\theta}_r)d\boldsymbol{\theta}_r \tag{17}$$

and $L_r(\boldsymbol{\theta}_r)$ is the likelihood function under model M_r, $\pi(\boldsymbol{\theta}_r)$ is the prior distribution of $\boldsymbol{\theta}_r$. A popular way for handling both the prior distribution problem and the integration problem is to compute an approximation to m_r that does not depend on the prior [49,50]. Let $\ell(\boldsymbol{\theta}) = \ln(L(\boldsymbol{\theta}))$, and let $\widehat{\boldsymbol{\theta}}$ denote the maximum likelihood estimator. The approximation to m_r is as follows:

$$\widehat{m}_r = \exp\left(\ell_r(\widehat{\boldsymbol{\theta}}_r) - \frac{d_r}{2}\ln n\right) \tag{18}$$

where d_r is the number of components of $\boldsymbol{\theta}_r$.

The mean and variance of t_p is then constructed as [25,51]:

$$E\left[t_p|D\right] = \sum_{k=1}^2 p(M_k|D)E\left[t_p|M_k, D\right] \tag{19}$$

$$Var\left(t_p|D\right) = \sum_{k=1}^K p(M_k|D)\left\{Var\left(t_p|M_k, D\right) + \left[E(t_p|M_k, D) - E(t_p|D)\right]^2\right\} \tag{20}$$

where $E\left[t_p|M_k, D\right]$ and $Var\left[t_p|M_k, D\right]$ are the expectation and variance under model M_k given data D, respectively; $p(M_k|D)$ is the posterior model probability of model M_k; The term $\left[E(t_p|M_k, D) - E(t_p|D)\right]^2$ is the between-model variability in the posterior mean of the quantity t_p. It is evident that the variance of the BMA probability distribution function contains two components: the within-model error variance and the between-model variance. These components indicate that the BMA model can provide a more reliable description of the total predictive uncertainty than the 'best' model based on some selection criterion.

Table 1
The details of the simulation scenarios.

Underlying distribution	Parameters
Lognormal	$\beta_0 = 25.8, \beta_1 = -2.4, \sigma = 1.0$
Weibull	$\beta_0 = 25.8, \beta_1 = -2.4, a = 2.1$
Log-logistic	$\beta_0 = 6.58 \times 10^4, \beta_1 = -268, k = 1.5$

4. Numerical experiments

4.1. Simulation data experiments

In contrast to the fatigue life property of practical engineering materials, where the underlying distribution may be unknown or a mixture of multiple distributions [19,20], we initially generated fatigue data with known life distributions by statistical simulation for evaluating the predictability and reliability of the proposed BMA model across different distribution scenarios. We considered three commonly employed probability distribution models in engineering: Lognormal, Weibull, and Log-logistic distributions (Appendix A). For each distribution, we predefined specified parameters, which are used to generate random fatigue life. The corresponding pre-specified parameters are provided in Table 1. The fatigue life is generated under five stress levels (120Mpa, 160Mpa, 200Mpa, 240Mpa, and 280Mpa).

Taking the Weibull distribution scenario as an example, the distribution of fatigue life N_S at a specific stress level S obeys the Weibull distribution characterized by a shape parameter a and a scale parameter b, that is $N_S \sim Weibull\ (a, b)$. According to Section 3.2, the stress level S is introduced by the scale parameter $b = \exp(\beta_0 + \beta_1 \ln(S))$, with considering the special Basquin model for the relationship between the expectation of fatigue life and the stress level. Then the scale parameter b is determined by β_0, β_1 and a specific stress level value S. Subsequently, the fatigue data can then be simulated by generating random numbers following the Weibull distribution $Weibull(a, \exp(\beta_0 + \beta_1 \ln(S)))$.

Similarly, the location parameter μ in the Lognormal distribution scenario is considered with stress level dependence as $\mu = \beta_0 + \beta_1 \ln(S)$. Then, the simulated fatigue life can be generated by generating random numbers following the Lognormal distribution $LN(\exp(\beta_0 + \beta_1 \ln(S)), \sigma^2)$. The scale parameter λ in the Log-logistic distribution scenario is considered with stress level dependence as $\ln\lambda = \beta_0 + \beta_1 \ln(S)$. Then, the simulated fatigue life can be generated by generating random numbers following the Log-logistic distribution $LG(k, \exp(\beta_0 + \beta_1 \ln(S)))$, where k is shape parameter.

For each simulation scenario, we generate a total of 20 data points representing fatigue lives at each stress level. The simulated fatigue data is illustrated in Fig. 2, with fatigue life represented by red circles and right-censored (runout) data indicated by green triangle symbols.

After generating simulated fatigue data, for Bayesian inference, vague prior distributions are adopted here, i.e., $\beta_0 \sim N(0, 1000^2)$, $\beta_1 \sim N(0, 1000^2)$, $\sigma \sim halfcauchy(25)$, $a \sim halfcauchy(25)$. In specific, by setting the mean to 0, we are not assuming any specific direction or magnitude for the coefficients. The large standard deviation of 1000 indicates a wide range of plausible values for the coefficients, allowing the data to have a stronger influence on the posterior distribution. Besides, Half-Cauchy distribution is commonly considered non-informative or weakly informative. For each Markov chain, 100,000 samples are obtained, and the first 3000 is discarded due to burn-in. Besides, to reduce the autocorrelation consecutive samples, every 20 sample in the chain is kept, while the others are discarded.

In compliance with the ASTM E739-91 standard, extensively employed in the construction of S-N or ε-N curves, it is imperative to restrict the curves within the bounds of available experimental data. Extrapolating these curves to predict fatigue life for particular stress or strain ranges, especially when targeting reliability levels above 95 % or failure likelihoods below 5 %, is not recommended [52]. Therefore, the posterior of the 0.05 log-life quantile ($\ln t_{0.05}$) is then simulated at a stress level of 180 MPa to evaluate the performance of the BMA model. Besides, the BMA model is compared with the Bayesian Lognormal regression model (LNR) and the Bayesian Weibull regression model (WBR).

The histograms of the predicted $\ln t_{0.05}$ are presented in Fig. 3. The prediction performance of the LNR, WBR, and BMA models on the simulated data with Lognormal distribution is shown in Fig. 3a, 3d, and 3 g, respectively. Comparing them to the true value of 119617, the mean posterior estimates obtained from the LNR, WBR, and BMA models are 156051, 77505, and 118212, respectively. Notably, the BMA model provides a posterior estimation of 118212, which is little lower than the true value, but exhibits the closest estimation to the true value. The LNR model did not perform optimally because the amount of data was insufficient to obtain accurate distributions and parameter estimates, which also confirms the necessity of considering model and parameter uncertainties. Besides, it is noteworthy that the LNR model overestimates reliability, which can be risky as it may result in a false sense of security. This can lead to inadequate safety measures, insufficient maintenance schedules, or improper design considerations, thereby increasing the likelihood of failures, breakdowns, or catastrophic events. On the contrary, the WBR model, which assumes that the data is generated from a Weibull distribution, demonstrates the weakest estimation capability as it deviates noticeably from the actual fatigue data with their inherent distributions. This significantly underestimation can lead to overdesign, excessive costs, and flawed decision-making. We also

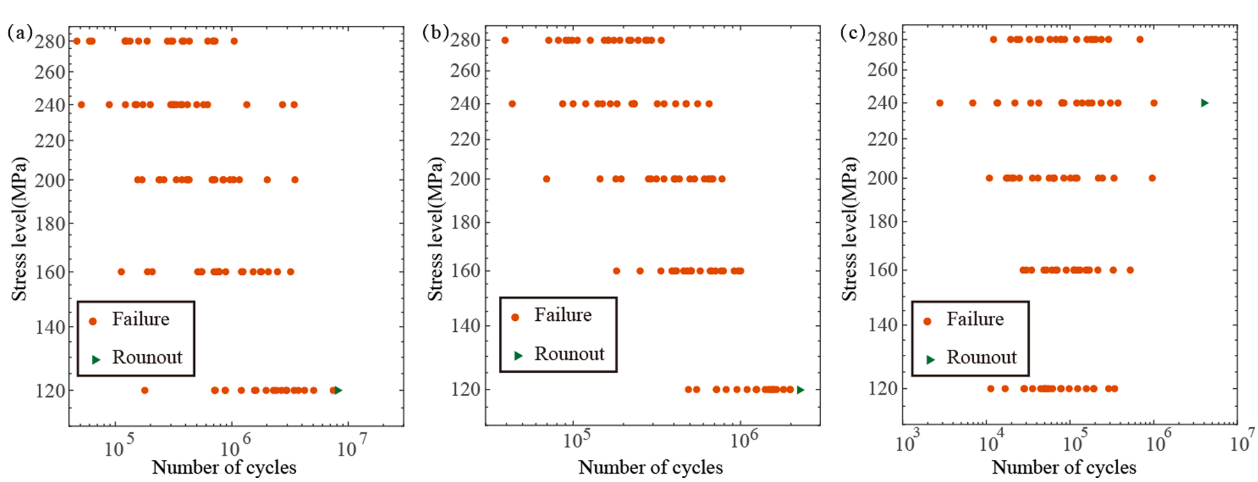

Fig. 2. Simulated fatigue data of the stress-cycle life relationship with different distribution models. The given three distribution underlying distributions are (a) Lognormal, (b) Weibull, and (c) Log-logistic, respectively.

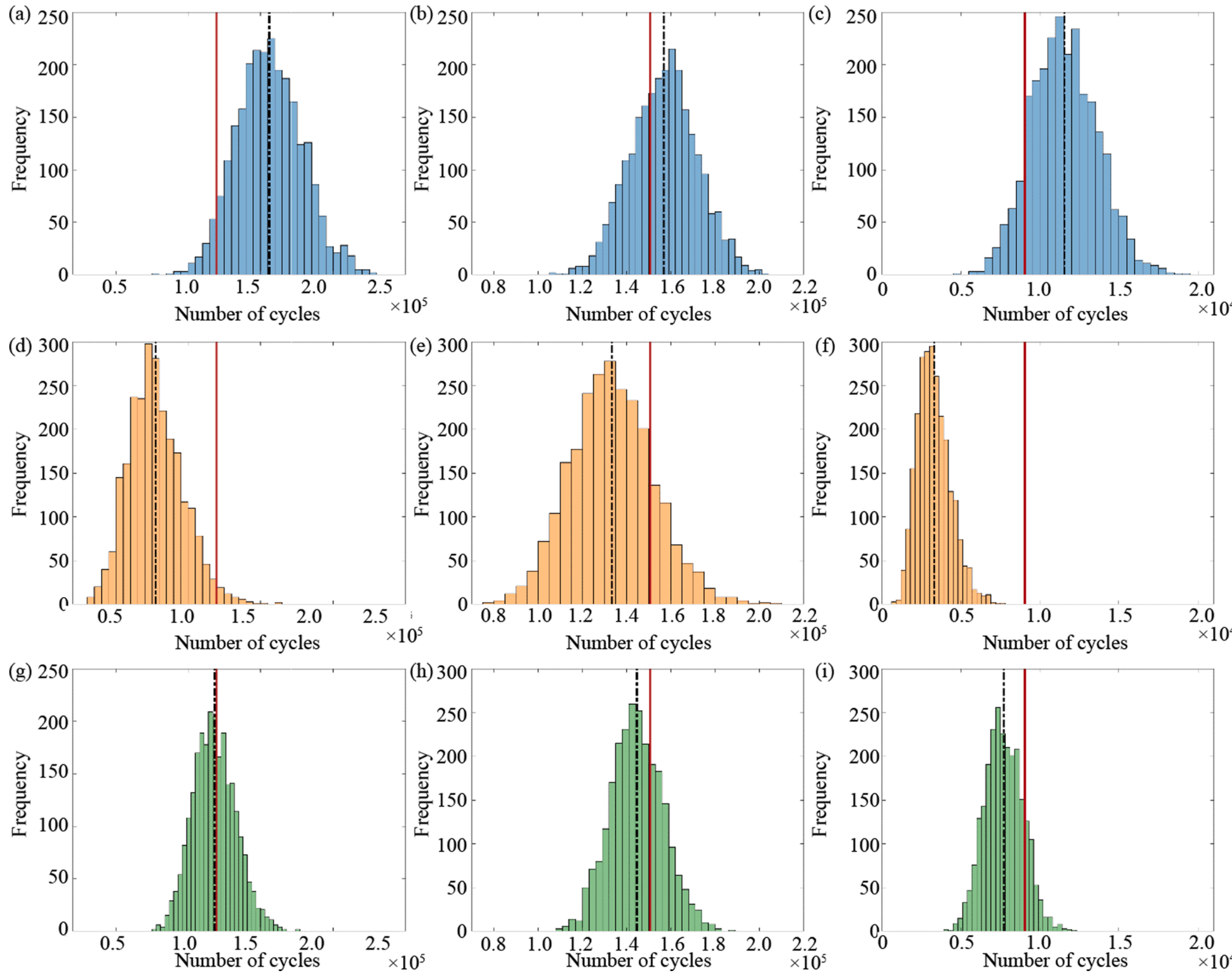

Fig. 3. Predictability of three different models on 0.05 quantile of log-life simulated from three different underlying distributions. Prediction results from three models are: (a ∼ c) the Bayesian Lognormal regression model, (d ∼ f) the Bayesian Weibull regression model, and (g ∼ i) the BMA model. The fatigue data simulated from three different underlying distribution models are: (a, d, g) the Lognormal distribution, (b, e, h) the Weibull distribution, and (c, f, i) the Log-logistic distribution. The solid red line represents the true value of the $\ln t_{0.025}$, and the black dotted line represents the mean of the posterior samples. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

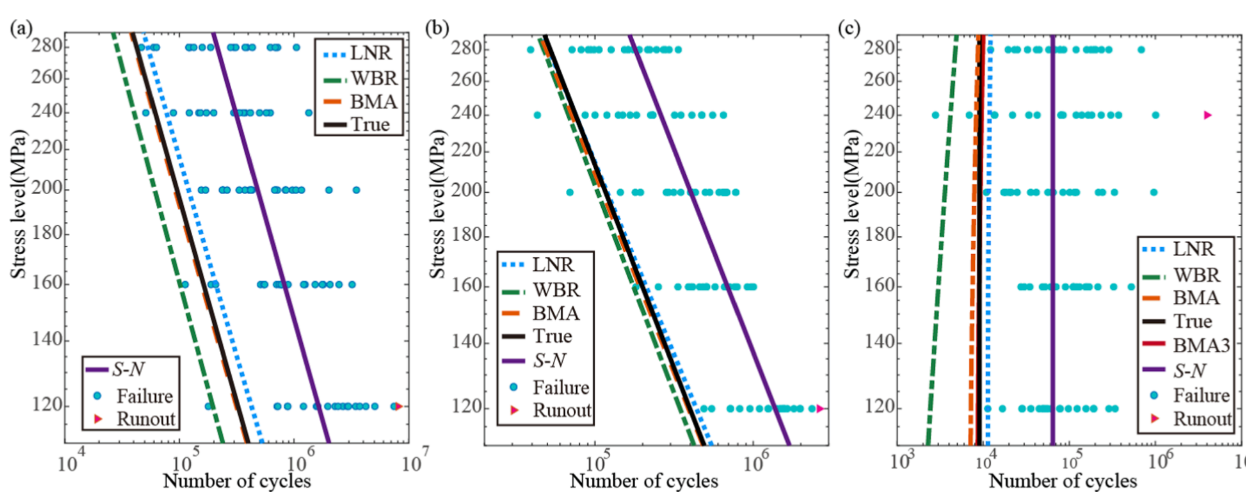

Fig. 4. Validation on the probability prediction of *P-S-N* curves (5% failure probability). Simulated data are generated from three different underlying distributions. Specifically, (a) the Lognormal distribution, (b) the Weibull distribution, and (c) the Log-logistic distribution.

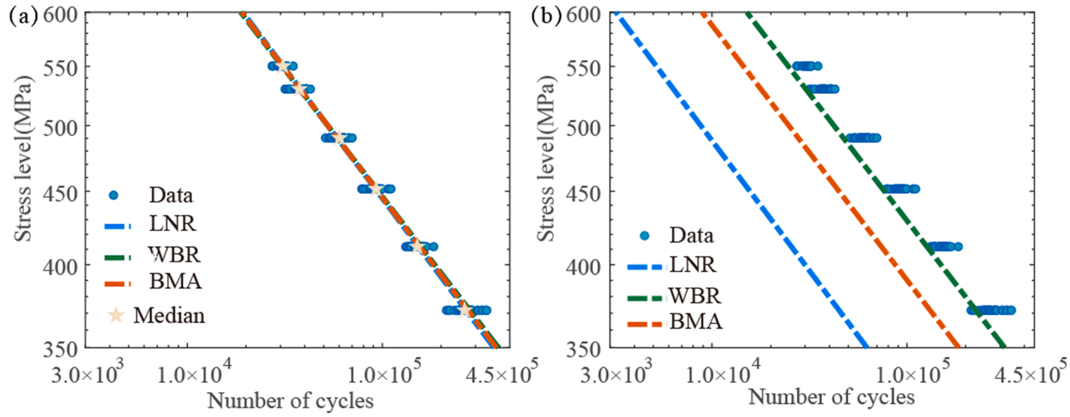

Fig. 5. Predictability to the probability *S-N* curves of the experimental fatigue data in the 2024-T4 aluminum alloy. (a) *S-N* curves, (b) *P-S-N* curves with 5% failure probability.

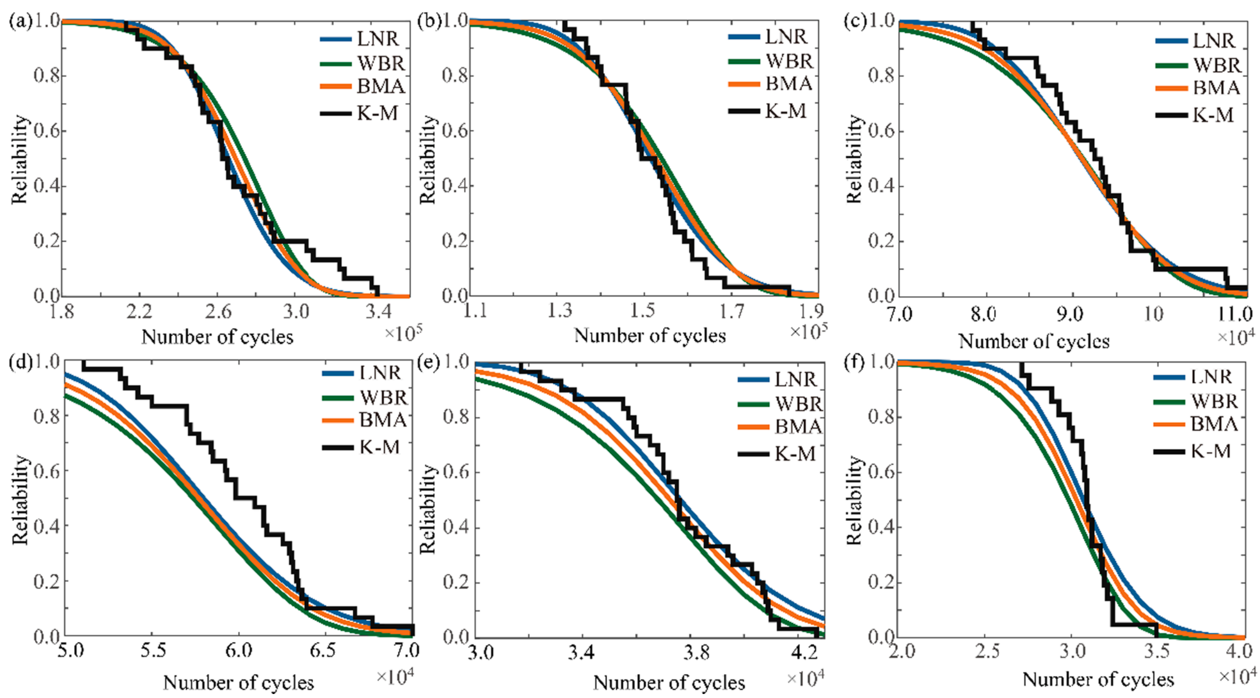

Fig. 6. Reliability of the 2024-T4 aluminum alloy at different stress levels. The corresponding stress levels are: (a) 371.7 MPa, (b) 411.7 MPa, (c) 451.7 MPa, (d) 490.3 MPa, (e) 530.3 MPa, and (f) 550.3 MPa, respectively.

applied different underlying distributions (Weibull and Log-logistic distributions) to check the predictability of the three regression models in Fig. 3. In the case of the Weibull underlying distribution, the true value and its corresponding three prediction results, as shown in Fig. 3b, 3e, and 3 h, are 150621, 156327, 133350, and 144400, respectively. As for the Log-logistic underlying distribution, the true value and its three prediction results, as shown in Fig. 3c, 3f, and 3i, are 9046, 11544, 3292, and 7709, respectively. Similarly, the LNR model demonstrates an overestimation, while the WBR model yields substantially lower estimations. The BMA model shows the highest robustness of the predictability, with predictive results relatively close to the true values and without overestimating reliability.

The prediction results on the 95 % probability of survival *P-S-N* curves are presented in Fig. 4. The subplots (Fig. 4a, 4b, and 4c) display the true *P-S-N* curves with a 5 % failure probability, along with the estimated *P-S-N* curve from each model on fatigue data generated from different underlying distributions. Specifically, for the fatigue data with the underlying Lognormal distribution form (Fig. 4a), the BMA model, which combines a weighted average of the Lognormal and Weibull regression models, provides a best evaluation of the failure probability compared with the LNR and WBR models. The LNR model overestimates the *P-S-N* curves with a 5 % failure probability. The WBR regression model shows the poorest predictability among the three models. Similarly, the BMA model can moderately evaluate the probability of the fatigue data with the underlying Weibull distribution in Fig. 4b.

For the simulation data generated from a Log-logistic distribution model, which falls outside the scope of the given models (LNR and WBR) in this paper, the estimated results from the LNR model are closer to the true values that those from the WBR model. This is because the Lognormal and Log-logistic distributions are very similar in shape [30]. To further evaluate the performance of the proposed model, the Log-logistic regression model is added to the model space of BMA3. In other words, the BMA3 model incorporates the Lognormal regression model, the Weibull regression model, and the Log-logistic regression model. The effectiveness of the BMA3 model is evident in Fig. 4c, where it provides more accurate prediction results for the *P-S-N* curve with 5 %

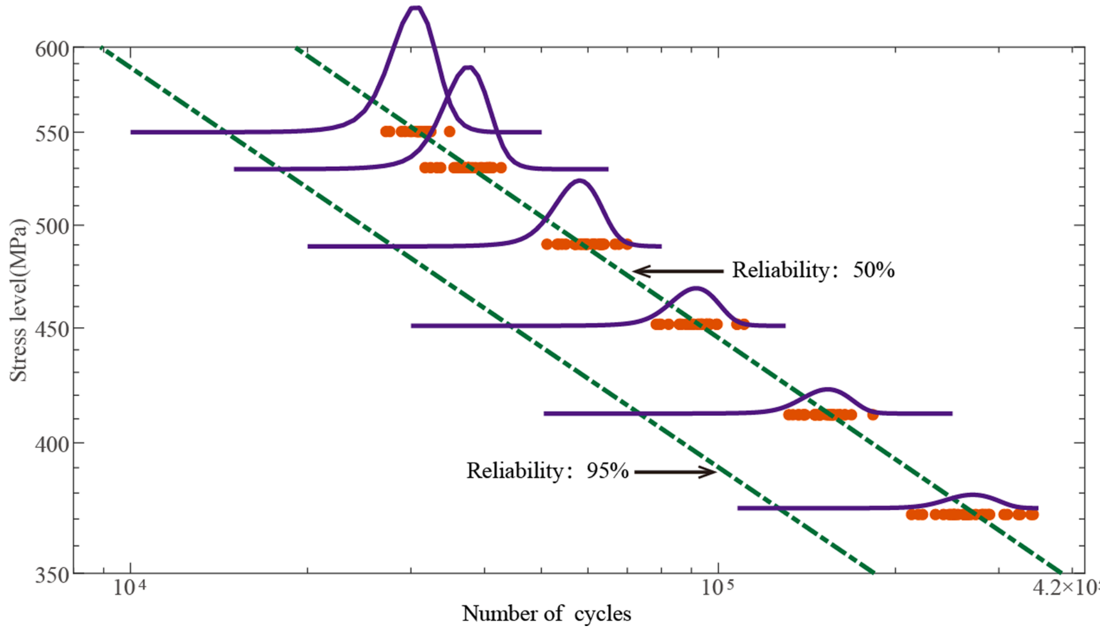

Fig. 7. Distributions of fatigue lives for the 2024-T4 aluminum alloy at different stress levels in double logarithmic coordinates. Red dots are tested fatigue data points, the purple line denotes the predicted probability density of fatigue life at each stress level from the BMA model, and green dotted lines are the predicted survival fatigue probability with the 50% and 95% reliability, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

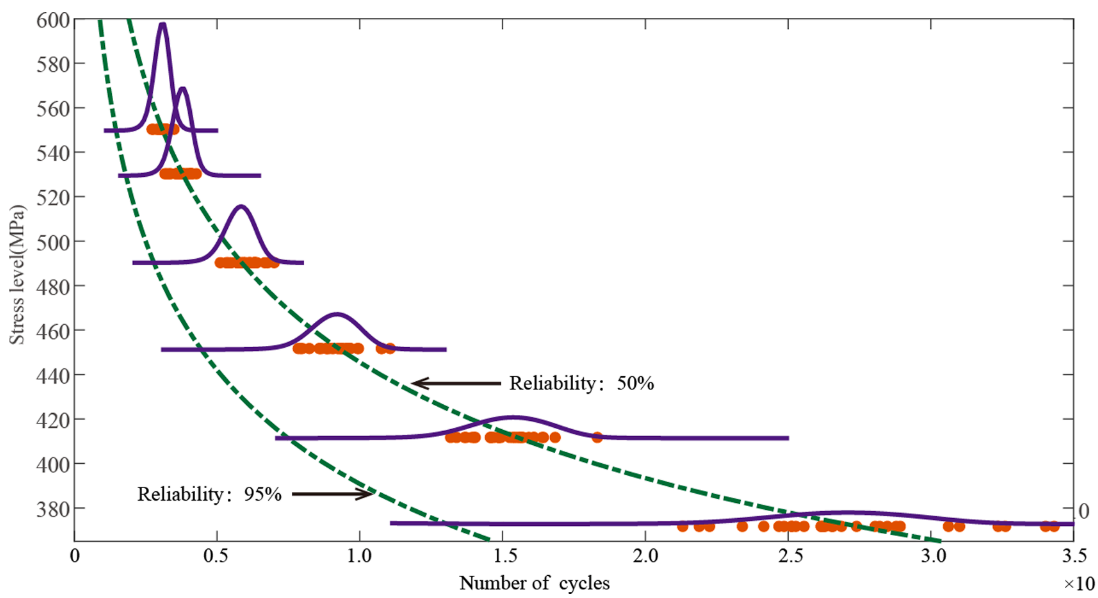

Fig. B1. Distributions of fatigue lives for the 2024-T4 aluminum alloy at different stress levels in linear coordinates. Red dots are tested fatigue data points, the purple line denotes the predicted probability density of fatigue life at each stress level from the BMA model, and green dotted lines are the predicted survival fatigue probability with the 50% and 95% reliability, respectively. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

failure probability compared to the BMA model. The BMA3 model demonstrates higher accuracy in capturing the characteristic of data simulated from the Log-logistic distribution, thereby improving the reliability of the predictions.

4.2. Verification from the experimental fatigue data

In this section, we assess the performance of the BMA model by predicting the failure probability of experimental fatigue data for the 2024-T4 aluminum alloy [53]. As depicted in Fig. 5, the dataset consists of 171 experimental data points at six different stress levels. Fig. 5a illustrates the *S-N* curves estimated by the LNR model, WBR model and BMA model. It can be observed that the estimated curves from these three models closely overlap with each other and align well with the medians of the experimental results. They all obeys the power law model (that is the *S-N* curve). The estimated *P-S-N* curves with 5 % failure probability (or 95 % survival probability) are presented in Fig. 5b. The estimation results obtained from the LNR model are significantly lower than all the experimental results, indicating a conservative estimation. Approximately five experimental data points are closely aligned with

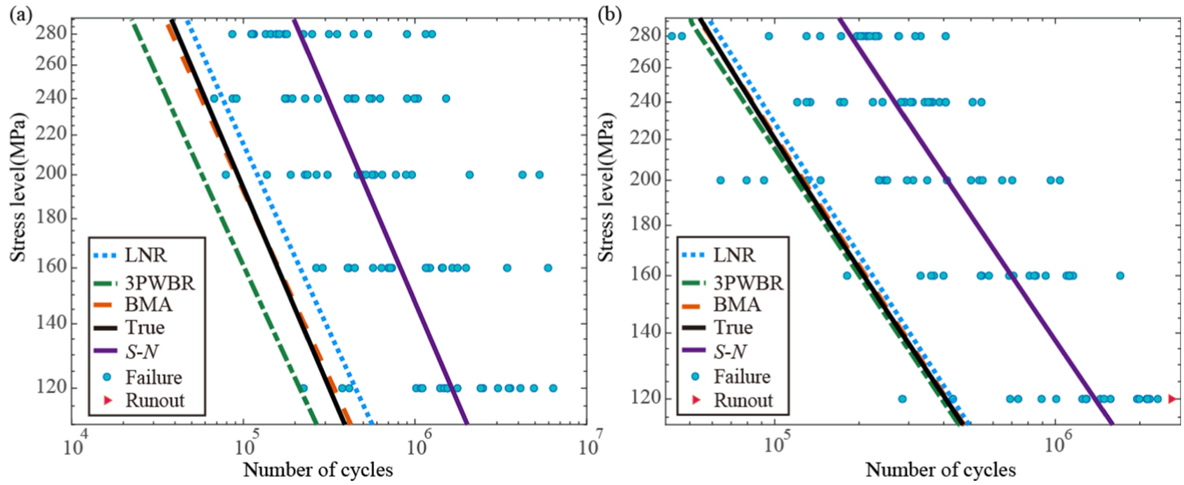

Fig. C1. Validation on the probability prediction of *P-S-N* curves (5 % failure probability). Simulated data are generated from two different underlying distributions: (a) fatigue data with the underlying Lognormal distribution, where $\sigma = 1.0, \mu = 25.8 - 2.4\ln(S)$ and (b) fatigue data with the underlying 3-parameter Weibull distribution, where shape parameter $a = 2.1$, scale parameter $b(S) = \exp(25.8 - 2.4\ln(S))$, and location parameter $\varsigma(S) = \exp(10 - 0.2\ln(S))$.

the *P-S-N* curve estimated by the WBR model, suggesting that the WBR model might be overly optimistic in its estimation. The BMA estimation, which is a weighted average of the two, is considered to be relatively appropriate.

To further validate the reliability of the BMA model on fatigue distribution property of the 2024-T4 aluminum alloy at different stress levels, we here used the Kaplan-Meier (K-M) estimator as the product limit reliability estimator for estimating the reliability function, also known as the survival function of a product's life [15,29]. What makes the K-M estimator particularly valuable is its ability to estimate the reliability without making any assumptions about the underlying distribution of failure times. Its operation involves dividing time into intervals and calculating survival probabilities for each interval. These interval-based probabilities are then multiplied to estimate the overall survival curve. As a result, it often generates a step-like, staircase pattern in its graph, reflecting changes in the survival or reliability function at observed event times.

The K-M estimator and the reliability estimates from each model across various stress levels are illustrated in Fig. 6. Overall, the reliability estimates from each model are relatively close to each other and closely resemble the K-M estimator, especially the BMA model. At high survival probabilities (or high reliability), the model estimates are relatively conservative compared to the Kaplan-Meier (K-M) estimator. However, at low survival probabilities, the model estimates tend to be more aggressive than the K-M estimator.

In Fig. 7, we showcase the comprehensive prediction performance of the BMA model in double logarithmic coordinates regarding the fatigue properties of the 2024-T4 aluminum alloy (Fig. B1 in Appendix B displays it in linear coordinates). As the applied stress level decreases within the tested materials, the corresponding fatigue life increases, and the scatter bandwidth of the fatigue life distributions tends to widen. Notably, the BMA model accurately captures the distribution evolution across different stress levels, and effectively represents the observed trend. Additionally, Fig. 7 presents the 50 % and 95 % reliability predictions, which closely align with experimental observations and findings from other studies [54–56]. These findings further support the robustness and reliability of the BMA model in assessing the fatigue behavior of the 2024-T4 aluminum alloy.

5. Discussions and conclusions

Reliability analysis of engineering components or structures heavily relies on accurately capturing the fatigue properties of materials. However, the fatigue life associated with stress exhibits significant variability, leading to wide scatter in the data. There exists considerable uncertainty regarding both the distribution form and value within the fatigue data. These make great challenges to construct a robust probability fatigue model for materials and accurately predict the service lives of components and structures. The 'classical' method typically involves fitting a single distribution to the observed fatigue life data. However, the predictability of the classical method heavily relies on the precise understanding of the fatigue mechanisms of practical engineering materials or a large amount of fatigue life data. Consequently, its applicability is relatively poor when dealing with different materials.

To address the uncertainty in model selection and the lack of consensus on the best representation of the data, we propose the use of Bayesian model averaging to incorporate model form uncertainty into the estimation of the probability density of fatigue life [46]. By considering a range of possible distributions, the BMA method offers a more flexible and robust approach that can better account for uncertainty and variability in the data. This is particularly when the true underlying distribution is not well-known or when there is significant variability in the fatigue life data. We demonstrate the performance of BMA through numerical experiments using simulated and real data. The results highlight the robustness and reliability of BMA compared to individual models, as it effectively incorporates model form uncertainty.

To further enhance the approach, it is valuable to broaden the scope of candidate models within the model space. The 3-parameter Weibull distribution represents another fitting option for fatigue life probability estimation [57,58]. It offers improved fitting capabilities to data exhibiting varying hazard rates over time, achieved through the inclusion of an additional location parameter when compared to the 2-parameter Weibull distribution. Subsequently, we proceed to validate the predictability of distinct models: the LNR model, the 3-parameter Weibull regression model (3PWBR), the BMA model containing both LNR model and 3PWBR model. For this validation, two sets of simulated fatigue data are utilized- one characterized by an underlying Lognormal distribution (Fig. C1a), and the other by a 3-parameter Weibull distribution (Fig. C1b). Fig. C1a illustrates that the 3PWBR model deviates significantly in predicting the 5 % failure probability for the fatigue data governed by the underlying Lognormal distribution. In contrast, the BMA model demonstrates precise prediction of the 5 % failure probability with high accuracy for the both sets of fatigue data with different underlying distributions in Fig. C1a and Fig. C1b. By considering various potential model forms and parameter configurations, BMA can comprehensively account for underlying uncertainties and variabilities. This expanded model space enables BMA to provide more comprehensive and robust model inference and predictions, effectively addressing the uncertainty associated with the true model.

CRediT authorship contribution statement

Qingrong Zou: Conceptualization, Methodology, Data curation, Writing – original draft. **Jici Wen:** Conceptualization, Visualization, Supervision, Writing – review & editing.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data that has been used is confidential.

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Appendix A:. Log-logistic distribution

The Log-logistic distribution with shape parameter $k > 0$ and scale parameter $\lambda > 0$ has probability density function

$$f(t|k,\lambda) = \frac{\frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}}{\left[1+\left(\frac{t}{\lambda}\right)^{k}\right]^{2}}$$

The probability distribution function is

$$F(t|k,\lambda) = 1 - \frac{1}{1+\left(\frac{t}{\lambda}\right)^{k}}$$

In Section 4.1, to generate simulation data, we introduce the stress using $\ln\lambda = \beta_0 + \beta_1 x$, where x represents the logarithm of the stress level.

Appendix B:. Distributions of fatigue lives for the 2024-T4 aluminum alloy in linear coordinates

(See Fig. B1)

Appendix C:. Modeling *S-N* curves with 3-parameter Weibull distribution

The three-parameter Weibull is often proposed in the fatigue literature as an appropriate model [57,58]. The cumulative distribution function has the form

$$F(t_i|a,b,\zeta) = 1 - \exp\left\{-\left[\frac{t_i - \zeta(S_i)}{b(S_i)}\right]^{a}\right\}. \tag{C.1}$$

where a is the shape parameter for fatigue life; $b(S_i)$ is the scale parameter describe by following relationship: $\ln(b(S_i)) = \alpha_0 + \alpha_1 \cdot \ln(S_i)$; $\zeta(S_i)$ is the location parameter describe by following relationship: $\ln(\zeta(S_i)) = \lambda_0 + \lambda_1 \ln(S_i)$. The 3-parameter Weibull appears to be attractive because the location parameter $\zeta(S_i)$ defines a non-zero lower bound on the sample space.

Considering the aforementioned 3-parameter Weibull regression model (3PWBR) as a candidate model within the BMA framework, the parameters can be estimated by Bayesian analysis, as introduced in Section 3.2. The weight of the 3PWBR model can be approximately calculated based on Eq. (16) and Eq. (18). Subsequently, the p quantile of the fatigue life can be obtained using Eq. (14).

(See Fig. C1).

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